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# ON WEYL FRACTIONAL INTEGRAL OPERATORS AND GENERALIZED VOIGT FUNCTIONS

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**ABSTRACT**

In a recent paper Pathan and Shahwan [Demonstratio Math. 39 (2006)] gave representations of generalized Voigt functions and their extensions in terms of series and integrals which are useful specially in situations when the parameters take on particular values. Explicit representations of these functions are given in terms of familiar special functions of one and more variables. In this paper we further extend these results and define the generalized (unified) Voigt functions of multivariables by means of an integral involving product of generalized hypergeometric functions ${}_1F^2$. We then establish a theorem exhibiting an interesting relationship existing between the Weyl fractonal integral of generalized Voigt functions and Whittaker transform of the product of exponential function and hypergeometric function ${}_1F^2$. It is also shown how the main theorem can be generalized to derive a number of known and new results involving Weyl fractional integrals, Laplace, Steiltjes and Bessel transforms. Some references of known results which follow as special cases of our theorems are also cited.

## 1. INTRODUCTION AND PRELIMINARIES

1.1 Voigt Function: Let

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{\nu+2n}}{\Gamma(n+1)\,\Gamma(\nu+n+1)}, \quad |z| < \infty, \tag{1.1}$$

be the Bessel function [22] of the first kind of order $\nu$. We note that $J_\nu(z)$ is the defining oscillatory kernel of Hankel's integral transform

$$(H_\nu f)(x) = \int_0^\infty f(t) J_\nu(xt)\, dt. \tag{1.2}$$

Furthermore, we have

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z, \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z. \tag{1.3}$$

Motivated by these relationships, Srivastava and Miller [23] and Srivastava and Chen [21], introduced and studied rather systematically a unification (and generalization) of the Voigt functions [19]

$$K(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-yt - \frac{t^2}{4}) \cos(xt)\, dt, \tag{1.4}$$

and

$$L(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-yt - \frac{t^2}{4}) \sin(xt)\, dt, \quad (x \in \mathbb{R};\ y \in \mathbb{R}^+), \tag{1.5}$$

in the form:

$$V_{\mu,\nu}(x,y) = \sqrt{\frac{x}{2}} \int_0^\infty t^\mu \exp(-yt - \frac{t^2}{4}) J_\nu(xt)\, dt, \quad (x, y \in \mathbb{R}^+; \mathrm{Re}(\mu+\nu) > -1). \tag{1.6}$$

Now from Eqs. (1.3)-(1.5), it follows that

$$K(x,y) = V_{\frac{1}{2},-\frac{1}{2}}(x,y) \text{ and } L(x,y) = V_{\frac{1}{2},\frac{1}{2}}(x,y). \tag{1.7}$$

The functions K(x, y) and L(x, y) were introduced by Voigt in 1899. Furthermore, the function K(x, y) + iL(x, y) is, except for a numerical factor, identical to the so-called plasma dispersion function, which is tabulated by Fried and Conte [6] and by Fettis et al. [5]. In any given physical problem, a numerical or analytical evaluation of the Voigt functions K(x, y) and L(x, y) (or of their aforementioned variants) is required. For a review of various mathematical properties and computational methods concerning the Voigt functions, see (for example) Reiche [19], Armstrong and Nicholls [2], Haubold and John [7], Srivastava and Miller [23], Klusch [9], Srivastava and Chen [21] and Yang [25]. These functions occur in great diversity in astrophysical spectroscopy, neutron physics, plasma physics and statistical communication theory, as well as in some areas in mathematical physics and engineering associated with multi-dimensional analysis of spectral harmonics. Recently, Pathan et al. [14] introduced and studied the multi-variable Voigt functions of the first kind, of the following form

$$K[x_1, x_2 \cdots, x_n, y] = (\pi)^{-n/2} \int_0^\infty t^{(1-n)/2} \exp(-yt - \frac{t^2}{4}) \prod_{j=1}^{n} (\cos(x_j t))dt \tag{1.8}$$

and

$$L[x_1, x_2 \cdots, x_n, y] = (\pi)^{-n/2} \int_0^\infty t^{(1-n)/2} \exp(-yt - \frac{t^2}{4}) \prod_{j=1}^{n} (\sin(x_j t))\, dt, \qquad (x_1, x_2, \cdots, x_n \in \mathbb{R};\ y \in \mathbb{R}^+). \tag{1.9}$$

The generalized (unified) multi-variable Voigt functions are defined by means of integral ([14]; p. 253 (2.4))

$$V_{\mu,\nu_1,\nu_2,\cdots,\nu_n}(x_1, x_2, \cdots, x_n, y) = \left(\frac{x_1}{2}\right)^{1/2} \left(\frac{x_2}{2}\right)^{1/2} \cdots \left(\frac{x_n}{2}\right)^{1/2} \times \int_0^\infty t^{\mu} \exp(-yt - \frac{t^2}{4}) \prod_{j=1}^{n} (J_{\nu_j}(x_j t))\, dt,$$

$$(\mu, y, x_1, x_2, \cdots, x_n \in \mathbb{R}^+;\ \mathrm{Re}(\mu + \sum_{j=1}^{n} \nu_j) > -1). \tag{1.10}$$

For n = 1, Eqs. (1.8)-(1.10) reduce to elementary integrals (1.4)-(1.6) respectively. We note the following relation ([14]; pp.253-254(2.7))

$$V\mu,\nu_1,\nu_2,\cdots,\nu_n(x_1, x_2, \cdots, x_n, y) = \frac{(2)^{\mu-\frac{1}{2}}\,(x_1)^{\nu_1+\frac{1}{2}}\,(x_2)^{\nu_2+\frac{1}{2}} \cdots (x_n)^{\nu_n+\frac{1}{2}}}{\Gamma(\nu_1+1)\,\Gamma(\nu_2+1)\cdots\Gamma(\nu_n+1)} \left\{ \Gamma(\frac{1}{2}(\mu + \sum_{j=1}^{n} \nu_i + 1)) \right.$$

$$\times \Psi_2^{(n+1)} \left[ \frac{1}{2}(\mu + \sum_{j=1}^{n} \nu_j + 1); \nu_1 + 1, \nu_2 + 1 \cdots, \nu_n + 1, \frac{1}{2}; -x_1^2, -x_2^2, \cdots, -x_n^2, y^2 \right]$$

$$-2y\Gamma(\frac{1}{2}(\mu + \sum_{j=1}^{n} \nu_j + 2))$$

$$\left. \times\ \Psi_2^{(n+1)} \left[ \frac{1}{2}(\mu + \sum_{j=1}^{n} \nu_j + 2); \nu_1 + 1, \nu_2 + 1, \cdots, \nu_n + 1, \frac{3}{2}; -x_1^2, -x_2^2, \cdots, -x_n^2, y^2 \right] \right\},$$

$$(x_1, x_2, \cdots, x_n \in \mathbb{R};\ \mu, y \in \mathbb{R}^+;\ \mathrm{Re}(\mu + \sum_{j=1}^{n} \nu_j) > -1) \tag{1.11}$$

where $\psi_2^{(n)}$ denotes Humbert's confluent hypergeometric function of *n* variables ([3],[22, p.62(11)])

$$\psi_2^{(n)}\ [a; c_1, c_2, \cdots, c_n; x_1, x_2, \cdots, x_n] = \sum_{m_1,m_2,\cdots,m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\cdots+m_n} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}}{(c_1)_{m_1} (c_2)_{m_2} \cdots (c_n)_{m_n}\, (m_1)!\, (m_2)! \cdots (m_n)!}, \tag{1.12}$$

$$(\max\{|x_1|, |x_2|, \ldots, |x_n| < \infty\})$$

1Here and throughout this work, the Pochhammer symbol $(a)_n$ is defined by

$$(a)_n = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1)(a+2)\cdots(a+n-1), & \text{if } n = 1,2,3\cdots. \end{cases} \quad (1.13)$$

Motivated by the contributions toward the unification (and generalization) of the Voigt functions see, for example [2], [5-9], [12], [14-16],[19,21,23,24] and due to the fact that this functions play a rather important role in several diverse fields of physics, we extend further the definition of Voigt functions.

A multiple generalized hypergeometric series is a hypergeometric series in two or more variables which reduces to the familiar Gaussian hypergeometric series. The generalized hypergeometric series [3,4,22]is defined as

$$\begin{aligned} {}_pF_q\left[\begin{array}{cc} \alpha_1,\alpha_2,\ldots,\alpha_p \; ; & \\ & z \\ \beta_1,\beta_2,\ldots,\beta_q \; ; & \end{array}\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\,(\alpha_2)_n\ldots(\alpha_p)_n\, z^n}{(\beta_1)_n\,(\beta_2)_n\ldots(\beta_q)_n\; n!} \\ &= {}_pF_q[\alpha_1,\alpha_2,\ldots,\alpha_p;\, \beta_1,\beta_2,\ldots,\beta_q;\, z] \end{aligned} \quad (1.14)$$

## 1.2 AN EXTENSION OF VOIGT FUNCTIONs $V_{\mu,\nu}(x,y)$

The function ( see,Pathan and Shahwan [15])

$$\Omega_{\mu,\alpha,\beta,\nu}(x,y) = \sqrt{\frac{x}{2}} \int_0^\infty t^\mu e^{-yt-t^2/4}\, {}_1F_2\left(\alpha;\, \beta, 1+\nu;\, -\frac{x^2t^2}{4}\right) dt$$

($\mu, y \in R+$, $x \in R$ and $Re(\mu + \nu) > -1$) (1.15)

defines an extension of Voigt function and exhibits the fact that

$$\Omega_{\mu,\alpha,\alpha,\nu}(x,y) = \Gamma(\nu+1)\left(\frac{2}{x}\right)^\nu V_{\mu-\nu,\nu}(x,y) \quad (1.16)$$

when $\alpha = \beta$. Infact, $J_\nu(x)$ defined by (1.1), ${}_1F_2$ and ${}_0F_1$ are contained as special cases, in the generalized hypergeometric function given by (1.14) .

The following special cases of (1.15) follow readily from the results [17, p.608(13) and (3)], respectively.

$$\Omega_{\mu,1,3/2,\beta-1}(x,y) =$$

$$\frac{\sqrt{\pi}}{2}\left(\frac{x}{2}\right)^{1-\beta}\Gamma(\beta)\int_0^\infty t^{\mu-\beta+\frac{1}{2}} e^{-yt-t^2/4}\, H_{\beta-3/2}(xt)\, dt \quad (1.17)$$

where $H_\nu(x)$ is Struve function [4]

$$\Omega_{\mu,\alpha,\alpha+1,\beta-1}(x,y) = 2^\beta \alpha \Gamma(\beta) x^{1-2\alpha} \int_0^\infty t^{\mu-2\alpha+1} e^{-yt-t^2/4}$$

$$\times [2\alpha J_{\beta-1}(xt) S_{2\alpha-\beta-1,\beta-2}(xt) - J_{\beta-2}(xt) S_{2\alpha-\beta,\beta-1}(xt)]\, dt \quad (1.18)$$

where $S_{\mu,\nu}(x)$ is Lommel function [4, p.44(13)].

## 1.3. REPRESENTATION OF $\Omega_{\mu,\alpha,\beta,\nu}(x, y)$

In the integrand of (1.15), expand ${}_1F_2$ and then integrating the resulting (absolutely convergent) series term by term using [4, p.416(24)] we get the following representation

$\Omega_{\mu,\alpha,\beta,\nu}(x, y) =$

$$\Gamma(\mu+1)\, x^{\frac{1}{2}} 2^{\mu/2} \sum_{m=0}^{\infty} \frac{(\mu+1)_m (\alpha)_m (-1)^m x^{2m}}{(\beta)_m (1+\nu)_m 2^m\, m!} e^{y^2/2}\, D_{-\mu-2m-1}(y\sqrt{2}) \quad (1.19)$$

where $D\mu(z)$ is parabolic cylinder function [4].

On the other hand, making use of the series representations of $e^{-t^2/4}$ and ${}_1F_2$ in (1.17) and then integrating the resultant (absolutely convergent) series term by term using the definition of gamma function, we get

$$\Omega_{\mu,\alpha,\beta,\nu}(x, y) = \frac{\Gamma(\mu+1)}{y^{\mu+1}} \sqrt{\frac{x}{2}} \sum_{m,n=0}^{\infty} \frac{(\mu+1)_{2m+2n} (\alpha)_m\, x^{2m} (-1)^{m+n}}{(\beta)_m (1+\nu)_m (2y)^{2m+2n}\, m!\, n!} \quad (1.20)$$

Now using $(a)_{2m+2n} = 2^{2m+2n} \left(\frac{a}{2}\right)_{m+n} \left(\frac{a+1}{2}\right)_{m+n}$, we get an alternative representation

$$\Omega_{\mu,\alpha,\beta,\nu}(x, y) = \frac{\Gamma(\mu+1)}{y^{\mu+1}} \frac{x^{\frac{1}{2}}}{\sqrt{2}} F^{2:1;0}_{0:2;0} \left[ \begin{matrix} \frac{\mu+1}{2}, \frac{\mu+2}{2} & : & \alpha & ;\_ & ; & \\ & & & & & -\frac{x^2}{y^2}, -\frac{1}{y^2} \\ \_\_\_\_ & : & \beta, 1+\nu & ;\_ & ; & \end{matrix} \right] \quad (1.21)$$

where $F^{2:1;0}_{0:2;0}$ denotes Kampé de Fériet series given by [22,p.63(16)].

## 1.4. INTEGRAL TRANSFORMS

Let $f(t)$ be a given function defined on an interval $[a, b]$, that belongs to a certain class of functions. An integral transform of f(t) is a mapping of the form,

$$T[f(t); s] \;=\; \bar{f}(s) \;=\; \int_a^b K(s,t) f(t)\, dt,$$

provided that the integral exists. $K(s, t)$ is a prescribed function, called the kernel of the transform. Among the well known transforms are the Laplace, Fourier, Hankel, Stieltjes, Whittaker and Mellin transforms. The most versatile of these, the Laplace transform has been widely used to solve differential equations and particularly problems related to heat transfer and electrical circuits. In the present work, we use the Laplace transform of $f(t)$ given by

$$L[f(t); p] = f(t) dt, \qquad \mathrm{Re}(p) > 0 \quad (1.22)$$

and the Whittaker transform

$$W[f(t);p] \;=\; \int_0^\infty e^{-\frac{1}{2}pt}(pt)^{\lambda-\frac{1}{2}}\,W_{k,m}(pt)\,f(t)\,dt \qquad (1.23)$$

where $W_{k,m}(z)$ is Whittaker function [4,18] defined by

$$W_{k,m}(z) \;=\; e^{-\frac{1}{2}z}\left[\frac{\Gamma(-2m)\,z^{\frac{1}{2}+m}}{\Gamma(\frac{1}{2}-m-k)}\,{}_1F_1\left[\frac{1}{2}+m-k;\,2m+1;\,z\right]\right.$$
$$\left.+\frac{\Gamma(2m)\,z^{\frac{1}{2}-m}}{\Gamma(\frac{1}{2}+m-k)}\,{}_1F_1\left[\frac{1}{2}-m-k;\,1-2m;\,z\right]\right] \qquad (1.24)$$

## 1.5. FRACTIONAL CALCULUS

To present some essential of fractional calculus (see [10,11,12]) we follow the following defintitions

**Definition 1**. If $\mu > 0$, the Riemann-Liouville fractional integral of order $\mu$ of the function $f(t)$ is defined by

$$I_\mu[f(t);\,y] \;=\; \frac{1}{\Gamma(\mu)}\int_0^y (y-t)^{\mu-1}\,f(t)\,dt$$

and the Weyl fractional integral of order $\mu$ of the function $f(t)$ is defined by

$$\omega_\mu[f(t);y] \;=\; \frac{1}{\Gamma(\mu)}\int_0^\infty (t-y)^{\mu-1}\,f(t)\,dt$$

In general $\mu$ and y are complex numbers.

**Definition 2**. Fractional derivatives of order $\mu$ of the function f(t) may be defined by

$$D_\mu[f(t)] \;=\; \frac{d^\mu}{dt^\mu}\,f(t) \;=\; \frac{d^n}{dt^n}\,I_{n-\mu}[f(x);t],$$
$$n-1<\mathrm{Re}(\mu)<n$$
$$=\; \frac{d^n}{dt^n}\,\omega_{n-\mu}[f(x);t], \qquad n-1<\mathrm{Re}(\mu)<n$$

so that tables of fractional integrals may be used to evaluate fractional derivatives.

## SOME THEOREMS ON WEYL FRACTIONAL INTEGRAL

First we recall a theorem by Pathan [13]

**Theorem 1.** If

$$\Phi[p] = L[f(t);p]$$

and

$$g(p) = L[h(t);p]$$

then

$$L[t^n g(t) f(t); p] = \int_p^\infty \Phi(t) h^n(t-p)\, dt$$
$$= \int_0^\infty \Phi(p+x) h^n(x)\, dx$$

provided that $f(t) \in L^2(0,\infty)$, $e^{-pt} t^n g(t) \in L^2(0,\infty)$ and $h^n(t)$ denotes the nth differential coefficient of $h(t)$ such that $h'(0) = h''(0) = \ldots = h^{n-1}(0) = 0$.

This theorem enables us to find the Laplace transform of the product of functions g(t) and f(t) in terms of Weyl fractional integral and Steiltjes transform. It is of utmost importance, therefore to obtain similar theorems to obtain a connection between Voigt function and various integral transforms. The first theorem we offer is the one we regard as the most fundamental in that it involves the generalization of Bessel function.

**Theorem 2.** If

$$\xi(x,p) = L\left[t^\mu e^{t^2/4}\, {}_1F_2\left[\alpha,\, \beta;\, 1+\nu;\, \frac{-x^2t^2}{4}\right]; p\right]$$

then

$$\omega_\sigma[t^{-\lambda}\xi(x,t); p] = \Gamma(\mu)\, p^{(\lambda-\sigma-1)/2} \int_0^\infty t^{\mu+\frac{\lambda-\sigma-1}{2}}\, e^{-\frac{1}{2}pt - t^2/4} W_{k,m}(pt)$$
$$\times\, {}_1F_2\left[\alpha,\, \beta;\, 1+\nu;\, \frac{-x^2t^2}{4}\right] dt$$

provided that Weyl fractional integral and Whittaker transform involved in (2.1) exist.

**Proof.** Consider [4, p.294(6)]

$$L[t^{\sigma-1}(t+a)^{-\lambda}; p] = \Gamma(\mu)\,(ap)^{(\lambda-\sigma-1)/2}\, e^{ap/2}\, W_{k,m}(ap)$$

where $k = \dfrac{1-\sigma-\lambda}{2}$, $m = \dfrac{\sigma-\lambda}{2}$ and $W_{k,m}$ is Whittaker function given by (1.26).

Then

$$e^{-ap}\Psi(p) = L[(t-a)^{\sigma-1}\, t^{-\lambda}\, H(t-a); p] \tag{2.2}$$

where H(t) is Heavisides unit function.

Also, we have

$$\xi(x,p) = L\left[t^\mu e^{-t^2/4}\, {}_1F_2\left[\alpha,\, \beta;\, 1+\nu;\, \frac{-x^2t^2}{4}\right]; p\right] \tag{2.3}$$

Applyiing Parseval theorem to pairs (2.2) and (2.3), we get

$$\int_a^\infty t^{-\lambda}(t-a)^{\sigma-1}\, \xi(xt)\, dt = \int_0^\infty t^\mu\, e^{-(at+t^2/4)}\, {}_1F_2\left[\alpha,\, \beta;\, 1+\nu;\, \frac{-x^2t^2}{4}\right] \psi(at)\, dt$$

which on replacing $a$ by $p$ gives (2.1).

(2.1) shows that Weyl fractional integral of Voigt function is Whittaker transform of

$$ {}_1F_2\left[\alpha,\, \beta;\, 1+\nu;\, \frac{-x^2t^2}{4}\right] $$

Choosing $\lambda = 0$ and $x = 0$ in Theorem 2, we get

**Theorem 3**. If

$$ g(p) \;=\; L[t^{\mu}\, e^{-t^2/4}\, f(t);\, p] \tag{2.4} $$

then

$$ \omega_{\sigma}[g(t);\, p] \;=\; \int_0^{\infty} t^{\mu-\sigma}\, e^{-pt-t^2/4}\, f(t)\, dt \tag{2.5} $$

provided that $\mathrm{Re}(p) > 0$, $\sigma > 0$, Weyl fractional integral of $|g(t)|$ exist and the integral involved is its R.H.S. is absolutely convergent.

**Remark**. If we choose $f(t) = J_{\nu}(xt)$, $x > 0$ in the above theorem then we find that R.H.S. of (2.5) is Voigt function $V_{\mu-\sigma}(x, p)$.

**Theorem 4.** If

$$ L[h(t);\, p] = g(p) \tag{2.6} $$

then

$$ L[e^{-a\sqrt{t}}\, h(t);\, p] \;=\; \frac{a}{2\sqrt{\pi}} \int_0^{\infty} e^{-a^2/4t}\, t^{-3/2}\; f(t)\, dt \tag{2.7} $$

provided that $\mathrm{Re}(p) > 0$, $\sigma > 0$, Weyl fractional integral of $|g(t)|$ exist and the integral involved is its R.H.S. of (2.7) is absolutely convergent.

**Remark (An Alternative Form of Voigt Integral).** Applying theorem 4, with $h(t) = t^{\mu-1} J_{\nu}(\sqrt{xt})$ and using the result [4, p.186(35)]

$$ L[t^{\mu-1} J_{\nu}(\sqrt{xt}); p] \;=\; \frac{\Gamma(\mu+\nu/2)\, t^{\nu/2}}{2^{\nu}\Gamma(nu+1)\, p^{\mu+\nu/2}}\, {}_1F_1\left[\mu+\nu/2;\, \nu+1;\, -\frac{x}{4p}\right], $$

$$ \mathrm{Re}(p) > 0, \quad \mathrm{Re}(\mu+\nu/2) > 0 $$

we get an alternative integral representation for Voigt functions in the form

$$ \int_0^{\infty} t^{2\mu-1} e^{-at-bt^2}\, J_{\nu}(xt)\, dt \;=\; \frac{a\Gamma(\mu+\nu/2)\, x^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+1)\, 2^{\nu+2}} $$

$$ \int_0^{\infty} e^{-a^2/4t}\, t^{-3/2}(t+b)^{-\mu-\nu/2}\, {}_1F_1\left[\mu+\nu/2;\, \nu+1;\, -\frac{x^2}{4(t+b)}\right]\, dt, $$

where $b,\, x \in R^{+}$ $\mathrm{Re}(a) > 0$, $\mathrm{Re}(\mu+\nu/2) > 0$,

For further investigation based on the discussions of this article we may use special integral operators as follows:

The Erdelyi fractional integral (see [20, Sec. 18.1]) for $\mathrm{Re}(\alpha) > 0$, $\sigma > 0$ $\eta \in \mathbb{C}$

and

$$
\begin{aligned}
(I^{\alpha}_{0+;\sigma,\eta}f)(x) &= \frac{\sigma x^{-\sigma(\sigma+\eta}}{\Gamma(\alpha)} \int_0^x (x^{\sigma} - t^{\sigma})^{\alpha-1}\, t^{\sigma\eta+\sigma-1}\, f(t)\, dt; \\
(I^{\alpha}_{-;\sigma,\eta}f)(x) &= \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^{\infty} (t^{\sigma} - x^{\sigma})^{\alpha-1}\, t^{\sigma(1-\alpha-\eta)-1}\, f(t)\, dt;
\end{aligned}
$$

the modified Hankel transform for $k \in R \setminus \{0\}$, $\mathrm{Re}(\eta) > -1$ and $x \in R_+$:

$$
(H_{k,\eta}f)(x) = \int_0^{\infty} (xt)^{1/k-1/2}\, J_{\eta}(|k|\,(xt)^{1/k})\, f(t)\, dt;
$$

and the modified Laplace transform for $k \in R \setminus \{0\}$, $\alpha \in C$ and $x \in R_+$:

$$
(L_{k,\alpha}f)(x) = \int_0^{\infty} (xt)^{-\alpha}\, e^{-|k|\,(xt)^{1/k})}\, f(t)\, dt.
$$

For $k = 1$, $\alpha = 0$, it reduces to Laplace transform. All these transforms are defined for continuous functions f with compact support in IR + for the range of parameter indicated.

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# ANGULAR TWIST IN A SHAFT AND MULTIVARIABLE FUNCTION

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## ABSTRACT

The aim of the present paper is to determine the product of generalized Kampé de Fériet function and the multivariable H-function in solving a solution of the following partial differential equation

$$\frac{\partial^2 U}{\partial t^2} = R^2 \frac{\partial^2 U}{\partial x^2} \qquad \text{(R is constant)}$$

of angular twist in a shaft. The result gives a number of known special cases on specializing the parameters and may prove useful in several interesting situations.

**Key word and phrases:** Kampé de Fériet function, multivariable H-function, orthogonality property of the cosine function.

## 1. INTRODUCTION

As an example of the application of the product of generalized Kampé de Fériet function and the multivariable H-function in applied mathematics, we shall consider the problem of determining angular twist U(x,t) in a shaft of circular section with its axis along the x-axis. If the ends $x = 0$ and $x = \gamma$ of the shaft are free, the twist due to initial displacement must satisfy the boundary value problem

$$\frac{\partial^2 U}{\partial t^2} = R^2 \frac{\partial^2 U}{\partial x^2} \qquad \ldots(1.1)$$

$$\frac{\partial}{\partial x} U(\theta, t) = 0, \frac{\partial}{\partial x} U(x, t) = 0, \frac{\partial}{\partial t} U(x,0) = 0 \qquad \ldots(1.2)$$

$$U(x,0) = f(x), \qquad \ldots(1.3)$$

where R is constant.

Here we shall consider

$$f(x) = \left(\sin\frac{\pi x}{2\gamma}\right)^{2\omega-\sigma-1}\left(\cos\frac{\pi x}{2\gamma}\right)^{\sigma-1}$$

$$\cdot\, S^{M:P;P'}_{N:Q;Q'}\left[\begin{matrix}[(m):\theta,\phi]:[(p):\alpha];[(p'):\beta];y\left(\frac{\tan\pi x}{2\gamma}\right)^{2h}\\ [(n):\in,t];\ [(q):\alpha'];[(q'):\beta'];z\left(\frac{\tan\pi x}{2\gamma}\right)^{2h'}\end{matrix}\right]$$

$$\cdot\, H^{0,0:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\\ [(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)};\\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\ldots,z_i\left(\tan\frac{\pi x}{2\gamma}\right)^{2k},\ldots z_r\right). \qquad \ldots(1.4)$$

The generalized Kampé de Fériet function ([21], 199) is defined as follows:

$$S^{M:P;P'}_{N:Q;Q'}\left[\begin{matrix}[(m):\theta,\phi]:[(p):\alpha];[(p'):\beta];\\ [(n):\in,t]\ :[(q):\alpha'];[(q'):\beta'];\end{matrix}\ y,z\right]$$

$$= \sum_{s,\rho=0}^{\infty} E_{s,\rho}\, y^s z^\rho \qquad \ldots(1.5)$$

where $E_{s,\rho}$ stands for the expression

$$\frac{\prod_{j=1}^{M}\Gamma[m_j+s\theta_j+\rho\phi_j]\prod_{j=1}^{P}\Gamma[p_j+\alpha_j s]\prod_{j=1}^{P'}\Gamma[p'_j+\beta_j\rho]}{\prod_{j=1}^{N}\Gamma[n_j+s\in_j+t_j\rho]\prod_{j=1}^{Q}\Gamma[q_j+\alpha'_j\rho)\prod_{j=1}^{Q'}\Gamma[q'_j+\beta'_j\rho]}, \qquad \ldots(1.6)$$

where, for convergence

$$T = 1 + \sum_{j=1}^{N} \epsilon_j + \sum_{j=1}^{Q} \alpha'_j - \sum_{j=1}^{M} \theta_j - \sum_{j=1}^{P} \alpha_j > 0,$$

$$= 1 + \sum_{j=1}^{N} \delta_j + \sum_{j=1}^{Q'} \beta'_j - \sum_{j=1}^{M} \phi_j - \sum_{j=1}^{P'} \beta_j > 0,$$

m stands for the sequence of M parameters $m_1, m_2, \ldots, m_M$ and, for the sake of brevity, the function in (1.5) will be denoted by $S^{M:P;P'}_{N:Q;Q'}\begin{bmatrix} y \\ z \end{bmatrix}$

The H-function of several complex variable [20] will be denoted by H $(z_1, z_r)$

$$= H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1,\ldots,z_r\right).$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\cdots\int_{L_r} U_1(s_1)\ldots U_r(s_r)\, V(s_1,\ldots,s_r)\, z_1^{s_1}\ldots z_s^{s_r}$$

$$ds_1 \ldots ds_r,\ \omega = \sqrt{-1}, \qquad \ldots(1.7)$$

where

$$U_i(s_i) = \frac{\prod_{j=1}^{u^{(i)}} \Gamma[(d_j^{(i)} - \delta_j^{(i)} s_i]\prod_{j=1}^{v^{(i)}} \Gamma[1 - b_j^{(i)} + \phi_j^{(i)} s_i)}{\prod_{j=1+u^{(i)}}^{D^{(i)}} \Gamma[1 - d_j^{(i)} + \delta_j^{(i)} s_i]\prod_{j=1+v^{(i)}}^{B^{(i)}} \Gamma[b_j^{(i)} - \phi_j^{(i)} s_i]},\ i = 1,\ldots,r$$

$$V(s_i,...,s_r)=\frac{\prod_{j=1}^{\lambda}\Gamma\left[1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}s_i\right]}{\prod_{j=\lambda+1}^{A}\Gamma\left[a_j-\sum_{i=1}^{r}\theta_j^{(i)}s_i\right]\prod_{j=1}^{C}\Gamma\left[1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}s_i\right]}$$

an empty product is interpreted as unity, the coefficient $\theta_j^{(i)}, j=1,...,A$; $\phi_j^{(i)}, j=1,...,B^{(i)}; \psi_j^{(i)}, j=1,...,C; \delta_j^{(i)}, j=1,...,D^{(i)};$ and $i=1,...,r,$ are positive numbers, and $\lambda, u^{(i)}, v^{(i)}, A, B^{(i)}, C, D^{(i)}$ are integers such that $0\le\lambda\le A, 0\le u^{(i)}\le D^{(i)}, C\ge 0$ and $0\le v^{(i)}\le B^{(i)}, i=1,...,r$. The contour $L_i$ in the complex plane $s_i$-plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $+\omega\infty$ with indentation, if necessary, in such a way, that all the poles of $\Gamma[d_j^{(i)}-\delta_j^{(i)}s_i], j=1,...,u^{(i)},$ are to the right and those of $\Gamma\{1-b_j^{(i)}+\phi_j^{(i)}s_i], j=1,...,v^{(i)}, \Gamma\left[1-a_j+\sum_{j=1}^{r}\theta_j^{(i)}s_i\right], j=1,...,\lambda$, to the left, of $L_i$ the various parameters being so restricted that these pole are all simple and none of them coincide; and with the points $z_i = 0$, $i = 1,...,r$, being tacitly excluded, the multiple integral in (1.7) converges absolutely if

$$|\arg(z_i)|<\frac{1}{2}T_i\,\pi,\ i=1,...,r,$$

where

$$T_i=\sum_{j=1}^{v^{(i)}}\phi_j^{(i)}-\sum_{i=1+v^{(i)}}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{u^{(i)}}\delta_j^{(i)}-\sum_{i=1+u^{(i)}}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=\in+1}^{A}\theta_j^{(i)}>0, i=1,...,r \quad ...(1.8)$$

## 2. THE MAIN INTEGRAL

The main integral to be derived here is

$$\int_0^{\gamma}\left(\cos\frac{\pi\omega x}{\gamma}\right)\sin\left(\frac{\pi x}{2\gamma}\right)^{2\omega-\sigma-1}\left(\cos\frac{\pi x}{2\gamma}\right)^{\sigma-1}$$

$$S^{M:P;P'}_{N:Q;Q'}\left(y\left(\tan\frac{\pi x}{2\gamma}\right)^{2h}, z\left(\tan\frac{\pi x}{2\gamma}\right)^{2h'}\right)$$

$$H\left(z_1,\dots,z_i\left(\tan\frac{\pi x}{2\gamma}\right)^{2k},\dots z_r\right)dx$$

$$=\sum_{s,\rho=0}^{\infty} E_{s,\rho}\frac{y^s z^\rho.\gamma^{2\omega-\sigma+2hs+2h'\rho}}{s!\ \rho!\sqrt{\pi}\,\Gamma(2\omega)}$$

$$H^{0,0:(u',v');\dots;(u^{(i)}+1,\,v^{(i)}+1);\dots;(u^{(r)},\,v^{(r)}}_{A,C:[B',D'];\dots;[B^{(i)}+2,\,D^{(i)}+1];\dots;[B^{(r)},D^{(i)}]}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]: \\ [(c):\psi',\dots,\psi^{(r)}]:\end{matrix}\right.$$

$$[(b'):\phi'];\dots;[1-\omega+\frac{\sigma}{2}-hs-h'\rho:k],[(b^{(i)}):\phi^{(i)}],$$

$$[(d'):\delta'];\dots;[\sigma\text{-}2hs-2h'\rho;2k],[(d^{(i)}):\delta^{(i)}],$$

$$\left.\begin{matrix}[\frac{1}{2}+\frac{\sigma}{2}-\omega-hs-h'\rho:k];\dots;[(b^{(r)}):\phi^{(r)}]; & z_1 \\ & \vdots \\ ;\dots;[(d^{(r)}):\delta^{(r)}]; & z_i\,4^k \\ & \vdots \\ & z_r\end{matrix}\right), \qquad \dots(2.1)$$

where $h>0, h'>0, k>0, 2\omega>\mathrm{Re}\left(\omega+\sum_{i=1}^{r}\ 2k\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$

$j=1, u^{(i)}$ and $E_{s,\rho}$, is given in (1.6).

**Proof:** To derive (2.1), express the generalized Kampé de Fériet function in series with the help of (1.5) and the multivariables H-function into the Mellin-Barnes type contour integral (1.7), then change the order of summations and integration, which is justified due to the absolute convergence of the integral involved in the process, and then evaluate the inner integral with the help of a result of Chaurasia and Sharma [10] eqn.(5), p.229).

Finally interpreting by vision of (1.7), we arrive at the desired result.

### 3. SOLUTION

$$u(x,t)=\frac{1}{2^{\sigma}\sqrt{\pi}}\sum_{\xi,s,\rho=0}^{\infty}E_{s,\rho}\frac{y^{s}z^{\rho}2^{2\xi+2hs+2h'\rho+1}}{s!\,\rho!\,\Gamma(2\xi)}$$

$$\cdot H^{0,0:(u',v');\ldots;(u^{(i)}+1,v^{(i)}+1);\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(i)}+2,D^{(i)}+1];\ldots;[B^{(r)},D^{(i)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\\ [(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\right.$$

$$[(b'):\phi'];\ldots;[1-\xi+\frac{\sigma}{2}-hs-h'\rho:k],[(b^{(i)}):\phi^{(i)}],$$

$$[(d'):\delta'];\ldots;[\sigma\text{-}2hs-2h'\rho;2k],[(d^{(i)}):\delta^{(i)}],$$

$$\left.\begin{matrix}[\frac{1}{2}+\frac{\sigma}{2}-\xi-hs-h'\rho:k];\ldots;[(b^{(r)}):\phi^{(r)}]; & z_1\\ & \vdots\\ ;\ldots;[(d^{(r)}):\delta^{(r)}]; & z_i\,4^{k}\\ & \vdots\\ & z_r\end{matrix}\right)$$

$$\cos\left(\frac{\xi\pi x}{\gamma}\right)\cos\left(\frac{\xi\pi Rt}{\gamma}\right). \qquad \ldots(3.1)$$

where $\text{Re}(\omega)>0$.

**Proof:** The solution of the problem can be written as [Churchill 1941, p.125(4)]:

$$u(x,t)=\frac{1}{2}a_0+\sum_{\xi=1}^{\infty}L_{\xi}\cos\frac{\pi\xi x}{\gamma}\cos\frac{\pi\xi Rt}{\gamma} \qquad \ldots(3.2)$$

where $L_{\xi}\,(\xi=0,1,2,\ldots)$ are the coefficients in the Fourier cosine series for f(x) in the interval (0,). If t = 0, then by virtue of (3.2), we have

$$\left(\sin\frac{\pi x}{2\gamma}\right)^{2\omega-\sigma-1}\left(\cos\frac{\pi x}{2\gamma}\right)^{\sigma-1}S^{M:P;P'}_{N:Q;Q'}\left(y\left(\tan\frac{\pi x}{2\gamma}\right)^{2h},z\left(\tan\frac{\pi x}{2\gamma}\right)^{2h'}\right)$$

$$H\left(z_1,\ldots,z_i\left(\tan\frac{\pi x}{2\gamma}\right)^{2k},\ldots,z_r\right)$$

$$=\frac{1}{2}a_0+\sum_{\xi=1}^{\infty}L_\xi\frac{\pi\xi x}{\gamma} \qquad \ldots(3.3)$$

Now multiplying both sides of (3.3) by $\frac{\cos\pi\xi x}{\gamma}$ and integrating with respect to x from 0 to $\gamma$, we find

$$\int_0^\gamma\left(\cos\frac{\pi\omega x}{\gamma}\right)\sin\left(\frac{\pi x}{2\gamma}\right)^{2\omega-\sigma-1}\left(\cos\frac{\pi x}{2\gamma}\right)^{\sigma-1}$$

$$S_{N:Q;Q'}^{M:P;P'}\left(y\left(\tan\frac{\pi x}{2\gamma}\right)^{2h},z\left(\tan\frac{\pi x}{2\gamma}\right)^{2h'}\right)$$

$$H\left(z_1,\ldots,z_i\left(\tan\frac{\pi x}{2\gamma}\right)^{2k},\ldots,z_r\right)dx$$

$$=\frac{1}{2}a_0\int_0^\gamma\cos\frac{\pi\omega x}{\gamma}dx+\sum_{\xi=1}^{\infty}L_\xi\int_0^\gamma\cos\frac{\pi\xi x}{\gamma}\cos\frac{\pi\omega x}{\gamma}dx \qquad \ldots(3.4)$$

Now using (2.1) along with orthogonality property of the cosine function, we have

$$L_\xi=\sum_{\xi,s,\rho=0}^{\infty}E_{s,\rho}\frac{y^s z^\rho\ 2^{2\xi-\sigma+hs+h'\rho+1}}{\sqrt{\pi}\,\Gamma(2\xi)}$$

$$H^{0,0:(u',v');\ldots;(u^{(i)}+1,\,v^{(i)}+1);\ldots;(u^{(r)},\,v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(i)}+2,D^{(i)}+1];\ldots;[B^{(r)},D^{(i)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\\ [(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\right.$$

$$[(b'):\phi'];\ldots;[1-\xi+\frac{\sigma}{2}-hs-h'\rho:k],[(b^{(i)}):\phi^{(i)}],$$

$$[(d'):\delta'];\ldots;[\sigma-2hs-2h'\rho;2k],[(d^{(i)}):\delta^{(i)}],$$

$$\left.\begin{matrix}[\frac{1}{2}+\frac{\sigma}{2}-\xi-hs-h'\rho:k];...;[(b^{(r)}):\phi^{(r)}]; & z_1 \\ & \vdots \\ ;...;[(d^{(r)}):\delta^{(r)}]; & z_i 4^k \\ & \vdots \\ & z_r\end{matrix}\right), \qquad ...(3.5)$$

With the help of (3.2) and (3.5), the solution (3.1) is obtained.

## 4. SPECIAL CASES

(i) For r = 2, the result in (3.1) reduces to result for the H-function of two variables (Mittal and Gupta [16]).

$$u(x,t)=\frac{1}{2^{\sigma}\sqrt{\pi}}\sum_{\xi,s,\rho=0}^{\infty}E_{s,\rho}\frac{y^s z^{\rho}.2^{2\xi+2hs+2h'\rho+1}}{s!\,\rho!\,\Gamma(2\xi)}$$

$$H^{0,0:(u'+1,\,v'+1);(u'',v''):}_{A,C:[B'+2,D'+1];[B'',D'];}\left[\begin{matrix}z_1\,4k \\ z_2\end{matrix}\middle|\begin{matrix}[(a):\theta',\theta''] \\ [(c):\psi',\psi'']:\end{matrix}\right.$$

$$\left.\begin{matrix}1-\xi+\frac{\sigma}{2}-hs-h'\rho:k],[(b'):\phi'];\left[\frac{1}{2}+\frac{\sigma}{2}-\xi-hs-h'\rho:k\right];[(b''):\phi''] \\ .\ [\sigma-2hs-2h'\rho;2k],[(d'):\delta']; \qquad ;\ [(d''):\delta'']\end{matrix}\right]$$

$$.\ \cos\left(\frac{\xi\pi x}{\gamma}\right)\cos\left(\frac{\xi\pi Rt}{\gamma}\right), \qquad ...(4.1)$$

where $\mathrm{Re}(\sigma)>0$.

(ii) Taking $h\to 0$, and $h'\to 0$ in (4.1), we find a known result of Chaurasia [7].

(iii) Letting $h\to 0, h'\to 0, A=C=v''=\beta''=d''=0, u''=D''=\delta''=\phi''=\delta'=1$ and making $z_2\to 0$, the result in (3.1) reduces to a known result of Bajpai [3] in view of a known formula Chaurasia [4] with $C_i = D_i = 1$.

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## 5. CONCLUSION

In view of the generality of the multivariable H-function and the generalized Kampé de Fériet function, the results obtained in this paper are of a general character and may prove to be useful in several interesting situation appearing in the literature on applied mathematics and mathematical physics.

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# ON THE EXACT SOLITON SOLUTIONS FOR THE KLEIN-GORDON EQUATION WITH CUBIC NONLINEARITY

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## ABSTRACT

The First integral method is used to obtain the exact travelling wave solutions for the Klein-Gordon equation with cubic nonlinearity. It is shown that the method is effective and direct method, based on the ring theory of commutative algebra.

***Keywords:*** Travelling wave, Klein-Gordon equation, First Integral Method.
**Mathematics Subject Classification (2000):** 35C08.81Q05.

## 1. INTRODUCTION

The soliton solutions of nonlinear partial differential equations (NPDEs) in mathematical physics has become an active area of research since past several decades, due to the development of symbolic computer softwares, such as Mathematica or Maple. Using these software's the users can do the tedious and complicated algebraic calculations on the computer.

A variety of ansatz methods, such as homogeneous balance method [10,22], inverse scattering method [1], modified extended tanh function method [8,9], bilinear transformation [16], tanh-coth method [23], exp-function method [3,14,25], Sine function method [13] have been used to obtain the exact travelling wave solutions of NPDEs. Hereman W, et al. [15], obtained the exact travelling wave solutions of Klein-Gordon equation with cubic nonlinearity by using direct algebraic method. Ye C and Zhang W [24], obtained exact travelling wave solutions of Klein-Gordon equation with cubic nonlinearity by using the bifurcation method and qualitative theory of dynamical systems. Dehghan M and Shokri Ali [6], obtained numerical solutions of Klein-Gordon equation with quadratic and cubic nonlinearity by using radial basis function and analyze the accuracy of their results with the analytical solutions. Jang B [17], obtained the travelling wave solutions of nonlinear Klein-Gordon equations. Jia-Min Zhu [18], investigated the simble algebraic transformation relation between the generalized variable coefficient KdV equation and a generalized cubic nonlinear Klein–Gordon equation. In a pioneering work Feng [12], proposed a new ansatz method, First Integral Method, based on the ring theory of commutative algebra. Recently, The First Integral Method have been used to obtain the exact solitary wave solutions of numerous nonlinear partial differential equations [2,20,21].

In this paper, we extend the application of First Integral Method to find the exact travelling wave solutions for the Klein-Gordon equation with cubic nonlinearity. In section 2, we proposed the analysis of the method for finding exact travelling wave solutions of NLPDEs. In section 3, we established the exact travelling wave solution for the Klein-Gordon equation with cubic nonlinearity. Finally, in section 4, conclusions of the analysis are given.

## 2. BASIC IDEA OF THE FIRST INTEGRAL METHOD

Let us recall the basic idea of the First Integral Method and division theorem [21].

Consider the nonlinear partial differential equation of the form

$$F(u,\ u_t,\ u_x,\ u_{xx},\ u_{tt},\ u_{xt},\ u_{xxx},\ \ldots) = 0 \tag{1}$$

where $u(x,\ t)$ is the solution of nonlinear partial differential equation (1). The transformations

$$u(x,\ t) = f(\xi),\ \text{where}\ \xi = x - ct \tag{2}$$

enables us to use the following changes

$$\frac{\partial}{\partial t}(.) = c\frac{d}{d\xi}(.);\frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.)\frac{\partial^2}{\partial x^2}(.) = \frac{d^2}{d\xi^2}(.)\ldots. \tag{3}$$

Using equation (3), we can transform equation (1) to the following equation

$$G(f, f_\xi, f_{\xi\xi}, \ldots) = 0 \tag{4}$$

Introducing new independent variables

$$X(\xi) = f(\xi), Y(\xi) = f_\xi(\xi) \tag{5}$$

we obtain the following system of ordinary differential equations(ODEs)

$$\begin{cases} X_\xi(\xi) = Y(\xi) \\ Y_\xi(\xi) = F(X(\xi), Y(\xi)) \end{cases} \tag{6}$$

From the qualitative theory of ODEs [7], if one can find the integrals of Eq. (6), then the general solution of Eq. (6) can be obtained directly. However, in general, it rarely happens, because for a given plane autonomous system, there is no systematic theory that tell us how to find its first integrals, also there is no logical way which tells what these first integrals are. So, we will apply the Division Theorem to obtain the first integral of equations (6) under various parameter conditions, which reduces (4) to a first order integrable ODE. An exact solution of (1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

**Division Theorem:** *Suppose that P(w, z) and Q(w, z) are polynomials of two variables w and z in domain C[w, z] and P(w, z) is irreducible in domain C[w, z]. If vanishes at all zero points of , then there exists a polynomial in domain such that*

$$Q(w,z) = P(w,z)G(w,z)$$

Feng [11], pointed out, that the Divisor Theorem follows immediately from Hilbert-Nullstellensatz Theorem [5] of commutative algebra.

### 3. APPLICATION TO KLEIN-GORDON EQUATION

Klein-Gordon equation with cubic nonlinearity, describes the propagation of dislocations within crystals, the Bloch wall motion of magnetic crystals, the propagation of a "splay wave" along a lied membrane, the unitary theory for elementary particles and the propagation of magnetic flux on a Josephson line, etc. [19, 26]. We apply the above method to Klein-Gordon equation with cubic nonlinearity of the form [4]

$$u_{tt} - u_{xx} + x - x^3 = 0 \tag{7}$$

Using Eq. (2) and (3) in Eq. (7), we receive

$$c^2 f'' - f'' + f - f^3 = 0 \tag{8}$$

Use of Eq. (5) in Eq. (8) leads to the following system of ODEs

$$\dot{X}(\xi) = Y(\xi) \tag{9a}$$

$$\dot{Y}(\xi) = \frac{x(\xi) - x^3(\xi)}{1 - c^2} \tag{9b}$$

According to the first integral method, suppose that X(ξ) and Y(ξ) are the nontrivial solutions of Eq. (9), and

$$q(X,Y) = \Sigma_i\, a_i(X)Y^i = 0$$

is an irreducible polynomial in the domain such that

$$q[X(\xi),Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0 \tag{10}$$

where, $a_i(X), (i = 0,1,...,m)$ are polynomials of X and $a_m(X) \neq 0$. Equation (10) is called the first integral of Eq. (9), due to the Division Theorem, there exists a polynomial g(X) + h(X)Y in the domain C[X, Y] such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial \xi}\frac{\partial x}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = (g(X + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i \tag{11}$$

Case I: Suppose that, $m = 1$ by equation the coefficients of $Y^i = (i = 0, 1, 2)$ on both sides of equation (11), we have

$$\dot{a}_1(X) = h(X)a_1(X) \tag{12a}$$

$$\dot{a}_1(X) = g(X)a_1(X) + h(X)a_0(X) \tag{12b}$$

$$a_1(X)\left(\frac{x-x^3}{1-c^2}\right)=g(X)a_0(X) \tag{12c}$$

Since $a_i(X)$ $(i=0,1)$ are polynomials, then from (12a) we conclude that $a_1(X)$ is constant and $h(X)=0$. For simplicity, let us take $a_1(X)=1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $deg(g(X)=1$ only. Suppose that $g(X)=A_1X+A_0$, where $A_1\neq 0$, then

$$a_0(X)=\frac{1}{2}A_1X^2+A_0X+B_0 \tag{13}$$

Substituting $a_0$(X), $a_1$(X) and $g$(X) in equation (12c) and setting all the coefficients of powers of X equal to zero, then we obtain a system of nonlinear algebraic equations and using Maple solving them, we obtain

$$A_0=0, B_0=\sqrt{\frac{2}{c-1}}, A_1=\sqrt{\frac{2}{c-1}} \tag{14}$$

Using Eq. (14) into Eq. (13) and (10), we have

$$Y=-\sqrt{\frac{2}{c-1}}(X^2+2) \tag{15}$$

combining Eq. (15) with (9), we obtain the exact solution of (8) and then the exact solution of Klein-Gordon equation (7) can be written as

$$f(\xi)=\sqrt{2}\tan\left(-\frac{\xi}{\sqrt{c-1}}+\alpha\right) \tag{16}$$

where $\alpha$ is an arbitrary constant. Thus the travelling wave solution of Klein-Gordon equation (7) can be written as

$$u(x,t)=\sqrt{2}\tan\left(-\frac{x-ct}{\sqrt{c-1}}+\alpha\right) \tag{17}$$

**Case II:** Suppose that, by equating the coefficients of on both sides of equation (11), we have

$$\dot{a}_1(X)=h(X)a_1(X) \tag{18a}$$

$$\dot{a}_1(X)=g(X)a_2(X)+h(X)a_1(X) \tag{18b}$$

$$\dot{a}_0(X)=-2a_2\left(\frac{x-x^3}{1-c^2}\right)+g(X)a_1(X)+h(X)a_0(X) \tag{18c}$$

$$a_1(X)\left(\frac{x-x^3}{1-c^2}\right)=g(X)a_0(X) \tag{18d}$$

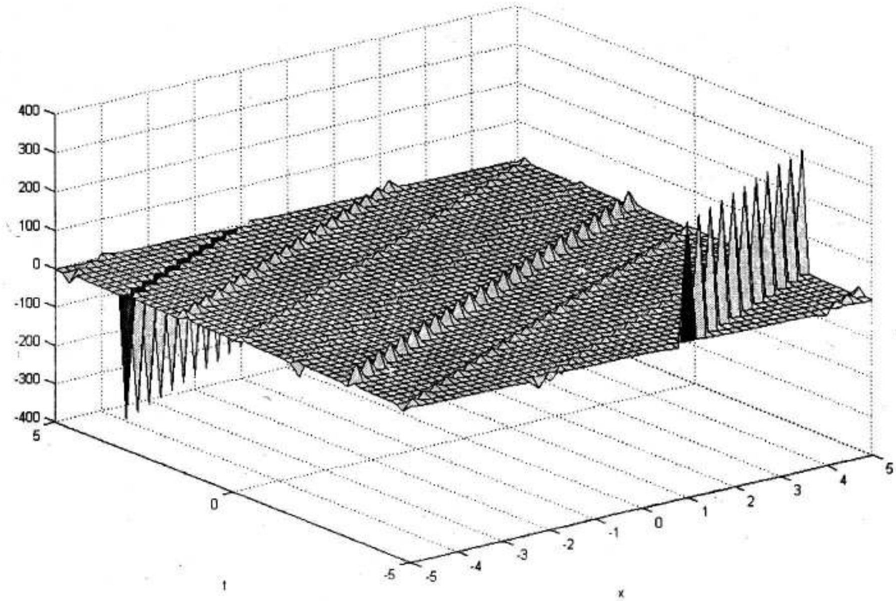


**Fig. 1**: Solution of eq. (17) for c = 2 and $\alpha = 0$

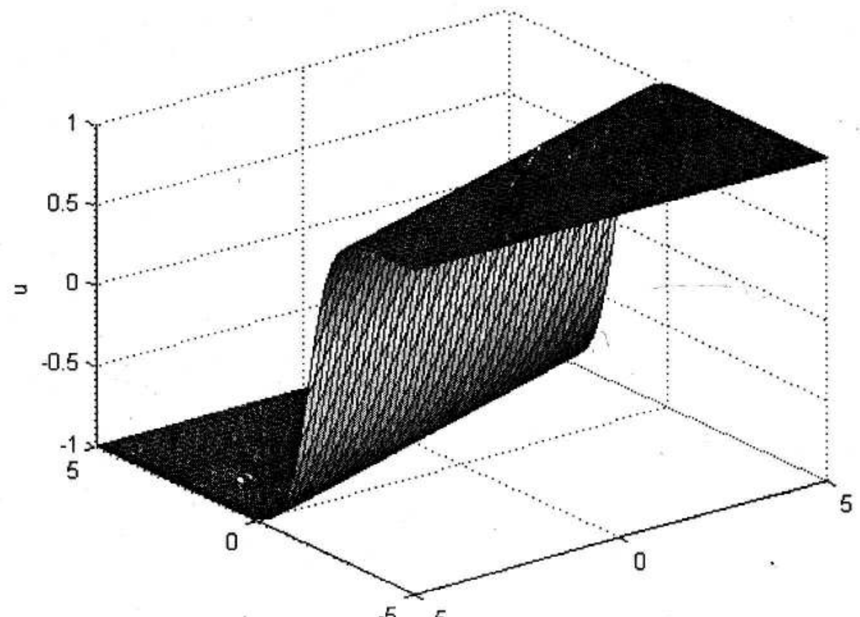


**Fig. 2**: Solution of eq. (17) for c = 3 and $\alpha = 0$

Since $a_i(X)$ $(i = 1, 2, 3)$ are polynomials, then from Eq. (18a) we deduce that $a_2(X)$ is constant and h(X) = 0. For simplicity, let us take $a_2(X) = 1$, and balancing the degrees of g(X), $a_i(X)$ and $a_0(X)$, we conclude that deg g(X) = 1 only. Suppose that, $g(X) = A_1X + A_0$ then

$$a_1(X) = \frac{1}{2}A_1X^2 + A_0X + B_0 \tag{19}$$

$$a_0 = d + A_0B_0x + \left(A_0^2 + A_1B_0 - \frac{2}{1-c^2}\right)\frac{x^2}{2} + \frac{A_1A_0}{2}x^3 + \left(\frac{A_1^2}{2} + \frac{2}{1-c^2}\right)\frac{x^4}{4} \tag{20}$$

Substituting $a_0(X)$, $a_1(X)$ and g(X) in equation (18d) and setting all the coefficients of powers of X equal to zero, then we obtain a system of nonlinear algebraic equations and using Maple solving them, we obtain

$$A_0 = 0, B_0 = -\frac{\sqrt{2}}{\sqrt{c^2-1}}, A_1 = \frac{2\sqrt{2}}{\sqrt{c^2-1}}, d = \frac{1}{2(c^2-1)} \tag{21}$$

Using Eq. (21) into Eq. (19), (20) and (10), we have

$$Y = \frac{1}{\sqrt{2(c-1)}}(1 - X^2) \tag{22}$$

combining Eq. (22) with (9), we obtain the exact solution of (8) and then the exact solution of Klein-Gordon equation (7) can be written as

$$f(\xi) = \tan h\left(\frac{\xi}{\sqrt{2(c-1)}} + \alpha\right) \tag{23}$$

where is an arbitrary constant. Thus the travelling wave solution of Klein-Gordon equation (7) can be written as

$$u(x,t) = \tan h\left(\frac{\xi}{\sqrt{2(c-1)}} + \alpha\right) \tag{24}$$

## 4. CONCLUSIONS

The First integral method was successfully applied to obtain the exact travelling wave solutions of the nonlinear Klein-Gordon equation with cubic nonlinearity. The method is easier and quicker than other traditional methods. Also, it is direct, concise and more specifically computerizable. The symbolic manipulation software Maple is used to solve complicated and tedious algebraic calculations. Therefore, the proposed method can be extended to solve the nonlinear problems which arise in the soliton theory and other areas.

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# A NEW MEASURE OF CSISZAR'S *f*-DIVERGENCE CLASS AND ITS BOUNDS

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## ABSTRACT

*A new symmetric divergence measure is proposed which is useful in comparing two probability distributions. This non-parametric measure belongs to the Csiszar's f divergence class. Its properties are studied and bounds are obtained in terms of some well known divergence measures. A numerical illustration based on the probability distribution is carried out.*

**MSC(2000)** : 94A17; 26D15

**Keywords** – Csiszar's *f* divergence, Information measure, Symmetric divergence.

## 1. INTRODUCTION

One problem in Probability theory which researchers has been trying to find is an appropriate measure of divergence (or distance or difference or discrimination or information) between two probability distributions. Non parametric measure give the amount of information supplied by the data for discriminating in favor of a probability distribution against another or for measuring the distance or affinity between two probability distributions. There are a number of divergence measures being proposed in literature which compare two probability distributions and have been applied in a variety of disciplines such as antropology, genetics, finance, economics and political science, biology, analysis of contingency of tables, approximations of probability distributions, signal processing and pattern recognition.

In this paper we present a non-parametric symmetric divergence measure which belongs to the class of Csiszar's $f$ divergences [2,3]. In section 2 we discuss the Csiszar's $f$ divergences and inequalities. New symmetric divergence measure is obtained in section 3. In section 4 we have derived some inequalities providing bounds for the new measure in terms of some well known divergence measures. The suggested measure is compared with other measures in section 5.Section 6 concludes the paper.

## 2. CSISZAR'S *f* DIVERGENCE

Let $\Omega = \{x_1, x_2, .....\}$ be a set with at least two elements and P the set of all probability distribution $P = (p(x) : x \in \Omega; p(x) \geq 0; \Sigma p(x) = 1)$. For a convex function $f: (0, \infty) \rightarrow$ , the *f-divergence* of the probability distributions P and Q by Csiszar, [2,3] and Ali & Silvey, [1] is defined as

$$C_f(P,Q) = \sum_{x \in \Omega} q(x) f\left(\frac{p(x)}{q(x)}\right) \tag{2.1}$$

Henceforth, for brevity we will denote $C_f(P,Q)$, p(x) .. ,q(x) and $\sum_{x \in \Omega}$ by C (P,Q), p, q and $\Sigma$ respectively.

Osterreicher [7] has discussed basic general properties of *f - divergences* including their axiomatic properties and some important classes. During the recent past, there has been a considerable amount of work providing different kind of bounds on the distance, information and divergence measures [9, 4,5 and 6]. Taneja and Kumar [8] unified and generalized three theorems studied by Dragomir [4, 5, 6] which provide bounds on C (P, Q). The main result in [8] is the following theorem:

**Theorem 2.1** Let $f: I \subset R_+ \rightarrow R$ be a mapping which is normalized, i.e., $f(1) = 0$ and suppose that

(i) *f is twice differentiable on (r,R),* $0 \leq r \leq 1 \leq R < \infty$, ($f'$ and $f''$ denote the first and second derivatives of $f$),
(ii) *there exist real constants m, M such that m < M and*

$m \leq x^{2-s} f''(x) \leq M, \forall x \in (r,R), s \in \mathbb{R}.$

if $P,Q \in P^2$ are discrete probability distributions with $0 < r \leq \frac{p}{q} \leq R \leq \infty$, then

$$m\Phi_s(P,Q) \leq C(P,Q) \leq M\Phi_s(P,Q), \tag{2.2}$$

and

$$m(\eta_s(P,Q) - \Phi_s(P,Q)) \leq C_\rho(P,Q) - C(P,Q) \leq M(\eta_s(P,Q) - \Phi_s(P,Q)), \tag{2.3}$$

where

$$\Phi_s(P,Q = \begin{cases} {}^2K_s(P,Q), & s \neq 0,1 \\ K(Q,P) & s = 0 \\ K(P,Q) & s = 1 \end{cases} \tag{2.4}$$

$${}^2K_s(P,Q) = [s(s-1)]^{-1} [\Sigma p^s q^{1-s} - 1], \qquad s \neq 0,1, \tag{2.5}$$

$$K(P,Q) = \Sigma p \ln\left(\frac{p}{q}\right),. \tag{2.6}$$

$$C\rho(P,Q) = C_{f'}\left(\frac{p^2}{Q}, P\right) - C_{f'}(P,Q) = \Sigma (p-q) f'\left(\frac{p}{q}\right), \tag{2.7}$$

and

$$\eta_s(P,Q) = C_{\varnothing' s}\left(\frac{p^2}{Q}, P\right) - C_{\varnothing' s}(P,Q) \tag{2.8}$$

$$= \begin{cases} (s-1)_s{}^{-1} \sum (p-q)\left(\frac{p}{q}\right)^{s-1}, & s \neq 1 \\ \sum (\mathrm{p}-\mathrm{q}) \ln \left(\frac{p}{q}\right), & s = 1 \end{cases}$$

The following information inequalities which are interesting from the information-theoretic point of view, are obtained from Theorem 2.1 and discussed in [8]

(i) The case s=2 provides the information bounds in terms of the Chi-square divergence

$\chi^2$ (P,Q):

$$\frac{m}{2}\chi^2\,(\mathrm{P,Q}) \leq C\,(P,Q) \leq \frac{\mathrm{M}}{2}\chi^2\,(\mathrm{P,Q}) \tag{2.9}$$

and

$$\frac{\mathrm{m}}{2}\chi^2\,(\mathrm{P,Q}) \leq \mathrm{C}_\rho\,(\mathrm{P,Q}) - \mathrm{C}\,(\mathrm{P,Q}) \leq \frac{\mathrm{M}}{2}\chi^2\,(\mathrm{P,Q}) \tag{2.10}$$

where

$$\chi^2\,(\mathrm{P,Q}) = \sum \frac{(p-q)^2}{q}. \tag{2.11}$$

(ii) For s=1, the information bounds in terms of the Kullback-Leibler divergence,

$$m\mathrm{K(P,Q)} \leq \mathrm{C\,(P,Q)} \leq \mathrm{MK\,(P,Q)}, \tag{2.12}$$

and

$$m\mathrm{K(Q,P)} \leq \mathrm{C}_\rho\,\mathrm{(P,Q)} - \mathrm{C(P,Q)} \leq \mathrm{MK\,(Q,P)}. \tag{2.13}$$

(iii) The case $s = \frac{1}{2}$ provides the information bounds in terms of the Hellinger's discrimination,

$$4mh(\mathrm{P,Q}) \leq \mathrm{C\,(P,Q)} \leq 4\mathrm{M}h\,\mathrm{(P,Q)}. \tag{2.14}$$

and

$$4m\left(\frac{1}{4}\eta_{1/2}\,(P,Q) - h\,(P,Q)\right) \leq C_\rho\,(P,Q) - C\,(P,Q) \tag{2.15}$$

$$\leq 4\mathrm{M}\left(\frac{1}{4}\eta_{1/2}\,(P,Q) - h\,(P,Q)\right)$$

where

$$h\,(P,Q) = \sum \frac{(\sqrt{p}-\sqrt{q})^2}{2} \tag{2.16}$$

(iv) for s = 0, the information bounds in terms of the Kullback- Leibler and $\chi^2$ -divergences:

$$\mathrm{mK\,(Q,P)} \leq \mathrm{C\,(P,Q)} \leq \mathrm{MK\,(Q,P)}, \tag{2.17}$$

and

$$\mathrm{m}\,[\chi^2\,(\mathrm{Q,P}) - \mathrm{K\,(Q,P)}] \leq \mathrm{C}_\rho\,(\mathrm{P,Q}) - \mathrm{C\,(P,Q)} \leq \mathrm{M}\,[\chi^2\,(\mathrm{Q,P}) - \mathrm{K\,(Q,P)}] \tag{2.18}$$

## 3. A SYMMETRIC DIVERGENCE MEASURE OF THE CSISZAR'S $f$-DIVERGENCE FAMILY

We consider the function $f(0, \infty) \to \mathbb{R}$ given by

$$f(u) = \frac{(u+1)^3(u-1)^4(u^2+1)}{u^4} \tag{3.1}$$

and thus the divergence measure

$$V^*(P,Q) = C_f(P,Q) = \sum \frac{(p+q)^3(p-q)^4(p^2+q^2)}{p^4q^4} \tag{3.2}$$

Since $f'(u) = \dfrac{(u+1)^2(u-1)^3[5u^4+5u^2+u^3+u+4]}{u^5}$ (3.3)

and

$$f''(u) = \frac{(u+1)(u-1)^2[15u^6+12u^5+12u^4+12u^3+13u^2+12u+20]}{u^6} \tag{3.4}$$

It follows that $f''(u) > 0$ for all $u > 0$. Hence $f(u)$ is convex for all $u > 0$ (Fig. 3.1).

Further $f(1) = 0$. Thus we can say that the measure is non-negative and convex in the pair of probability distribution $(P,Q) \in \Omega$.

Its corresponding non-symmetric function is given by

$$f(u) = \frac{(u+1)^2(u-1)^4(u^2+1)}{u^4}$$

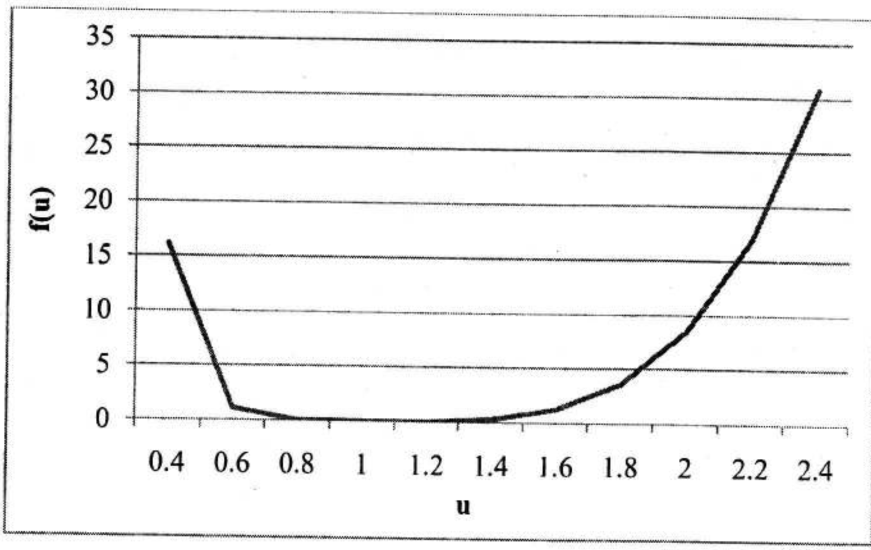


**Fig.3.1 Graph of the convex function $f$ (u)**

and

$$f''(u)=\frac{(u-1)^2(12u^6+12u^5+12u^4+12u^3+16u+20)}{u^6}$$

which is convex for $u>0$.

We now derive information divergence inequalities providing bounds for V* (P,Q) in terms of the well known divergence measures in the following propositions.

First we derive the information bounds in terms of $\chi^2$ divergences.

## 4. BOUNDS FOR V* (P,Q)

**Proposition 4.1**:

Let $\chi^2$ (P,Q) and V* (P,Q) be defined as in (2.11) and (3.2) respectively for (P,Q) $\in P^2$ and $0<r\le\frac{p}{q}\le R<\infty$ we have

$$\frac{(R+1)(R-1)^2}{2R^6}\;[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20]\;\chi^2(P,Q)\le V^*(P,Q)\le$$

$$\frac{(r+1)(r-1)^2}{2r^6}\;[15r^6+12r^5+12r^4+12r^3+13r^2+12r+20]\;\chi^2(P,Q)\qquad(4.1)$$

and

$$\frac{(R+1)(R-1)^2}{2R^6}\;[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20]\chi^2(P,Q)\le$$

$$V^*_\delta(P,Q)-V^*(P,Q)$$

$$\le\frac{(r+1)(r-1)^2}{2r^6}\;[15r^6+12r^5+12r^4+12r^3+13r^2+12r+20]\;\chi^2(P,Q)\quad(4.2)$$

$$\text{Where, } V^*_\delta(P,Q)=\frac{(p-q)^4(p+q)^2}{p^5q^4}[5p^4+5p^2q^2+pq(p^2+q^2)+4q^4]\qquad(4.3)$$

Proof : From (3.4) we have

$$f''(u)=\frac{(u+1)(u-1)^2}{u^6}[15u^6+12u^5+12u^4+12u^3+13u^2+12u+20]$$

If $u\in[a,b]\subset(0,\infty)$ then

$$\frac{(b+1)(b-1)^2}{b^6}[15b^6+12b^5+12b^4+12b^3+13b^2+12b+20]\le f''(u)\le$$

$$\frac{(a+1)(a-1)^2}{a^6}[15a^6+12a^5+12a^4+12a^3+13a^2+12a+20] \qquad (4.4)$$

Or accordingly

$$\frac{(R+1)(R-1)^2}{R^6}[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20] \le f''(u) \le$$

$$\frac{(r+1)(r-1)^2}{r^6}[15r^6+12r^5+12r^4+12r^3+13r^2+12r+20] \qquad (4.5)$$

Where r, R are defined above

In view of (2.9), (2.10) and (4.5) we get inequalities (4.1) and (4.2) respectively. The information bounds in terms of the Kullback-Leibler divergence K(P,Q) follow:

**Proposition 4.2**

Let K(P,Q), V*(P,Q) and $V^*_\delta$(P,Q) be defined as in (2.6), (3.2) and (4.3) respectively.

For $(P,Q) \in \mathbb{P}^2$ and $0 < r \le \frac{p}{q} \le R < \infty$ then

$$0 \le V^*(P,Q) \le \frac{(R+1)(R-1)^2}{R^5}[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20]\ K(P,Q) \qquad (4.6)$$

and

$$0 \le V^*_\delta(P,Q) - V^*(P,Q) \le$$

$$\frac{(R+1)(R-1)^2}{R^5}[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20]\ K(P,Q) \qquad (4.7)$$

**Proof** : Let the function of $g : [r, R] \to R$ be such that

$$g(u) = uf''(u) = \frac{(u+1)(u-1)^2}{u^5}[15u^6+12u^5+12u^4+12u^3+13u^2+12u+20] \quad (4.8)$$

Then

$$0 \le g(u) \le$$

$$\frac{(R+1)(R-1)^2}{R^5}[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20] u \in (r,R) \qquad (4.9)$$

The inequalities (4.6) and (4.7) follow from (2.12), (2.13) using (4.8) and (4.9).

The following proposition provides the information bounds in terms of the hellinger's discrimination h(P,Q) and $\eta_{1/2}$ (P,Q).

**Proposition 4.3**

Let ħ (P,Q), $\eta_{1/2}$(P,Q), V*(P,Q), and $V^*_\delta$(P,Q) be defined as in (2.16), (2.8), (3.2) and (4.3) respectively.

For (P,Q) $\in P^2$ and $0 < r \leq R < \infty$)

$$\frac{4(R+1)(R-1)^2}{R^{9/2}}\,[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20]h(P,Q) \leq V^*(P,Q)$$

$$\frac{4(r+1)(r-1)^2}{r^{9/2}}[15r^6+12r^5+12r^4+12r^3+13r^2+12r+20]h(P,Q) \qquad (4.10)$$

and

$$\frac{4(R+1)(R-1)^2}{R^{9/2}}[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20][\frac{1}{4}\eta_{1/2}(P,Q)-h(P,Q)]$$
$$\leq V_\delta^*(P,Q) - V^*(P,Q) \leq$$
$$\frac{4(r+1)(r-1)^2}{r^{9/2}}[15r^6+12r^5+12r^4+12r^3+13r^2+12r+20]h(P,Q)\,[\frac{1}{4}\eta_{1/2}(P,Q)-h(P,Q)]$$

(4.11)

**Proof** : For *f(u)* in (3.1) we have *f''(u)* given by (3.4). Let the function of g : *[r, R]* $\to$ R be such that

$$g(u) = u^{3/2}f''(u) = \frac{(u+1)(u-1)^2}{u^{9/2}}[15u^6+12u^5+12u^4+12u^3+13u^2+12u+20] \quad (4.12)$$

Then

$$\inf_{u\in[r,R]} g(u) = \frac{(R+1)(R-1)^2}{R^{9/2}}[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20] \quad (4.13)$$

$$\sup_{u\in[r,R]} g(u) = \frac{(r+1)(r-1)^2}{r^{9/2}}[15r^6+12r^5+12r^4+12r^3+13r^2+12r+20] \quad (4.14)$$

Thus (4.9) and (4.10) follow from (2.14), (2.15), (4.13) and (4.14).

Next follows the information bounds in terms of the Kullback-Leibler and $\chi^2$ Divergences.

**Proposition 4.4**

Let K(P, Q), $\chi^2$(P, Q),V*(P, Q) and$V^*_\delta$ (P, Q) be defined as in (2.6), (2.11), (3.2) and (4.3) respectively. If (P, Q), $\in P^2$ and $0 < r \leq R < \infty$ then

$$0 \leq V^*(P,Q) \leq \frac{(R+1)(R-1)^2}{R^4}\,[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20]K(Q,P)$$

(4.15)

and

$$0 \leq V_\delta^*(P,Q) - V^*(P,Q) \leq$$
$$\frac{(R+1)(R-1)^2}{R^4}\,[15R^6+12R^5+12R^4+12R^3+13R^2+12R+20]K(Q,P) \quad (4.16)$$

**Proof** : From the expression (3.1) we have $f''(u)$ as given in (3.4). Let the function of $g : [r, R] \to R$ be such that

$$g(u) = u^2 f''(u) = \frac{(u+1)(u-1)^2}{u^4}[15u^6 + 12u^5 + 12u^4 + 12u^3 + 13u^2 + 12u + 20] \quad (4.17)$$

Then

$$0 \le g(u) \le \frac{(R+1)(R-1)^2}{R^4}[15R^6 + 12R^5 + 12R^4 + 12R^3 + 13R^2 + 12R + 20] \quad (4.18)$$

Thus (4.15) and (4.16) follow from (2.17), (2.18) and (4.18)

## 5. NUMERICAL ILLUSTRATION

In this section we have numerically verified the bounds obtained in section 4.

We consider an example of symmetrical probability distribution. We calculate measures $\Psi(P,Q)$, $\chi^2(P, Q)$, $J(P, Q)$ and $V^*(P,Q)$ and Compare bounds.

Here $J(P,Q) = K(P,Q) + K(Q,P) = \Sigma(p-q)\ln\frac{p}{q}$ is the Kullback-Leibler divergence and

$\Psi(P,Q) = \chi^2(P,Q) + \chi^2(Q,P) = \Sigma\frac{(p-q)^2(p+q)}{pq}$ is the Symmetric Chi-square Divergence.

5.1 Let P be the binomial probability distribution for the random variable X With parameter (n=8 p=0.5) and Q its approximated normal probability distribution.

**Table 1 Binomial Probability Distribution**

(n=8 p=0.5)

| X | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| p (x) | .004 | .031 | .109 | .219 | .274 | .219 | .109 | .031 | .004 |
| q (x) | .005 | .030 | .104 | .220 | .282 | .220 | .104 | .030 | .005 |
| $\frac{p(x)}{q(x)}$ | .774 | 1.042 | 1.0503 | .997 | .968 | .997 | 1.0503 | 1.042 | .774 |

The measures $\Psi(P,Q)$, $\chi^2(P, Q)$, $J(P, Q)$ and $V^*(P,Q)$ are

$\Psi(P,Q) = 0.00305063$ $J(P,Q) = 0.00151848$

$\chi^2(P, Q) = 0.00145837$ $V^*(P,Q)=0.00039323$

$$r = (0\cdot774179933) \le \frac{p}{q} \le R(1.050330018)$$

The lower bound for $V^*(P, Q)$ based on $\chi^2(P, Q)$ divergence from (4.1)

Lowerbound =

$$\frac{(R+1)(R-1)^2}{2R^6}[15R^6 + 12R^5 + 12R^4 + 12R^3 + 13R^2 + 12R + 20]\chi^2(P,Q) = 0.00031288$$

and thus 0.00031288 < V*(P,Q) = 0.000393233.

The upper bound for V*(P,Q) based on $\chi^2$(P, Q) from (4.1) is

Upper bound =

$$\frac{(r+1)(r-1)^2}{2r^6}[15r^6+12r^5+12r^4+12r^3+13r^2+12r+20]\chi^2(P,Q) = 0.0163936$$

Therefore 0.00031288 <V*(P,Q) = 0.000393233 < 0.0163936.

The length of the interval is 0.016000.

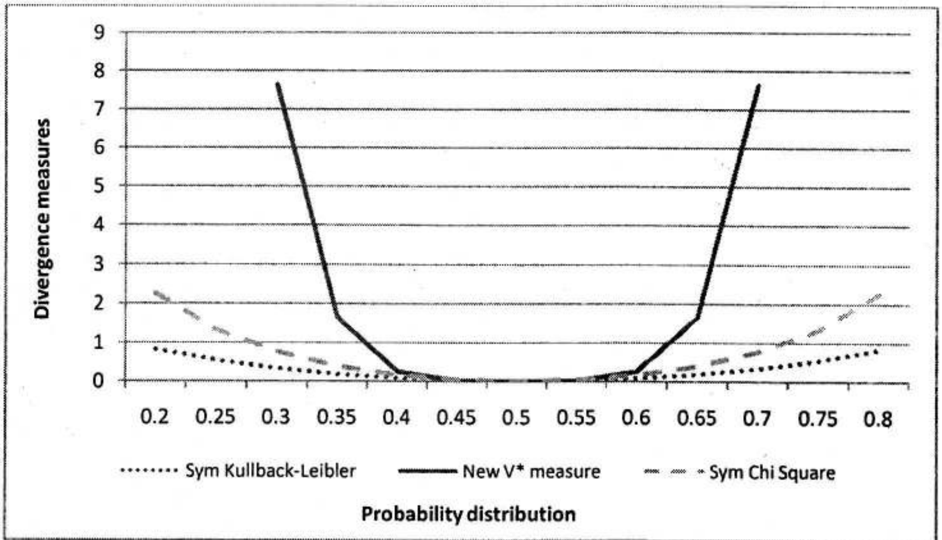


**Fig. 5.1 New V*(P,Q), sym Chi square and sym K-L**

Fig 5.1 shows the behavior of V*(P,Q), Ψ(P,Q) (sym- Chi square) and J(P,Q) (sym Kullback-Leibler).

We have considered $p = (a, 1 - a)$ and $q = (1 - a, a) \in (0, 1)$.

It is clear from Figure 5.1 that the new measure has a steeper slope than Ψ(P,Q) and J(P,Q).

## 6. CONCLUDING REMARKS

The Csiszar's *f*-divergence is a general class of divergence measures, which includes several divergences used in measuring the distance or affinity between two probability distribution. This class is introduced by using a convex function f, defined on (0, ∞). An important property of this divergence is that many known divergence can be obtained by appropriately defining the convex function *f*. Non-parametric measures for the Csiszar's *f-divergences* are also available. We have introduced a new symmetric divergence measure by considering a convex function and have investigated its properties.

Further we have established its bounds in terms of known divergence measures. Work on one parametric generalization of this measure is in progress and will be reported elsewhere.

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# CONSTRUCTION OF SEMI-REGULAR GD DESIGNS

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## ABSTRACT

*In this paper, we propose some new methods of constructing new families of semi-regular group divisible (SRGD) designs by using normalized Hadamard matrices. Further, new non-isomorphic solutions for some known group divisible (GD) designs are also given together with useful group divisible (GD) designs not listed in Clatworthy [2].*

***Key words & Phrases:*** BIBD, self complementary design, truly self complementary, trivial BIBD, Semi-Regular Group Divisible (SRGD) Design, Hadamard matrix.

**Mathematics Subject Classification (2000):** 62K10

## 1. INTRODUCTION

Aim of present study is to obtain new methods of constructing new families of Semi-Regular Group Divisible (SRGD) designs by using normalized Hadamard matrices, Incidence matrices of BIBD's, self complementary BIBD's and trivial BIBD's. These are defined below:

The celebrated class of PBIB designs with m-associate classes introduced by Bose and Shimamoto [1] and its classification for m=2 are well-known.

Group divisible designs have been further classified into three types (c f. Raghavarao [5]); (i) singular, if $r-\lambda_1=0$ (ii) semi-regular, if $r-\lambda_1>0$ and $rk-v\lambda_2=0$ and (iii) regular, if $r-\lambda_1>0$ and $rk-v\lambda_2>0$.

A BIBD is an arrangement of v treatments (elements) in b blocks (sets) such that (a) each block contains k distinct treatments ($k < v$), (b) each treatment occurs in exactly r different blocks, (c) every pair of distinct treatments appears together in $\lambda$ blocks.

A design D is said to be self complementary if it's complementary design $D^{*}$ has precisely the same parameters as the parent design D. The designs D and $D^{*}$ may or may not be isomorphic. For a design to be self complementary, it is necessary that $v = 2k$ implies $b = 2r$.

If a design is self complementary and if D and its complementary design $D^{*}$ are isomorphic, we say design D is truly self complementary.

An SBIBD whose incidence matrix is identity matrix is called trivial SBIBD.

A square matrix $H_n$ of order n with elements +1 and –1 is called a Hadamard matrix if its distinct rows are pair-wise orthogonal (two distinct rows are said to be orthogonal if inner product between them is zero). That is, $H_n H_n' = nI_n$, also $H_n' H_n = nI_n$ holds.

If all elements of first row as well as column of $H_n$ matrix are +1, then it is called normalized Hadamard matrix.

It is known that from a normalized Hadamard matrix $H_{4t}$, if we remove the first row and first column, we obtain core of the H-matrix of order $(4t-1)\times(4t-1)$. Now replacing –1 by 0 in the core of the H-matrix we get incidence matrix of a $(v, k, \lambda)$-configuration, namely, $(4t-1, 2t-1, t-1)$. This is a Hadamard design $(4t-1, 2t-1, t-1)$.

It is to be noted that the method discussed above is reversible i. e. given incidence matrix of an SBIBD $(4t-1, 2t-1, t-1)$ by reversing the steps given above we can get our $H_{4t}$.

## 2. CONSTRUCTION OF SEMI-REGULAR GD DESIGNS

In this section, we shall obtain new families of semi-regular group divisible designs.

**Theorem 2.1:** If N is an incidence matrix of order $v \times b$ of a BIBD ($v = 2k$, $b = 2r$, r, k, $\lambda$), then the incidence pattern

$$S = \begin{pmatrix} N & \overline{N} & N & \overline{N} & N & \overline{N} & N & \overline{N} \\ N & N & \overline{N} & \overline{N} & N & N & \overline{N} & \overline{N} \\ N & \overline{N} & \overline{N} & N & N & \overline{N} & \overline{N} & N \\ N & N & N & N & \overline{N} & \overline{N} & \overline{N} & \overline{N} \\ N & \overline{N} & N & \overline{N} & \overline{N} & N & \overline{N} & N \\ N & N & \overline{N} & \overline{N} & \overline{N} & \overline{N} & N & N \\ N & \overline{N} & \overline{N} & N & \overline{N} & N & N & \overline{N} \end{pmatrix} \qquad (2.1)$$

is the incidence matrix of a SRGD design with parameters

$$v^* = 7v,\ b^* = 8b,\ r^* = 8r,\ k^* = 7k,\ \lambda_1^* = 8\lambda, \lambda_2^* = 4r;\ m = 7, n = v. \qquad (2.2)$$

**Proof :** Evidently S is an incidence matrix of order $v^* \times b^*$ having row (column) sum $r^*(k^*)$. Then incidence matrix S of order $v^* \times b^*$ of the design satisfy the matrix equation

$$SS' = \begin{bmatrix} A & B & B & \cdots & B \\ B & A & B & \cdots & B \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B & B & B & \cdots & A \end{bmatrix}$$

where

$$\begin{aligned} A &= 4NN' + 4\overline{N}\,\overline{N}', \\ &= 4NN' + 4(J - N)(J' - N'), \\ &= 8NN' + 4bJ - 8rJ, \\ &= 8(r - \lambda)I + 8\lambda J + 4bJ - 8rJ, \\ &= 8(r - \lambda)I + 8\lambda J. \qquad (\because b = 2r) \end{aligned} \qquad (2.3)$$

and

$$\begin{aligned} B &= 2NN' + 2\overline{N}\,\overline{N}' + 2\overline{N}N' + 2N\overline{N}', \\ &= 2NN' + 2(J - N)(J' - N') + 2(J - N)N' + 2N(J' - N'), \\ &= 2JJ', \\ &= 4rJ\,.\ (\because b = 2r) \end{aligned} \qquad (2.4)$$

Hence from (2.3) and (2.4), we have,

$$\lambda_1^* = 8\lambda, \quad \lambda_2^* = 4r$$

Other parameters of the design are obvious. From the matrix structure SS' it reveals that S is an incidence matrix of a GD design with seven groups each of size n = v = 2k having the parameters as in (2.2).

We note that $r^* > \lambda_1^*$, since this is equivalent to $r > \lambda$ in a BIBD. Also since $r^*k^* - v^*\lambda_2^* = 0$, therefore the resultant design is SRGD design. Hence the theorem.

As an illustration of Theorem 2.1, consider N the incidence matrix of a BIBD (4, 6, 3, 2, 1). We obtain SRGD design with parameters

$v^* = 28, b^* = 48, r^* = 24, k^* = 14, \lambda_1^* = 8, \lambda_2^* = 12; m = 7, n = 4, n_1 = 3, n_2 = 24$

which seems to be new design since it is not available in Parihar[3], Parihar and Shrivastava[4].

Also talking a trivial BIBD (2, 2, 1, 1, 0) having $N = I_2$ in Theorem 2.1, we get a GD design with parameters

$$v^* = 14, b^* = 16, r^* = 8, k^* = 7, \lambda_1^* = 0, \lambda_2^* = 4; m = 7, n = 2.$$

Theorem 2.1 can be generalized as follows:

**Theorem 2.2:** If there exists a normalized Hadamard matrix of order 8t (t e" 1) and a BIBD ($v = 2k, b = 2r, r, k, \lambda$) with N as its incidence matrix then SRGD design with parameters

$$v^* = (8t-1)v, b^* = 8tb, r^* = 8tr, k^* = (8t-1)k, \lambda_1^* = 8t\lambda, \lambda_2^* = 4tr, m^* = 8t-1, n^* = v.$$

always exists.

**Proof:** By deleting first row of a normalized Hadamard matrix of order 8t, replacing +1 by N and –1 by $\overline{N}$, the incidence matrix of complementary design of a BIBD (2k, 2r, r, k, $\lambda$) in the remaining matrix of order $(8t-1) \times 8t$, we obtain matrix S of order $(8t-1)v\times 8tb$. Rest part of the proof is done on the similar line as of Theorem 2.1.

As an illustration of Theorem 2.2, if we start with t = 2 and BIBD (4, 6, 3, 2, 1), then we get SRGD design with parameters

$$v^* = 60, b^* = 96, r^* = 48, k^* = 30, \lambda_1^* = 16, \lambda_2^* = 24, m^* = 15, n^* = 4.$$

Also taking a trivial BIBD (2, 2, 1, 1, 0) i. e. $N = I_2$ and t = 3 in the above theorem, we get SRGD design with parameters

$$v^* = 46, b^* = 48, r^* = 24, k^* = 23, = 0, = 12, m^* = 23, n^* = 2.$$

**Theorem 2.3:** If there exists a normalized Hadamard matrix of order 8t (t e□ 1) and a self complementary balanced incomplete block design with N as its incidence matrix, then there exists a SRGD design with parameters

$$v^* = (8t-2)v, b^* = 8tb, r^* = 8tr, k^* = (8t-2)k, \lambda_1^* = 8t\lambda, \lambda_2^* = 4tr, m^* = 8t-2, n^* = v.$$

**Proof :** By deleting the first row and any other row of a normalized Hadamard matrix of order 8t and replacing +1 by N and –1 by $\overline{N}$ in the remaining matrix of order $(8t-2)\times 8t$ (where $\overline{N}$ is the incidence matrix of self complementary BIB design), we get matrix S of order $(8t-2)v\times 8tb$. Proof is obvious.

As an illustration of Theorem 2.3, we take t = 1, and trivial BIBD (2, 2, 1, 1, 0), then the resulting design is a semi-regular GD design with parameters

$$v^* = 12,\ b^* = 16,\ r^* = 8,\ k^* = 6,\ \lambda_1^* = 0,\ \lambda_2^* = 4,\ m^* = 6,\ n = 2.$$

The solution of the design is resolvable, in which blocks are written as columns:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 |
| 3 | 4 | 3 | 4 | 4 | 3 | 4 | 3 | 3 | 4 | 3 | 4 | 4 | 3 | 4 | 3 |
| 5 | 6 | 6 | 5 | 6 | 5 | 5 | 6 | 5 | 6 | 6 | 5 | 6 | 5 | 5 | 6 |
| 7 | 8 | 8 | 7 | 7 | 8 | 8 | 7 | 8 | 7 | 7 | 8 | 8 | 7 | 7 | 8 |
| 9 | 10 | 9 | 10 | 10 | 9 | 10 | 9 | 10 | 9 | 10 | 9 | 9 | 10 | 9 | 10 |
| 11 | 12 | 12 | 11 | 11 | 12 | 11 | 12 | 12 | 11 | 11 | 12 | 11 | 12 | 12 | 11 |

The groups of the GD association scheme are given as

| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 | 11 |
| 2 | 4 | 6 | 8 | 10 | 12 |

The SRGD design constructed by us is listed as SR69 in Clatworthy [2]. Our solution is non-isomorphic to the solution 1 mentioned in Clatworthy [2].

**Theorem 2.4:** Let there exists a normalized Hadamard matrix of order 8t (t e" 1) and a BIBD (v = 2k, b = 2r, r, k, λ), then SRGD design with parameters

$$v^* = (8t-3)v,\ b^* = 8tb,\ r^* = 8tr,\ k^* = (8t-3)\,k,\ \lambda_1^* = 8t\lambda,\ \lambda_2^* = 4tr\ ;\ m^* = 8t-3,\ n^* = v.$$

always exists.

**Proof :** By deleting first row and any other two rows of a normalized Hadamard matrix of order 8t and replacing +1 by N the incidence matrix of BIBD and −1 by $\overline{N}$ in the remaining matrix of order (8t − 3) × 8t, we get matrix S of order (8t–3)v× 8tb. Matrix S generates the required SRGD design. Rest proof is obvious.

As an illustration of Theorem 2.4, we take t = 1 and trivial BIBD (2, 2, 1, 1, 0), then we get SRGD design with parameters

$$v^* = 10,\ b^* = 16,\ r^* = 8,\ k^* = 5,\ \lambda_1^* = 0,\ \lambda_2^* = 4\ ;\ m^* = 5,\ n^* = 2.$$

The solution of the design is resolvable and is given below in which columns are blocks:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 |
| 3 | 4 | 3 | 4 | 4 | 3 | 4 | 3 | 3 | 4 | 3 | 4 | 4 | 3 | 4 | 3 |
| 5 | 6 | 6 | 5 | 6 | 5 | 5 | 6 | 5 | 6 | 6 | 5 | 6 | 5 | 5 | 6 |
| 7 | 8 | 8 | 7 | 7 | 8 | 8 | 7 | 8 | 7 | 7 | 8 | 8 | 7 | 7 | 8 |
| 9 | 10 | 9 | 10 | 10 | 9 | 10 | 9 | 10 | 9 | 10 | 9 | 9 | 10 | 9 | 10 |

The groups of the GD association scheme are given as

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 5 | 7 | 9 |
| 2 | 4 | 6 | 8 | 10 |

The SRGD design thus constructed is listed as SR54 in Clatworthy [2]. Solution obtained by us is non-isomorphic to the solutions 1 and 3 mentioned in Clatworthy [2].

To further illustrate Theorem 2.4, consider t = 2, and BIBD (4, 6, 3, 2, 1). We obtain SRGD design with parameters

$$v^* = 52, b^* = 96, r^* = 48, k^* = 26, \lambda_1^* = 16, \lambda_2^* = 24; m^* = 13, n^* = 4.$$

Also if we consider t = 2, 3, 4 and BIBD (2, 2, 1, 1, 0) in Theorem 2.4, we obtain SRGD designs with parameters respectively as given below:

(i) $v^* = 26, b^* = 32, r^* = 16, k^* = 13, \lambda_1^* = 0, \lambda_2^* = 8; m^* = 13, n^* = 2.$

(ii) $v^* = 42, b^* = 48, r^* = 24, k^* = 21, \lambda_1^* = 0, \lambda_2^* = 12; m^* = 21, n^* = 2.$

(iii) $v^* = 58, b^* = 64, r^* = 32, k^* = 29, \lambda_1^* = 0, \lambda_2^* = 16; m^* = 29, n^* = 2.$

The above obtained SRGD designs seem to be new, since they are not available in the existing lists.

**Theorem 2.5:** The existence of a Hadamard matrix of order 4t (t e" 1) implies the existence of a symmetrical SRGD design with parameters

$$v^* = 8t = b^*, r^* = 4t = k^*, \lambda_1^* = 0, \lambda_2^* = 2t; m^* = 4t, n^* = 2.$$

**Proof:** Suppose there exists a Hadamard matrix of order 4t ($t \geq 1$). Replacing its elements +1 by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and −1 by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ yields a (0, 1) matrix of order $8t \times 8t$. Obviously, the number of rows is $4t\times 2 = 8t$ as well as number of columns is $4t \times 2 = 8t$ in the (0, 1) matrix. We identify (0, 1) matrix of order $8t \times 8t$ as an incidence matrix of SRGD design with $v^* = 8t = b^*$, $r^* = 4t = k^*$. In order to find the values of $\lambda_1^*$ and we proceed as follows. Two treatments $(\theta, \varphi)$ are first associates if they lie in the i-th and (i +1)-th rows otherwise they are second associates. It is clear that in the i-th (i should be odd) and (i +1)-th rows, there are column vectors of type $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, hence $\theta$ and $\varphi$ can not occur as a pair in the design yielding $\lambda_1^* = 0$. Using the fact that $n_1\lambda_1 + n_2\lambda_2 = r(k-1)$. We have $\lambda_2^* = 2t$. Also $m^* = 4t$, $n^* = 2$, which completes the proof of the theorem.

As an illustration of Theorem 2.5, consider Hadamard matrix of order 4t in which take t = 1, replacing +1 by $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and −1 by $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ yields SRGD design with parameters

$$v^* = 8 = b^*,\ r^* = 4 = k^*,\ \lambda_1^* = 0,\ \lambda_2^* = 2;\ m^* = 4,\ n^* = 2.$$

The $H_4$ matrix is as under:

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1 \end{bmatrix}$$

The solution of the SRGD design is written as follows in which columns are blocks:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| 3 | 4 | 4 | 3 | 3 | 4 | 4 | 3 |
| 5 | 6 | 5 | 6 | 6 | 5 | 6 | 5 |
| 7 | 8 | 8 | 7 | 8 | 7 | 7 | 8 |

The groups of the GD association scheme are given as

| | | | |
|---|---|---|---|
| 1 | 3 | 5 | 7 |
| 2 | 4 | 6 | 8 |

In fact for t = 1, 2, 3, 4, 5, 6 we get symmetrical SRGD design in the useful range of $v \le 50$.

| t | v | b | r | k | $\lambda_1$ | $\lambda_2$ | m | n |
|---|---|---|---|---|---|---|---|---|
| 1 | 8 | 8 | 4 | 4 | 0 | 2 | 4 | 4 |
| 2 | 16 | 16 | 8 | 8 | 0 | 4 | 8 | 2 |
| 3 | 24 | 24 | 12 | 12 | 0 | 6 | 12 | 2 |
| 4 | 32 | 32 | 16 | 16 | 0 | 8 | 16 | 2 |
| 5 | 40 | 40 | 20 | 20 | 0 | 10 | 20 | 2 |
| 6 | 48 | 48 | 24 | 24 | 0 | 12 | 24 | 2 |

**Corollary 2.5.1:** Interchanging the scheme of replacement for elements +1 and -1 of a Hadamard matrix in Theorem 2.5 again gives us SRGD with same set of parameters

However the solution of Corollary 2.5.1 may be same as or isomorphic to the solution of the Theorem 2.5.

**Remark 2.5.1:** SRGD design constructed in Theorem 2.5 and hence in Corollary 2.5.1 are always resolvable and hence affine resolvable.

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# SUMMATION FORMULAE INVOLVING GENERALIZED VOIGT FUNCTIONS, PSEUDO LAGUERRE POLYNOMIAL AND PARABOLIC CYLINDER FUNCTION

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**ABSTRACT**

*The paper contains three series expansion formulae. The first gives a series expansion involving pseudo-Laguerre polynomial and Voigt functions in terms of Lauricella function. The second involves pseudo-Laguerre polynomial and Voigt functions in terms of Humbert's Confluent hypergeometric function. The third presents pseudo Laguerre polynomial and parabolic cylinder function in terms of H-function of two variables. One special case of each of the above main results which is also interesting and believed to be new has also been given.*

**Keywords:** Bessel function, Lauricella function, parabolic cylinder function, pseudo-*Laguerre* polynomial, Voigt functions.

**Mathematics Subject Classification (2010): 33C47, 33C60, 33C65, 33E20, 85A99**

## 1. INTRODUCTION

The Voigt functions K(x,y) and L(x,y) occur frequently in a wide variety of problems of physics such as astrophysical spectroscopy, transfer of radiation in heated atmosphere, and also in the theory of neutron reactions [1]. Furthermore, the function

$$K(x,y)+iL(x,y)$$

is, to a numerical factor, identical to the so-called 'plasma dispersion function', which is tabulated by Fried and Conte [4]and several representations of the Voigt functions have been given by a number of workers, for examples, Exton[2], Riche[8], Fettis [3], Srivastava and Miller [13] Srivastava, Pathan and Kamarujjama [14] and Pathan, Kamarujjama and Alam [6] etc. First of all, we recall here the following integral representations due to Reiche [8]:

$$K(x,y)=(\pi)^{-(1/2)}\int_0^\infty e^{\left(-yt-\frac{1}{4}t^2\right)}\cos(xt)\,dt \tag{1.1}$$

and

$$L(x,y)=(\pi)^{-(1/2)}\int_0^\infty e^{\left(-yt-\frac{1}{4}t^2\right)}\sin(xt)\,dt \quad (-\infty<x<\infty, y>0), \tag{1.2}$$

so that

$$K(x,y)\pm iL(x,y)=(\pi)^{-(1/2)}\int_0^\infty e^{\left(-(y\mp ix)t-\frac{1}{4}t^2\right)}dt$$

$$= \exp[(y\mp ix)^2]\{1-\operatorname{erf}(y\mp ix)\}, \tag{1.3}$$

where erf(w) denotes the error function[12, p.28]

Srivastava and Miller [13, p.113, equation.(8)] introduced and studied systematically a unification (and generalization) of the Voigt functions K(x, y) and L(x, y) in the form:

$$V_{\mu,\upsilon}(x,y) = \left(\frac{x}{2}\right)^{1/2}\int_0^\infty t^\mu \exp\left(-yt-\frac{1}{4}t^2\right)J_\upsilon(xt)\,dt, \tag{1.4}$$

$$\left(x,y\in R^+, \operatorname{Re}(\mu+\upsilon)>-1\right)$$

where $K(x,y)=V_{1/2,-1/2}(x,y)$ and $L(x,y)=V_{1/2,1/2}(x,y)$,

and the Bessel function $J_\upsilon(z)$ defined by

$$J_\upsilon(z) = \frac{(z/2)^\upsilon}{\Gamma(\upsilon+1)}\,{}_0F_1\left[-;\upsilon+1;-z^2/4\right]=\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{\upsilon+2n}}{n!\,\Gamma(\upsilon+m+1)}, \quad |z|<\infty. \tag{1.5}$$

Further representation of the unified Voigt functions $V_{\mu,\upsilon}(x,y)$ is given in Srivastava and Miller [13, p.113, equation (11)] in terms of $\Psi_2$,

$$V_{\mu,\upsilon}(x,y) = \frac{2^{\mu-1/2}\,x^{\upsilon+1/2}}{\Gamma(\upsilon+1)}$$

$$\left\{\Gamma\left(\frac{1}{2}(\mu+\upsilon+1)\right)\psi_2\left[\frac{1}{2}(\mu+\upsilon+1);\upsilon+1,1/2;-x^2,y^2\right]\right.$$

$$\left.-2y\,\Gamma\left((\mu+\upsilon+2)/2\right)\psi_2\left[\frac{1}{2}(\mu+\upsilon+2);\upsilon+1,3/2;-x^2,y^2\right]\right\}, \quad (1.6)$$

$$\operatorname{Re}(\mu+\upsilon)>-1$$

where $\psi_2$ denotes one of Humbert's confluent hypergeometric function **of two** variables defined in the treatise by Srivastava & Manocha [12,p.59,equation (42)]

$$\psi_2[a;c_1,c_2;x,y]=$$

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\quad;\qquad \max\{|x|,|y|\}<\infty \qquad (1.7)$$

**Pseudo-Laguerre polynomial**

Srivastava [10] introduced the general class of polynomial, defined as follows:

$$S_V^U[x]=\sum_{r=0}^{[V/U]}\frac{(-V)_{Ur}}{r!}A_{V,r}\,(x)^r$$

taking U=1,

$$A_{V,r}=\frac{(a+V)_V}{V!}\frac{1}{(a+V)_r}$$

$S_V^U[x]$ reduces to pseudo-Laguerre polynomial $R_V(a,x)$ [7,p.298,**equation** (1)],

$$\text{where}\quad R_V(a,x)=\frac{(a)_{2V}}{V!(a)_V}\,{}_1F_1\left(\begin{matrix}-V;\\ a+V;\end{matrix}x\right)$$

## 2. MAIN RESULTS

**Formula 1:**

$$\sum_{n=0}^{\infty}\frac{R_n(a,x)}{(2)^{3n}\left(\frac{a+1}{2}\right)_n}V_{\alpha+2n-\beta,\beta-1}(z,y)=\left(\frac{z}{2}\right)^{\beta-1/2}\frac{\Gamma(\alpha)}{\Gamma(\beta)y^{\alpha}}$$

$$F^{1:0;0;0}_{1:1;0;0}\left[\begin{matrix}-\left(z^2/4y^2\right)\\ \left(2^{-6}y^{-4}\right)\\ -\left(x/8y^2\right)\end{matrix}\left|\begin{matrix}(\alpha;2,4,2) & :- & ;-;- \\ \left(\frac{(a+1)}{2};0,1,1\right) & :(\beta,1); & -;-\end{matrix}\right.\right]. \quad (2.1)$$

$$\left(z,y\in R^{+},\mathrm{Re}(\alpha)>0\right)$$

**Proof:**

To prove the Formula 1, we take the following Toscano's generating function [15] obtained by Shively [9] for Pseudo-Laguerre polynomial $R_n(a,x)$ [7, p.298, equation (4)] in slightly different form obtainable with the help of result [11, p.28, equation (30)]

$$e^{-2t}\sum_{n=0}^{\infty}\frac{R_n(a,x)t^n}{((a+1)/2)_n}=F^{0:0:0}_{1:0:0}\left[\begin{matrix}- & :-;-; & \\ \frac{(a+1)}{2} & :-;-; & t^2,-xt\end{matrix}\right] \quad (2.2)$$

Now, in the equation (2.2), we replace t by $t^2/8$, multiply by $e^{-yt}t^{\alpha-1}{}_0F_1\left[-;\beta;-\left(z^2t^2\right)/4\right]$ and integrate the resulting equation with respect to t from 0 to $\infty$. Next in the left hand side, we replace ${}_0F_1$ by Bessel function with the help of (1.5) and change the order of summation and integration; we get the following equation,

$$\sum_{n=0}^{\infty}\frac{R_n(a,x)}{(2)^{3n}((a+1)/2)_n}\int_0^{\infty}e^{-\left(yt+t^2/4\right)}t^{\alpha+2n-1}\frac{\Gamma(\beta)}{(zt/2)^{\beta-1}}J_{\beta-1}(zt)\,dt$$

$$= \int_0^\infty e^{-yt} t^{\alpha-1} {}_0F_1\left[-;\beta;\left(z^2t^2\right)/4\right]$$

$$F^{0:0:0}_{1:0:0}\left[\begin{array}{c} - \quad :-;-; \\ \frac{(a+1)}{2}:-;-; \end{array} \frac{t^4}{64}, -\frac{xt^2}{8}\right]dt \qquad (2.3)$$

Further, we express ${}_0F_1$ and Kampe de Feriet function occurring in right hand side in the form of contour integral, change the order of contour integrals and t-integral, and evaluate the t-integral. Now on reinterpreting the resulting contour integrals in terms of Lauricella function and expressing the integral involved in the left hand side of the above (2.3), in terms of Voigt function (1.4), we easily arrive at the Formula 1 after a little simplification.

**Formula 2:**

$$\sum_{n=0}^{\infty} \frac{R_n(a,x)z^n}{((a+1)/2)_n} V_{n+\mu,\upsilon}(k,y) = \sum_{s,r=0}^{\infty} \frac{(-1)^r (zx)^r (z)^{2s}}{(a+1/2)_{s+r}\, s!r!} \frac{2^{\mu+2s+r-1/2} k^{\upsilon+1/2}}{\Gamma(\upsilon+1)}$$

$$\left[\Gamma\left(\frac{1}{2}(\mu+2s+r+\upsilon+1)\right)\left\{\psi_2\left(\frac{1}{2}(\mu+2s+r+\upsilon+1);\upsilon+1,\frac{1}{2};-k^2,(y-2z)^2\right)\right\}\right.$$

$$\left.-2(y-2z)\Gamma\left(\frac{1}{2}(\mu+2s+r+\upsilon+2)\right)\left\{\psi_2\left(\frac{1}{2}(\mu+2s+r+\upsilon+2);\upsilon+1,\frac{3}{2};-k^2,(y-2z)^2\right)\right\}\right].$$

(2.4)

$$\left(k, y \in R^+, \mathrm{Re}(\mu+\upsilon) > -1\right)$$

**Proof:**

To establish the Formula 2, we start with the following slightly altered form of (2.2)

$$\sum_{n=0}^{\infty} \frac{R_n(a,x)t^n}{((a+1)/2)_n} = F^{0:0:0}_{1:0:0}\left[\begin{array}{c} - \quad :-;-; \\ \frac{(a+1)}{2}:-;-; \end{array} t^2, -xt\right]e^{2t} \qquad (2.5)$$

now, we express Kampe de Feriet function occurring in right hand side of (2.5) in series form with the help of [11, p.27, eq. (28)], next in both sides of (2.5) replacing t by tz, multiplying by $t^{\mu}e^{-yt-t^2/4}J_{\upsilon}(kt)$, and integrating with respect to t from the limit 0 to $\infty$. Now change the order of summation and integration and express integrals in terms of Voigt function with the help of (1.4) both sides, we get following equation

$$\sum_{n=0}^{\infty}\frac{R_n(a,x)z^n}{((a+1)/2)_n}V_{n+\mu,\upsilon}(k,y) = \sum_{s,r=0}^{\infty}\frac{(-x)^r(z)^{2s+r}}{((a+1)/2)_{s+r}\,s!r!}V_{\mu+2s+r,\upsilon}(k,y-2z). \tag{2.6}$$

Finally, on using result (1.6) in the right hand side of (2.6), we get the Formula 2 after a little simplification.

**Formula 3:**

$$\sum_{n=0}^{\infty}\frac{R_n(a,x)\Gamma(\alpha+2n)}{(2)^{2n}((a+1)/2)_n}D_{-\alpha-2n}\left(\sqrt{2}y\right) = \Gamma\left(\frac{a+1}{2}\right)\frac{1}{\left(\sqrt{2}y\right)^{\alpha}}e^{-\left(y^2/2\right)}$$

$$H^{0,1:1,0;1,0}_{1,1:0,1;0,1}\left[\begin{matrix}-\left(2^{-6}y^{-4}\right)\\ \left(x/8y^2\right)\end{matrix}\middle|\begin{matrix}(1-\alpha;4,2) & : & - & ;-\\ \left(\frac{(1-a)}{2};1,1\right) & : & (0,1) & ;(0,1)\end{matrix}\right]. \tag{2.7}$$

$(\alpha>0, y>0)$

**Proof:**

To evaluate the Formula 3, again we use (2.2), replace t by $t^2/8$, multiply by $e^{-yt}t^{\alpha-1}$ and integrate with respect to t within the limit 0 to $\infty$ both sides. After changing the order of summation and integration in the left hand side, we get

$$\sum_{n=0}^{\infty}\frac{R_n(a,x)}{2^{3n}((a+1)/2)_n}\int_0^{\infty}e^{-\left(yt+t^2/4\right)}t^{\alpha+2n-1}\,dt$$

$$=\int_0^{\infty}e^{-yt}t^{\alpha-1}\;F^{0:0:0}_{1:0:0}\left[\begin{matrix}- & :-;-; \\ \frac{(a+1)}{2} & :-;-;\end{matrix}\;\frac{t^4}{64},-\frac{xt^2}{8}\right]dt \tag{2.8}$$

Now we express integral involved in the left hand side of the above equation(2.8), in terms of Parabolic cylinder function with the help of result [5, p.337,equation 3.462(1)]. In the right hand side of (2.8) we express the Kampe de Feriet function in contour form, change the order of contour integrals and t-integral, and evaluate the t-integral. Reinterpreting the contour integrals in terms of H-function of two variables, we arrive at the Formula 3 after a little simplification.

## 3. SPECIAL CASES

**1.** In the Formula 1, if we take, $a=1, \beta=1/2, z=y$, we get an interesting formula involving Laguerre polynomial $L_n^{(\alpha)}$, given below which is believed to be new:

$$\sum_{n=0}^{\infty} \frac{L_n^{(n)}(x)}{(2)^{3n}\, n!} V_{\alpha+2n-1/2,-1/2}(y,y) = \frac{\sqrt{\pi}}{\left(\sqrt{2}\, y\right)^{\alpha}}$$

$$H^{0,1\,:1,0;1,0}_{2,2:0,1;0,1}\left[\begin{array}{c|c} -\left(1/2^8 y^4\right) & (1-\alpha;4,2),((2-\alpha)/4;1,1/2):\ -\ ;\ - \\ \left(x/2^4 y^2\right) & (0;1,1),((2-\alpha)/4;1,1/2):(0,1);(0,1) \end{array}\right]. \tag{3.1}$$

$$\left(y \in R^+, \mathrm{Re}(\alpha) > 0\right)$$

**2.** If in the Formula 2, put y = 2z, we get the following interesting expansion formula involving ${}_1F_1$

$$\sum_{n=0}^{\infty} \frac{R_n(a,x)z^n}{((a+1)/2)_n} V_{n+\mu,\upsilon}(k,2z) =$$

$$\sum_{s,r=0}^{\infty} \frac{(-1)^r (zx)^r (z)^{2s}}{((a+1)/2)_{s+r}\, s!r!} \frac{2^{\mu+2s+r-1/2} k^{\upsilon+1/2}}{\Gamma(\upsilon+1)} \Gamma\left(\frac{\mu+2s+r+\upsilon+1}{2}\right)$$

$${}_1F_1\left[\frac{\mu+2s+r+\upsilon+1}{2};\upsilon+1;-k^2\right] \tag{3.2}$$

**3.** If in the Formula 3, if we put x = -1, $y = 1/\sqrt{2}$ we easily get the following formula involving H-function of one variable

$$\sum_{n=0}^{\infty}\frac{R_n(a,-1)\Gamma(\alpha+2n)}{(2)^{2n}((a+1)/2)_n}D_{-\alpha-2n}(1) = \frac{\Gamma((a+1)/2)}{\sqrt{\pi}}e^{-(1/4)}2^{(a-\alpha-1)}$$

$$H_{2,2}^{2,1}\left[\left(-\frac{1}{64}\right)\middle|\begin{matrix}(1-\alpha,4),(a-\alpha,2)\\(0,1),\left(\frac{a}{2}-\alpha,3\right)\end{matrix}\right] \tag{3.3}$$

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# A TWO WAREHOUSE INVENTORY MODEL FOR DETERIORATING ITEMS WITH DEMAND DEPENDENT PRODUCTION

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**ABSTRACT**

*In this paper a two warehouse inventory model for deteriorating items with demand dependent production is studied. Deterioration rate is constant. Holding cost is a linear increasing function of time. Most researchers on inventory models do not consider simultaneously all these phenomena with two-warehouse. Cost minimization technique is applied to solve the model. Numerical examples are implemented to illustrate the inventory model.*

**Keywords:** Demand Dependent Production, Deterioration, Two Warehouses.

## 1. INTRODUCTION

First, in case of constant demand, in (1983) **Sarma** developed a retailing inventory model where the transportation cost of inventory between warehouses is taken into account with no shortage allowed. In (1985) **Murdeshwar** and **Sathe** extended **Sarma's** models work to the case of production of non-perishable items. **Sarma** (1987) proposed a two-warehouse inventory model for deteriorating items with infinite replenishment rate. In case of time dependent demand, **Pakkala** and **Achary** (1992) proposed a similar model with finite replenishment rate. Later **Bunia** and **Maiti** (1998) developed a two-warehouse inventory model in which items are deteriorating and delivered in a continuous release pattern and **Yang** (2004) developed a two warehouse inventory model with shortage and inflation. **Malik, Singh and Gupta** (2008) developed a two warehouse inventory model for deteriorating items under FIFO dispatching policy and time dependent demand. **Singh** and **Malik** (2009) considered a two warehouses nventory model with inflation induced demand under the credit period.

In this study we consider an inventory model on two warehouses for deteriorating items with constant demand. Production rate is demand dependent and deterioration rate is constant. Linear holding cost is also considered. Our objective is to find the excess stock to be kept in RW so as to minimize the total system cost.

## 2. NOTATION AND ASSUMPTIONS:

### ASSUMPTIONS

1. The OW has limited capacity of w units and the RW has unlimited capacity.
2. Inventory decreases due to demand and deterioration.
3. Deterioration units can't be repaired or replaced.
4. The RW is located near the OW and thus the transportation cost between them is negligible.
5. The inventory cost (including carrying cost and deterioration cost) in RW is higher than that in OW.

### NOTATIONS

$pd$ Demand dependent production

$d$ Demand rate

H Total planning horizon

w Fix inventory level

$\alpha$ Constant deterioration rate of inventory items in OW, >0

Constant deterioration rate of inventory items in RW, >0

A category of production cycle that only OW is used within the cycle

A category of production cycle that both OW and RW are used within the cycle

n Number of production cycles during the entire horizon H

m The index of production cycle whose type switches from to or from to.

The time at the beginning of the $i^{th}$ production cycle belonging to

The time at which the inventory level in OW first reaches w units within the $i^{th}$ production cycle

The time at the end of production of the $i^{th}$ production cycle

The time at which all inventory units in RW are depleted within the $i^{th}$ production cycle

$I_{i1}$ Inventory level in OW at time t with $t \in [t_{i0}, t_{i1}]$

$I_{i2}$ Inventory level in RW at time t with $t \in [t_{i1}, t_{i2}]$

$I_{i3}$ Inventory level in OW at time t with $t \in [t_{i2}, t_{i3}]$

$I_{i4}$ Inventory level in OW at time t with $t \in [t_{i3}, t_{i+1,0}]$

$I_{i5}$ Inventory level in OW at time t with $t \in [t_{i1}, t_{i3}]$

The time at the beginning of the j[th] production cycle belonging to

The time at the beginning of the j[th] production cycle belonging to

The time at the end of production for the j[th] production cycle

Inventory level in OW at time t with

Inventory level in RW at time t with

The maximum inventory level during the j[th] production cycle

The quantity of deteriorated items during the i[th] production cycle

Setup per production run

Cost of a deteriorated unit

Carrying cost per inventory unit held in OW per unit time

Carrying cost per inventory unit held in RW per unit time

TC total system cost during H

## 3. MATHEMATICAL FORMULATION AND SOLUTION

The inventory level during a production cycle in which both OW and RW are used with any arbitrary production cycle i belonging to $L_2$, the cycle starts from $t_{i0}$ and ends at $t_{i+1,0}$. Production, demand and deterioration occur simultaneously at. During items accumulate from 0 up to w units in OW. Any production quantity exceeding w will be stored in RW after $t_{i1}$.

$$I_{i1}'(t) + \alpha I_{i1}(t) = (p-1)d, \qquad t_{i0} \le t \le t_{i1} \qquad \dots (1.1)$$

$$I_{i2}'(t) + \beta I_{i2}(t) = (p-1)d, \qquad t_{i1} \le t \le t_{i2} \qquad \dots (1.2)$$

$$I_{i3}'(t) + \beta I_{i3}(t) = -d, \qquad t_{i2} \le t \le t_{i3} \qquad \dots (1.3)$$

$$I_{i4}'(t) + \alpha I_{i4}(t) = -d, \qquad t_{i3} \le t \le t_{i+1,0} \qquad \dots (1.4)$$

$$I_{i5}'(t) + \alpha I_{i5}(t) = 0, \qquad t_{i1} \le t \le t_{i3} \qquad \dots (1.5)$$

With the boundary conditions $I_{i1}(t_{i0})=0$, $I_{i2}(t_{i1})=0$, $I_{i3}(t_{i3})=0$, $I_{i4}(t_{i+1,0})=0$, and $I_{i5}(t_{i1})=w.$

The solutions of the above equations are:

$$I_{i1}(t) = \frac{(p-1)d}{\alpha}\left[1 - e^{\alpha(t_{i0}-t)}\right], \quad \dots (1.6)$$

$$I_{i2}(t) = \frac{(p-1)d}{\beta}\left[1 - e^{\beta(t_{i1}-t)}\right], \quad t_{i1} \le t \le t_{i2} \quad \dots (1.7)$$

$$I_{i3}(t) = \frac{d}{\beta}\left[e^{\beta(t_{i3}-t)} - 1\right], \quad t_{i2} \le t \le t_{i3} \quad \dots (1.8)$$

$$I_{i4}(t) = \frac{d}{\alpha}\left[e^{\alpha(t_{i+1,0}-t)} - 1\right], \quad t_{i3} \le t \le t_{i+1,0} \quad \dots (1.9)$$

$$I_{i5}(t) = we^{\alpha(t_{i1}-t)}, \quad t_{i1} \le t \le t_{i3} \quad \dots (1.10)$$

The inventory level in RW can be derived as:

$$I_{RW,i} = [\int_{t_{i1}}^{t_{i2}} I_{i2}(t)dt + \int_{t_{i2}}^{t_{i3}} I_{i3}(t)dt]$$

$$= \left[\frac{(p-1)d}{\beta}\left\{t_{i2} - t_{i1} + \frac{e^{\beta(t_{i1}-t_{i2})}}{\beta} - \frac{1}{\beta}\right\} + \frac{d}{\beta}\left\{t_{i2} - t_{i3} + \frac{e^{\beta(t_{i3}-t_{i2})}}{\beta} - \frac{1}{\beta}\right\}\right] \quad \dots (1.11)$$

In addition, the following relation exists:

$$I_{i2}(t_{i2}) = I_{i3}(t_{i3}) \Rightarrow \frac{(p-1)d}{\beta}\left[1 - e^{\beta(t_{i1}-t_{i2})}\right] = \frac{d}{\beta}\left[e^{\beta(t_{i3}-t_{i3})} - 1\right]$$

$$\Rightarrow \quad t_{i1} = t_{i2} \quad \dots (1.12)$$

The inventory level in OW can be derived as:

$$I_{OW,i} = [\int_{t_{i0}}^{t_{i1}} I_{i1}(t)dt + \int_{t_{i3}}^{t_{i+1,0}} I_{i4}(t)dt + + \int_{t_{i1}}^{t_{i3}} I_{i5}(t)dt]$$

$$= \left[\frac{(p-1)d}{\alpha}\left\{t_{i1} - t_{i0} + \frac{e^{\alpha(t_{i0}-t_{i1})}}{\alpha} - \frac{1}{\alpha}\right\} + \frac{d}{\alpha}\left\{t_{i3} - t_{i+1,0} + \frac{e^{\alpha(t_{i+1,0}-t_{i3})}}{\alpha} - \frac{1}{\alpha}\right\}\right.$$

$$\left. + W\left\{-\frac{e^{\alpha(t_{i1}-t_{i3})}}{\alpha} + \frac{1}{\alpha}\right\}\right] \quad \dots (1.13)$$

Accordingly the following relation exists:

$I_{i4}(t_{i3}) = I_{i5}(t_{i3}) \Rightarrow \dfrac{d}{\alpha}\left[e^{\alpha(t_{i+1,0}-t_{i3})} - 1\right] = we^{\alpha(t_{i1}-t_{i3})}$, **neglecting the second and higher degree terms, we get**

$$t_{i3} = \frac{w(1+\alpha t_{i1}) - d\, t_{i+1,0}}{\alpha w - d} \qquad \text{... (1.14)}$$

$I_{i1}(t_{i1}) = I_{i5}(t_{i1}) \Rightarrow \quad \dfrac{(p-1)d}{\alpha}\left[1 - e^{\alpha(t_{i0}-t_{i1})}\right] = w$, **neglecting the second and higher degree terms, we get**

$$t_{i0} = t_{i1} - \frac{1}{\alpha}\log\left\{1 - \frac{\alpha w}{(p-1)d}\right\} \qquad \text{... (1.15)}$$

**The quantity of deteriorated items during the production cycle i is:**

$$D_i = \alpha I_{OW,i} + \beta I_{RW,i}$$

$$= \left[(p-1)d\left\{t_{i1} - t_{i0} + \frac{e^{\alpha(t_{i0}-t_{i1})}}{\alpha} - \frac{1}{\alpha}\right\} + d\left\{t_{i3} - t_{i+1,0} + \frac{e^{\alpha(t_{i+1,0}-t_{i3})}}{\alpha} - \frac{1}{\alpha}\right\} + W\left\{1 - e^{\alpha(t_{i1}-t_{i3})}\right\}\right]$$

$$+\left[(p-1)d\left\{t_{i2} - t_{i1} + \frac{e^{\beta(t_{i1}-t_{i2})}}{\beta} - \frac{1}{\beta}\right\} + d\left\{t_{i2} - t_{i3} + \frac{e^{\beta(t_{i3}-t_{i2})}}{\beta} - \frac{1}{\beta}\right\}\right] \qquad \text{... (1.16)}$$

**The previous results can be simplified to express the inventory level in the one-warehouse model. For any arbitrary production cycle j belonging to $L_1$, $I_{j1}$ is similar to $I_{i1}$ in behavior, so is $I_{j2}$ to $I_{i4}$. Therefore, by using the analogous derivations of and, and can be stated by**

$$I_{j1}(t) = \frac{(p-1)d}{\alpha}\left[1 - e^{\alpha(t_{j0}-t)}\right] \qquad t_{j0} \le t \le t_{j1} \qquad \text{... (1.17)}$$

$$I_{j2}(t) = \frac{d}{\alpha}\left[e^{\alpha(t_{j+1,0}-t)} - 1\right] \qquad t_{j1} \le t \le t_{j+1,0} \qquad \text{... (1.18)}$$

**The inventory levels over the cycle j can be expressed as**

$$I_{RW,j} = 0 \qquad \text{... (1.19)}$$

$$I_{OW,j}=\left[\int_{t_{j0}}^{t_{j1}} I_{j1}(t)dt+\int_{t_{j1}}^{t_{j+1,0}} I_{j2}(t)dt\right]$$

$$=\left[\frac{(p-1)d}{\alpha}\left\{t_{j1}-t_{j0}+\frac{e^{\alpha(t_{j0}-t_{j1})}}{\alpha}-\frac{1}{\alpha}\right\}+\frac{d}{\alpha}\left\{t_{j1}-t_{j+1,0}+\frac{e^{\alpha(t_{j+1,0}-t_{j1})}}{\alpha}-\frac{1}{\alpha}\right\}\right] \quad \ldots (1.20)$$

The amount of deteriorated items during the production cycle j is:

$$D_j=\alpha\, I_{OW,j}=\left[(p-1)d\left\{t_{j1}-t_{j0}+\frac{e^{\alpha(t_{j0}-t_{j1})}}{\alpha}-\frac{1}{\alpha}\right\}+d\left\{t_{j1}-t_{j+1,0}+\frac{e^{\alpha(t_{j+1,0}-t_{j1})}}{\alpha}-\frac{1}{\alpha}\right\}\right] \quad \ldots (1.21)$$

The following relation exists

$$I_{j1}(t_{j1})=I_{j2}(t_{j1})\Rightarrow \frac{(p-1)d}{\alpha}\left[1-e^{\alpha(t_{j0}-t_{j1})}\right]=\frac{d}{\alpha}\left[e^{\alpha(t_{j+1,0}-t_{j1})}-1\right]$$

Neglecting the second and higher degree terms, we get

$$t_{j1}=\frac{1}{p}\left\{t_{j+1,0}-(p-1)t_{j0}\right\} \quad \ldots (1.22)$$

Obviously, the maximum inventory level within a cycle j occurs at $t_{j1}$ and then

$$U_j=I_{j1}(t_{j1})=\frac{(p-1)d}{\alpha}\left[1-e^{\alpha(t_{j0}-t_{j1})}\right] \quad \ldots (1.23)$$

The total system cost consist of set up cost, holding cost and deterioration cost incurred in each production cycle within the planning horizon H and can be expressed as:

$$TC=nC_1+(C_{RW}+\phi t)\sum_i I_{RW,i}+(C_{OW}+\gamma t)\sum_i I_{OW,i}+(C_{OW}+\gamma t)\sum_j I_{OW,j}+C_2\sum_i D_i+C_2\sum_j D_j \quad \ldots (1.24)$$

To minimize the total cost per unit time, the optimal values of $t_{i0}$ and $t_{i+1,0}$ can be obtained by solving the following system of equations simultaneously

$$\frac{\partial TC}{\partial t_{i0}}=0 \quad \text{and} \quad \frac{\partial TC}{\partial t_{i+1,0}}=0 \quad \ldots (1.25)$$

Provided, they satisfy the following conditions:

And
$$\left.\begin{aligned}&\frac{\partial^2 TC}{\partial t_{i0}^2}>0,\frac{\partial^2 TC}{\partial t_{i+1,0}^2}=0>0\\&\left(\frac{\partial^2 TC}{\partial t_{i0}^2}\right)\left(\frac{\partial^2 TC}{\partial t_{i+1,0}^2}\right)-\left(\frac{\partial^2 TC}{\partial t_{i0}\partial t_{i+1,0}}\right)^2>0\end{aligned}\right\} \quad \ldots (1.26)$$

To derive the optimal solutions, the classical optimization technique can be applied.

## 4. NUMERICAL ILLUSTRATION

The Optimal replenishment policy to minimize the total present value cost is derived by using the methodology. To analyze the problem, the following parameters are assumed.

$d$=250, $w=350, C_{RW}=10, \gamma=0, \phi=0, C_{OW}=6, C_2=60, C_1=2600, \alpha=0.03, \beta=0.02,$

p=2000, $H=12$

From the set of parameters we observed that the result is reasonable for which the fraction of stocks stored in RW will decrease when the production lot becomes smaller.

## 5. CONCLUSION

Today's, companies have recognized that besides maximizing profit, customer satisfaction plays a great importance for getting and keeping a successful position in a competitive market. This general model can be applied to the inventory problem of either time increasing or time decreasing market demand. A production inventory model is developed for deteriorating items and demand dependent production rate with two warehouses. In both warehouse, deterioration rate is constant. Linear holding cost is also considered. The proper inventory level should be depending on the relationship between the investment in inventory and the service level. The proposed model can be extended in numerous ways. For example, we may extend the constant demand to be a more generalized demand pattern that fluctuates with time or stock dependent and stochastic demand.

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# TRANSIENT THERMO ELASTIC PROBLEM OF A THIN ANNULAR DISC BY INTEGRAL TRANSFORM METHOD

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## ABSTRACT

In this paper an attempts is made to solve problem of thermoelasticity. We have to determine the unknown temperature, displacement and stress function of a thin annular disc occupying the space D: $a \le r \le b$, $0 \le z \le h$ applying finite hankel transform, finite fourier sine transform and Laplace transform techniques.

**Keywords and Phrases:** Annular disc, unsteady-state problem, thermoelastic problem, finite hankel transform, finite fourier sine transform and Laplace transform.

**2001 Mathematics Subject Classification**: 73

## 1. INTRODUCTION

Deshmukh and Wankhede [1], and Grysa and Kozlowski [2] have studied one dimensional transient thermo elastic problems and derived the heating temperature. Khobragade and Khalsa [3] have discussed transient thermo elastic problem of a thin annual disc by using Marchi and Zgrablich transform.

We have determine the temperature displacement and stress function of a thin annular disc of thickness h occupying the space D: $a \le r \le b$, $0 \le z \le h$ by using finite Hankel transform, finite Fourier Sine transform and Laplace transform with different boundary conditions stated in the problem.
which are given by Sneddon [4,5] finite Hankel integral transform

$$\bar{f}_p(n) = \int_a^b x f(x) S_p(K_1, K_2, \mu_n x)\,dx \qquad \ldots(1.1)$$

and its inversion is given by

$$f(x)=\sum_{n=1}^{\infty}\frac{\bar{f}_p(n)S_p(K_1,K_2,\mu_n x)}{dn} \qquad ...(1.2)$$

where

$$dn=\frac{2}{\pi^2 h^2}\Big[(J_p(a\mu_n)+K_1\mu_n J_p'(a\mu_n))^2.(1+\mu_n^{\,2}(1-(\frac{p}{b\mu_n})^2)^{K_2^2}$$

$$-(J_p(b\mu_n)+K_2\mu_n J_p'(b\mu_n))^2.(1+\mu_n^{\,2}(1-(\frac{p}{a\mu_n})^2)^{K_1^2}\Big]$$

$$\frac{1}{(J_p(b\mu_n)+K_2\mu_n J_p'(b\mu_n))^2} \qquad ...(1.3)$$

and operational property

$$\int_a^b x(\frac{\partial^2 f}{\partial x^2}+\frac{1}{x}\frac{\partial f}{\partial x}-\frac{p^2}{x^2}f)S_p(K_1,K_2\mu_n x)dx$$

$$=\frac{2}{\pi}\frac{[(J_p(a\mu_n)+K_1\mu_n J_p'(a\mu_n)]}{[(J_p(b\mu_n)+K_2\mu_n J_p'(b\mu_n)]}.\Big[f(b)+K_2 f'(b)\Big]$$

$$-\frac{2}{\pi}[f(a)+K_1 f'(a)]-\mu_n^{\,2}\bar{f}(\mu_n) \qquad ...(1.4)$$

and

$$\int_a^b x(\frac{\partial^2 f}{\partial x^2}+\frac{1}{x}\frac{\partial f}{\partial x}-\frac{p^2}{x^2}f)S_p(K_1,K_2,\mu_n x)dx$$

$$=\frac{2}{\pi}\frac{J_p^{\bullet}(a\mu_n)}{J_p(b\mu_n)}f(b)-\frac{2}{\pi}f(a)-\mu_n^{\,2}\bar{f}(\mu_n) \qquad ...(1.5)$$

the finite fourier sine transform of f (x) $0<x<l$ is defined as

$$\bar{f}_s(m)=\int_0^l f(x)\sin\frac{m\pi x}{l}dx, \qquad ...(1.6)$$

where m is an integer.

The inverse finite fourier sine transform of $\bar{f}$ (m) is given by

$$f(x)=\frac{2}{l}\sum_{m=1}^{\infty}\bar{f}(m)\sin\frac{m\pi x}{l} \qquad ...(1.7)$$

## 2. STATEMENT OF THE PROBLEM

Consider a thin annular isotropic disc of thickness h occupying the space $a \le r \le b$, $0 \le z \le h$. The differential equation governing the displacement function U (r, z, t) as

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = (1+\nu)a_t T \,, \qquad ...(2.1)$$

where $U_r = 0$ at $r = a$ and $r = b$, ...(2.2)

$\nu$ and $a_t$ are the Poission's ratio and the linear coefficient of thermal expansion of the material of the disc respectively and T (r, z, t) is the temperature of the disc satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{K}\frac{\partial T}{\partial t} \qquad ...(2.3)$$

subject to the initial condition

$$T(r,z,0) = T_0 \,. \qquad ...(2.4)$$

The boundary conditions

$$T(a,z,t) + K_1 \frac{\partial}{\partial r} T(a,z,t) = f_1(z,t) \qquad ...(2.5)$$

and

$$T(b,z,t) + K_2 \frac{\partial}{\partial r} T(b,z,t) = f_2(z,t) \,, \qquad ...(2.6)$$

where K is the thermal diffusivity and $K_1$, $K_2$ are the radiation constant on the two curved surfaces of the material of the disc and

$$T(r,z,t) \text{ at } z = h = f_3(t) \text{ and} \qquad ...(2.7)$$

$$T(r,z,t) \text{ at } z = 0 = f_4(t) \,. \qquad ...(2.8)$$

The stress function $\sigma_{rr}$ and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -\frac{2\mu}{r}\frac{\partial U}{\partial r} \text{ and} \qquad ...(2.9)$$

$$\sigma_{\theta\theta} = -2\,\mu \frac{\partial^2 U}{\partial r^2} \,, \qquad ...(2.10)$$

here $\mu$ is thae Lame's constants, while each of the stress functions $\sigma_{rz}$, $\sigma_{zz}$ and $_{\theta z}$ are zero within the disc in the plane state of stress.

## 3 SOLUTION OF THE PROBLEM

Applying finite hankel transform to equation (2.3) defined by the equation (1.1) with p = 0 and boundary condition given by (2.4), (2.5) and (2.6), we get

$$\frac{2}{\pi}\frac{[(J_0(a\mu_n)+K_1\mu_n J_0'(a\mu_n)]}{[(J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n)]}f_2(z,t)-\frac{2}{\pi}f_1(z,t)-\mu_n^{\,2}\overline{T}(\mu_n,z,t)+\frac{\partial^2\overline{T}}{\partial z^2}=\frac{1}{K}\frac{\partial\overline{T}}{\partial t}, \quad \ldots(3.1)$$

where $\overline{T}(\mu_n, z, t)$ represents finite hankel transform of T (r, z, t) .

Now applying finite fourier sine transform given by the equation (1.6) and boundary conditions (2.7) and (2.8), we get

$$\frac{2}{\pi}\frac{[(J_0(a\mu_n)+K_1\mu_n J_0'(a\mu_n)]}{[(J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n)]}\bar{f}_2(m,t)-\frac{2}{\pi}\bar{f}_1(m,t)-\mu_n^{\,2}\overline{\overline{T}}(\mu_n,m,t)$$

$$-\frac{m\pi}{h}(-1)^m f_3(t)+\frac{m\pi}{h}f_4(t)-\frac{m^2\pi^2}{h^2}\overline{\overline{T}}(\mu_n,m,t)=\frac{1}{K}\frac{\partial\overline{\overline{T}}}{\partial t}, \quad \ldots(3.3)$$

Where $\bar{f}_1(m, t)$, $\bar{f}_2(m, t)$, $\overline{\overline{T}}(\mu_n, m, t)$ represent finite fourier sine transform of $f_1(z, t)$, $f_2(z, t)$, $\overline{T}(\mu_n, z, t)$, respectively.

Now using Laplace transform with boundary condition (2.4), we get

$$\overline{\overline{\overline{T}}}(\mu_n,m,s)=\frac{1}{(\mu_n^{\,2}+\frac{m^2\pi^2}{h^2}+\frac{s}{K})}$$

$$\Big[\frac{T_0}{K}+\frac{2}{\pi}\frac{[(J_0(a\mu_n)+K_1\mu_n J_0'(a\mu_n)]}{[(J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n)]}\bar{\bar{f}}_2(m,s)$$

$$\frac{2}{\pi}\bar{\bar{f}}_1(m,s)-\frac{m\pi}{h}(-1)^m\bar{f}_3(s)+\frac{m\pi}{h}\bar{f}_4(s)\Big], \quad \ldots(3.4)$$

where $\bar{\bar{f}}_1(m, s)$, $\bar{\bar{f}}_2(m, s)$, $\overline{\overline{\overline{T}}}(\mu_n, m, s)$, $\bar{f}_3(s)$, $\bar{f}_4(s)$ represent Laplace transform f $\bar{f}_1(m, t)$, $\bar{f}_2(m, t)$, $\overline{\overline{T}}(\mu_n, m, t)$, $f_3(t)$, $f_4(t)$ respectively.

Now using inverse Laplace transform and convolution theorem, we get

$$\overline{\overline{T}}(\mu_n,m,t)=T_0 e^{-K(\mu_n^{\,2}+\frac{m^2\pi^2}{h^2})t}+\frac{2}{\pi}\Big[K\frac{J_0(a\mu_n)+K_1\mu_n J_0'(a\mu_n)}{J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n)}$$

$$\int_0^t \bar{f}_2(m,u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}du - K\int_0^t \bar{f}_1(m,u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}du\Big]$$

$$-\frac{m\pi}{h}(-1)^m K\int_0^t f_3(u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}du$$

$$+\frac{m\pi}{h}K\int_0^t f_4(u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)du} \quad ...(3.5)$$

Now using inverse finite fourier sine transform given by the equation (1.7), we get

$$\bar{T}(\mu_n,z,t)=\frac{2T_0}{h}\sum_{m=1}^{\infty}e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})t}\sin\frac{m\pi z}{h}+\frac{2}{\pi}\Big[K\frac{J_0(a\mu_n)+K_1\mu_n J_0'(a\mu_n)}{J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n)}\frac{2}{h}$$

$$\cdot\sum_{m=1}^{\infty}\int_0^t \bar{f}_2(m,u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du$$

$$-\frac{2K}{h}\sum_{m=1}^{\infty}\int_0^t \bar{f}_1(m,u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du\Big]$$

$$-\frac{2K\pi}{h^2}\sum_{m=1}^{\infty}(-1)^m m\int_0^t f_3(u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du$$

$$+\frac{2K\pi}{h^2}\sum_{m=1}^{\infty}m\int_0^t f_4(u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du. \quad ...(3.6)$$

Now again using inverse finite hankel transform given by equation (1.2), we get

$$T(r,z,t)=\sum_{n=1}^{\infty}\Big[\frac{2T_0}{K}\sum_{m=1}^{\infty}e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})t}\sin\frac{m\pi z}{h}+\frac{2}{\pi}\Big(K\frac{J_0(a\mu_n)+K_1\mu_n J_0'(a\mu_n)}{J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n)}$$

$$\frac{2}{h}\sum_{m=1}^{\infty}\bar{f}_2(m,u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du$$

$$-\frac{2K}{h}\sum_{m=1}^{\infty}\int_0^t \bar{f}_1(m,u)\, e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du\Big)$$

$$-\frac{2K\pi}{h^2}\sum_{m=1}^{\infty}(-1)^m m\int_0^t f_3(u)\, e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du$$

$$+\frac{2K\pi}{h^2}\sum_{m=1}^{\infty} m\int_0^t f_4(u)\, e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du\ \Big]\frac{S_0(K_1,K_2,\mu_n r)}{dn}, \qquad \ldots(3.7)$$

where

$$dn=\frac{2}{\pi^2\mu_n^2}\Big[(J_0(a\mu_n)+K_1\mu_n J_0'(a\mu_n))^2(1+\mu_n^2K_2^2)-(J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n))^2$$

$$(1+\mu_n^2K_1^2)\Big]\frac{1}{(J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n))^2} \qquad \ldots(3.8)$$

## 4. DETERMINATION OF THERMO-ELASTIC DISPLACEMENT

Using finite hankel transform (1.1)with p=0 equation (2.1) with boundary condition given by equation (2.2) and operational property (1.5), we get

$$-\mu_n^2\bar{U}(\mu_n,z,t)=(1+\nu)a_t\bar{T} \quad .$$

Now using inverse finite hankel transform (1.2) and (1.3), we get

$$U(r,z,t)=-(1+\nu)a_t\sum_{n=1}^{\infty}\frac{\bar{T}(\mu_n,z,t)S_0(K_1,K_2,\mu_n r)}{\mu_n^2 dn}, \qquad \ldots(4.1)$$

where $\bar{T}$ ($\mu_n$, z, t) is given by the equation (3.6) and dn is given by equation (3.8)

## 5. DETERMINATION OF STRESS FUNCTIONS

Using (4.1) in (2.7) and (2.8), the stress functions are obtained as

$$\sigma_{rr}=\frac{2\mu}{r}(1+\nu)a_t\sum_{n=1}^{\infty}\frac{\bar{T}(\mu_n,z,t)S_0'(K_1,K_2,\mu_n r)}{\mu_n dn} \quad \text{and} \qquad \ldots(5.1)$$

$$\sigma_{\theta\theta}=\frac{2\mu}{r}(1+\nu)a_t\sum_{n=1}^{\infty}\frac{\bar{T}(\mu_n,z,t)S_0''(K_1,K_2,\mu_n r)}{dn} \qquad \ldots(5.2)$$

## 6. SPECIAL CASE

When

$$[T(r,z,t)]_{at\ t=0} = 0\ , \qquad ...(6.1)$$

$$[T(r,z,t) + K_1 T'(r,z,t)]_{at\ r=a} = 0\ , \qquad ...(6.2)$$

$$[T(r,z,t) + K_2 T'(r,z,t)]_{at\ r=b} = 0\ , \qquad ...(6.3)$$

the equation (3.6), reduces to

$$\overline{T}(\mu_n, z, t) = -\frac{2\pi}{h^2}\sum_{m=1}^{\infty}(-1)^m m\int_0^t f_3(u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du$$

$$+\frac{2K\pi}{h^2}\sum_{m=1}^{\infty} m\int_0^t f_4(u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du \qquad ...(6.4)$$

From equation (1.2), we have

$$T(r,z,t) = \sum_{n=1}^{\infty}\frac{\overline{T}(\mu_n, z, t)S_0(K_1, K_2, \mu_n r)}{dn} \quad \text{and} \qquad ...(6.5)$$

Using equation (6.4) using in equation (6.5)

$$T(r,z,t) = \sum_{n=1}^{\infty}\Big[-\frac{2\pi}{h^2}\sum_{m=1}^{\infty}(-1)^m m\int_0^t f_3(u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du$$

$$+\frac{2K\pi}{h^2}\sum_{m=1}^{\infty} m\int_0^t f_4(u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du\ \Big]\frac{S_0(K_1, K_2, \mu_n r)}{dn}. \qquad ...(6.6)$$

## 7. EXAMPLE

For

$$[T(r,z,t)]_{at\ z=h} = e^{-C_1 t} \qquad ...(7.1)$$

and

$$[T(r,z,t)]_{at\ z=0} = e^{-C_2 t} \qquad ...(7.1)$$

we get by (3.7)

$$T(r,z,t) = \sum_{n=1}^{\infty}\Big[\frac{2T_0}{h}\sum_{m=1}^{\infty} e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})t}\sin\frac{m\pi z}{h} + \frac{2}{\pi}\Big(K\frac{J_0(a\mu_n)+K_1\mu_n J_0'(a\mu_n)}{J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n)}$$

$$\frac{2}{h}\sum_{m=1}^{\infty}\int_0^t f_2(m,u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du$$

$$-\frac{2K}{h}\sum_{m=1}^{\infty}\int_0^t f_1(m,u)e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)}\sin\frac{m\pi z}{h}du\Big)$$

$$-\frac{2K\pi}{h^2}\sum_{m=1}^{\infty}(-1)^m m\frac{e^{-C_1 t}-e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})t}}{K(\mu_n^2+\frac{m^2\pi^2}{h^2})-c_1}\sin\frac{m\pi z}{h}$$

$$+\frac{2K\pi}{h^2}\sum_{m=1}^{\infty}m\frac{e^{-C_2 t}-e^{-K(\mu_n^2+\frac{m^2\pi^2}{h^2})t}}{K(\mu_n^2+\frac{m^2\pi^2}{h^2})-c_2}\sin\frac{m\pi z}{h}\Big]\frac{S_0(K_1,K_2,\mu_n r)}{dn} \quad ...(7.3)$$

Where dn is given by

$$dn=\frac{2}{\pi^2\mu_n^2}\Big[(J_0(a\mu_n)+K_1\mu_n J_0'(a\mu_n))^2(1+\mu_n^2K_2^2)-J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n))^2$$

$$(1+\mu_n^2K_1^2)\Big]\frac{1}{(J_0(b\mu_n)+K_2\mu_n J_0'(b\mu_n))^2} \quad ...(7.4)$$

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# ON THE RATIO OF EXPONENTIATED EXPONENTIAL AND GAMMA RANDOM VARIABLES

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## ABSTRACT

*The distributions of two independent random variables arise in many applied problems and have been extensively studied by many researchers. This article is related to the distribution of ratio Z = X/Y when X and Y are independently distributed as exponentiated exponential and two parameter gamma random variables respectively. The pdf, cdf and moments are also derived for the obtained distribution. Graphs for pdf and cdf of Z are also provided for different combinations of the parameters involved in the expression.*

**Keywords:** Exponentiated exponential distribution, two parameter gamma distribution, ratio of random variables, probability density function (pdf), cumulative density function (cdf).

**Mathematics Subject Classification (2000):** 60E05.82E15.

## 1. INTRODUCTION

For given random variables X and Y, the distribution of the ratio X/Y arises in many applied problems of biological and physical sciences and other field of knowledge viz: genetics, medicine, engineering, hydrology, number theory, order statistics, economics and psychology etc. Some of the examples are; Mendelian inheritance ratios in genetics, inventory ratios in genetics mass to energy ratios in nuclear physics, etc. When X and Y are independently distributed random variables and belongs to the same family, the ratio, X/Y has been extensively studied by many researchers, among them few are; Marsaglia.[8],Korthenen and Narula. [2] who studied ratio of random variables for normal family, Press[9]studied for Student's -t family, Basu and Lochner[1]studied for weibull family, Provost [10] and Lee,Hollandand Flueck [7], studied for gamma family, Pham-Gia [8] studied for beta family . However, there is relatively little work on ratio X/Y when X and Y belong to different families. Some of them are the works of Nadarajah [5] Nadarajah and Gupta [9] Nadarajah and Kotz [10]

In this paper we derive the exact distribution of X/Y when X and Y are independent random variables having exponentiated exponential distribution and two-parameter gamma distribution with pdfs as mentioned below:

$$f_X(x) = \lambda/\theta[1-\exp(-x/\theta)]^{\lambda-1}\exp(-x/\theta) \text{ ; for } x>0 \text{ ; } \lambda and \theta > 0 \quad (1.1)$$

$$f_Y(y) = \frac{a^m}{\Gamma m}\exp(-ay)y^{m-1}\text{; for } y>0\text{; a and m} > 0 \quad (1.2)$$

## 2. PDF AND CDF RATIO OF X AND Y

*Theorem 2.1:* Suppose X and Y are distributed according to (1.1) and (1.2) respectively. The cdf and pdf of Z =X/Y can be expressed as ;

$$F_Z(z) = \sum_{k=0}^{\lambda}(-1)^k\,{}^{\lambda}C_k\left[1+kz/a\theta\right]^{-m} \quad (2.1)$$

and

$$f_Z(z) = \frac{m}{a\theta}\sum_{k=0}^{\lambda}k(-1)^{k+1}\,{}^{\lambda}C_k\left[1+kz/a\theta\right]^{-(m+1)} \quad (2.2)$$

where

$z \geq 1-m^{1/(m+1)}$

*Proof*:

Let X/Y + Z , then the cdf of Z can be followed by

$$F_Z(z) = \Pr(X/Y \leq z)$$

$$= \Pr(X \leq zy)$$

$$= \int_0^{\infty} F_X(zy) f_Y(y)dy$$

$$= \frac{a^m}{\Gamma m}\int_0^{\infty}\left[1-\exp(-zy/\theta)\right]^{\lambda}\exp(-ay)y^{m-1}dy$$

and differentiating the above expression we get

We see that is zero and is one, satisfying the property of cdf. We also obtained hazard rate function of Z, which is given in section3.

## 3. HAZARD RATE FUNCTION

The hazard rate function is an important quantity, characterizing life phenomena defined by

h(x) = f(x)/(1-F(x) and for the variable Z it is given by

$$h_Z(z)=\left[(m/a\theta)\sum_{k=0}^{\lambda}k(-1)^{k+1}\,{}^{\lambda}C_k\{1+kz/a\theta\}^{-(m+1)}\right]/\left\{1-\sum_{k=0}^{\lambda}(-1)^{k}\,{}^{\lambda}C_k(1+kz/a\theta)^{-m}\right\} \quad (3.1)$$

## 4. MOMENTS

For the r.v. Z = X/Y whose pdf is given by (2.2) the moments can be obtained as follows

$$E\left(z^r\right)=\left(m/a\theta\right)\sum_{k=0}^{\lambda}k(-1)^{k+1}\,{}^{\lambda}C_k\int_0^{\infty}(1+kz/a\theta)^{-(m+1)}z^r dz$$

$$=m\sum_{k=1}^{\lambda}(-1)^{k+1}\,{}^{\lambda}C_k(a\theta/k)^r B(m-r,r+1) \quad (4.1)$$

where

$$\text{B(p,q)} = \Gamma p\Gamma q/\Gamma(p+q)$$

By taking r =1,2.3.4(m>r) we can find all the four moments and ascertain the shape using these moments.

## 5. PARTICULAR CASES

***CASE - I:***

For $a=1;\theta=1$, the pdf and cdf of Z is represented by:

$$f_Z(z)=m\sum_{k=0}^{\lambda}k(-1)^{k+1}\,{}^{\lambda}C_k(1+kz)^{-(m+1)} \quad (5.1)$$

$$F_Z(z)=\sum_{k=0}^{\lambda}(-1)^{k}\,{}^{\lambda}C_k(1+kz)^{-m} \quad (5.2)$$

***CASE - II:***

For $a=\theta=1$ and $\lambda=1$ the pdf and cdf of Z is represented by:

$$f_Z(z)=m/(1+z)^{m+1} \quad (5.3)$$

$$F_Z(z)=1-\left(1/(1+z)^m\right) \quad (5.4)$$

***CASE - III:***

For $a = \theta = 1$ and = $m$ the pdf and cdf of Z is represented by:

$$F_z(z) = 1/(1 + z)^2 \tag{5.5}$$

$$F_Z(z) = z/(1+z) \tag{5.6}$$

***CASE - IV:***

For $a = \theta = 1/2$ , the pdf and cdf of Z is represented by:

$$f_Z(z) = 4m\sum_{k=0}^{\lambda} k(-1)^{k+1}\,{}^{\lambda}C_k(1+4kz)^{-(m+1)} \tag{5.7}$$

$$F_Z(z) = \sum_{k=0}^{\lambda}(-1)^k\,{}^{\lambda}C_k(1+4kz)^{-m} \tag{5.8}$$

***CASE - V:***

For $a = 1/2, \theta = 4$ and $a = 4.\theta = 1/2$, such that, $a\theta = 2$ the pdf and cdf of Z is represented by

$$f_Z(z) = m/2\sum_{k=0}^{\lambda} k(-1)^{k+1}\,{}^{\lambda}C_k(1+kz/2)^{-(m+1)} \tag{5.9}$$

$$F_Z(z) = \sum(-1)^k\,{}^{\lambda}C_k(1+kz/2)^{-m} \tag{5.10}$$

***CASE - VI:***

For $\lambda = 1$, $a$= 1 and $m$=*1 the* pdf specified by (1.1) and (1.2) reduces to pdf of simple exponential distribution. In this case the distribution.of Z = X/Y as specified in (2.1)and (2.2) reduces to:

$$f_Z(z) = \theta/(z+\theta)^2 \tag{5.11}$$

$$F_Z(z) = z/(z+\theta) \tag{5.12}$$

Examining the equation (5.11) we see that distribution of Z for θ = 1 reduces to rectangular hyperbola. The graph for (5.11) shown by Fig (7) also exhibits this.

## 6. SHAPE OF THE DISTRIBUTION

The shape of the derived distribution are studied by graphs drawn for different values of a, θ and m.

The graph of cdf given by (2.1) are plotted for different values of a, θ and m and are shown in Fig (2), Fig (4) and Fig (5). The graph for pdf given by (2.2) are plotted for different values of a, θ and m and are shown in Fig (1), Fig (2) and Fig (3).

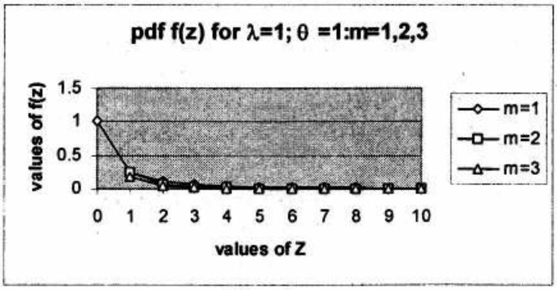


**Fig. 1**

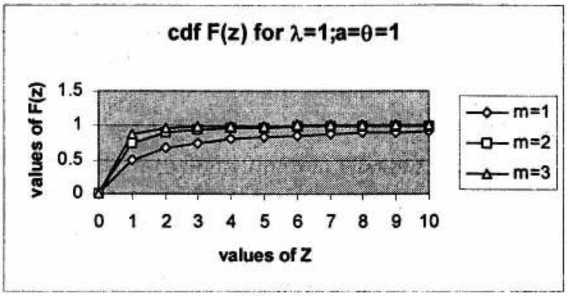


**Fig. 2**

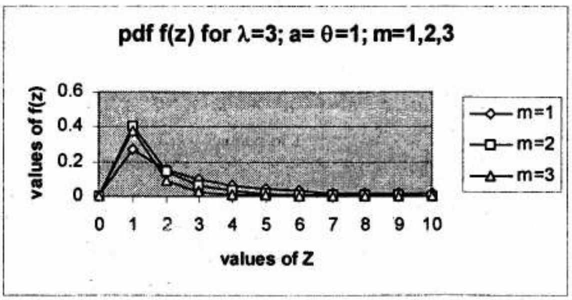


**Fig. 3**

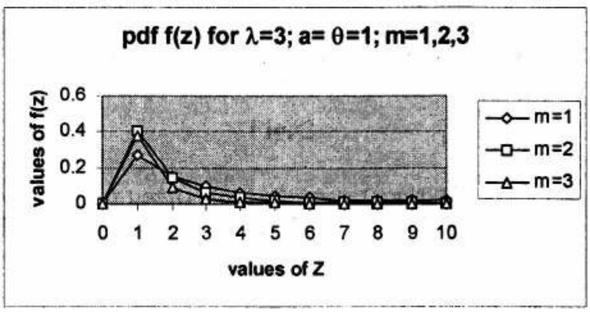


**Fig. 4**

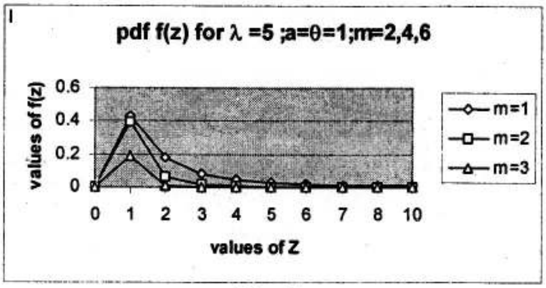


Fig. 5

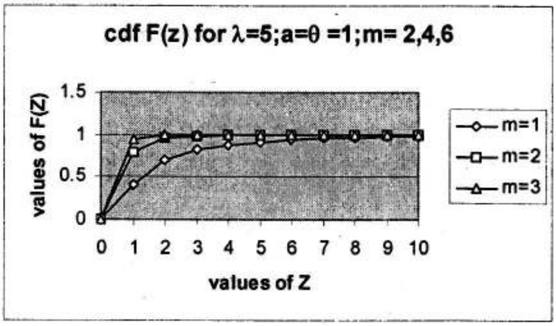


Fig. 6

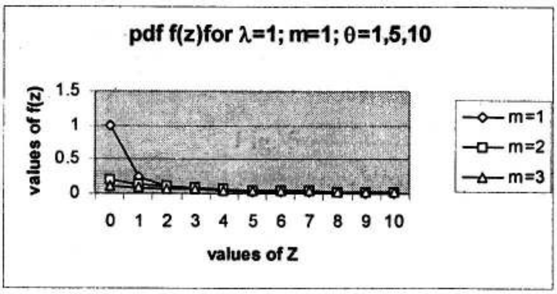


Fig. 7

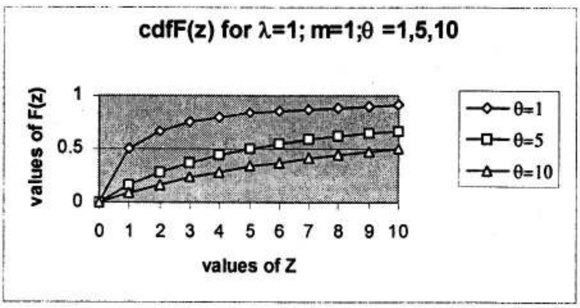


Fig. 8

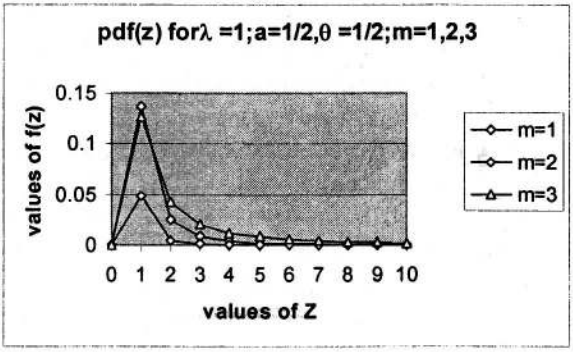


**Fig. 9**

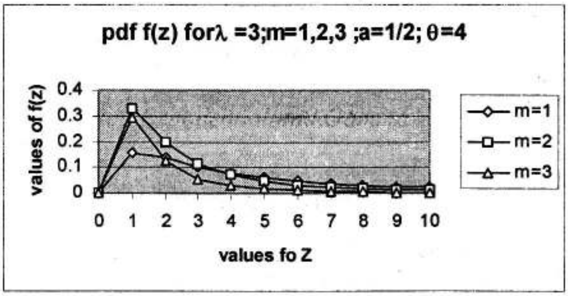


**Fig. 10**

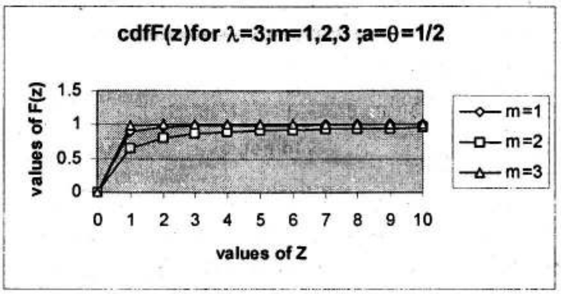


**Fig. 11**

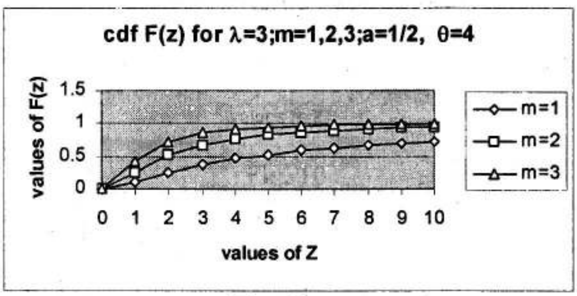


**Fig. 12**

## 7. CONCLUSIONS

As is evident on examining the expression of cdf F(z) and pdf f(z) represented by (2.1) and (2.2) respectively that the role of parameter $\lambda$ in the derived distribution is to increase or decrease the number of terms.

The shapes of pdf are more or less same for all $\lambda$. The curve is positively skewed rises at Z=0 to Z=1and approaches X-axis as Z increases. It has also been observed that for Z=0 the value of f(z) is smaller for smaller value of m but for Z e" 1 f(z) decreases with increase in value of m.

Regarding cdf, the curve rises gradually for lower value of $\lambda$ and m as compared to higher value of $\lambda$ and m. The curve of F(z) first rises and become parallel to X-axis as Z increases.

Mode of the derived distribution as depicted graphically is approx 1.

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# A STUDY ON STATE DESIGNATOR FOR GENERALIZED FUZZY MARKOV MODEL

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## ABSTRACT

In this paper we have defined generalized fuzzy Markov model using vague set theory and studied its state designator. We have given a numerical example for finding state designator for generalized fuzzy Markov model.

**Keywords:** Generalized Fuzzy Markov Model, State Designator, Triangular Vague Set.

**Mathematical Subject Classification:** 03E72, 60J20.

## 1. INTRODUCTION

In the real world there are vaguely specified data values in many applications. Fuzzy set theory is proposed to handle such vagueness by generalizing the notion of membership in a set. Essentially, in a fuzzy set (FS) each element is associated in a point value selected from a unit interval [0, 1], which is termed as the grade of membership in the set. A vague set (VS) is a generalized fuzzy set in which interval based membership is used and this interval based membership captures data for many real time situations [2, 3]. The grade of membership of an element $x$ in a vague set is represented by a vague value

$[t_x, 1-f_x]$ in [0, 1], where $t_x$ indicates the degree of truth, $f_x$ indicates the degree of false such that $0 \leq t_x \leq 1-f_x \leq 1$ and $t_x + f_x \leq 1$.

In this paper, we have defined Generalized Fuzzy Markov Model *(GFMM)* using vague set theory which consists of set of states together with vague transition possibilities between the states. We have also studied the state designator for *GFMM* using vague set theory. Since in many real life systems, the data source involves imprecise and inexact data, we see that the above proposed model can capture exactly the vagueness of data. Throughout the paper, we have taken vague transition possibilities as triangular vague set.

The paper is organized as follows: section 2 and 3 gives the basic concepts of vague set theory and arithmetic operations of triangular vague set respectively. In section 4, we discuss about *GFMM*. Section 5 completely deals with the study on state designator for *GFMM* and a numerical example for finding state designator for *GFMM* is given in section 6. The conclusion is discussed in section 7.

## 2. BASIC CONCEPTS OF VAGUE SET

Let U be the universe of discourse, $U = \{u_1, u_2 ... u_n\}$. A vague set $\breve{A}$ [1-3] in U is characterized by a truth- membership function $t_{\breve{A}}$, $t_{\breve{A}} : U \rightarrow [0, 1]$, and a false-membership function $f_{\breve{A}}$, $f_{\breve{A}} : U \rightarrow [0, 1]$, where $t_{\breve{A}}(u_i)$ is a lower bound of the grade of membership of $u_i$ derived from the evidence for $u_i$, $f_{\breve{A}}(u_i)$ is a lower bound on the negation of $u_i$ derived from the evidence against $u_i$, such that $t_{\breve{A}}(u_i) + f_{\breve{A}}(u_i) \leq 1$. The grade of membership of $u_i$ in the vague set $\breve{A}$ is bounded by a subinterval $[t_{\breve{A}}(u_i), 1-f_{\breve{A}}(u_i)]$. For example, a vague set $\breve{A}$ in the universe of discourse U is shown in Figure 1.

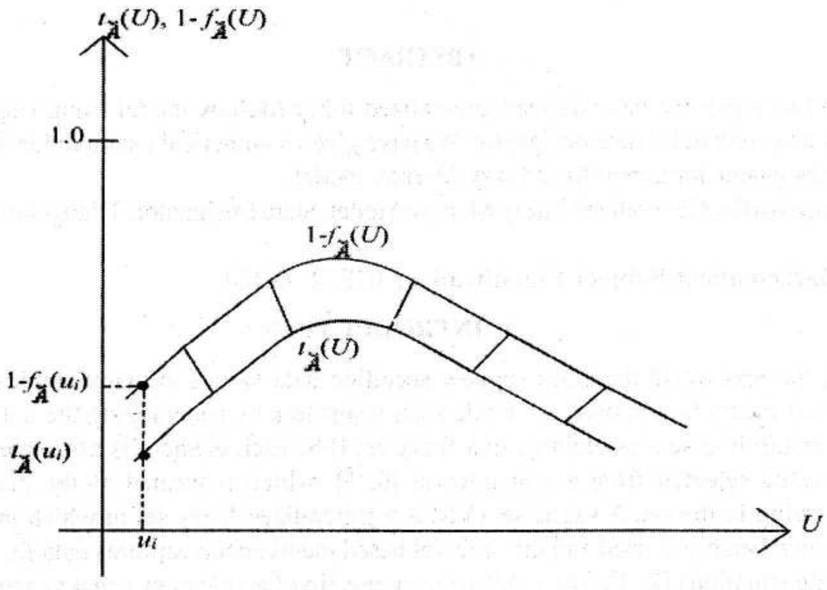


**Figure 1: Vague Set**

**Definition 2.1.**

Let $\breve{A}$ be a vague set of the universe of discourse U with truth membership function $t_{\breve{A}}$ and the false membership function $f_{\breve{A}}$ respectively. The vague set $\breve{A}$ is convex [1], if and only if for all $u_i$ in U,

$t_{\breve{A}}(\lambda u_1 + (1-\lambda)u_2) \geq \text{Min}(t_{\breve{A}}(u_1), t_{\breve{A}}(u_2))$

$1-f_{\breve{A}}(\lambda u_1 + (1-\lambda)u_2) \geq \text{Min}(1-f_{\breve{A}}(u_1), 1-f_{\breve{A}}(u_2))$, where $\lambda \in [0, 1]$.

**Definition 2.2.**

A vague set $\breve{A}$ of the universe of discourse U is called a normal vague set [1], if $u_i \in U$, $1-f_{\breve{A}}(u_i) = 1$. That is, $f_{\breve{A}}(u_i) = 0$.

**Definition 2.3.**

A vague number [1] is a vague subset in the universe of discourse U that is both convex and normal.

In the following, we present the arithmetic operations of triangular vague sets.

## 3. ARITHMETIC OPERATIONS OF TRIANGULAR VAGUE SETS

Let us consider the triangular vague set $\breve{A}$, where the triangular vague set $\breve{A}$ can be parameterized by a tuple $< [(a, b, c); \mu_1], [(a, b, c); \mu_2]>$, where $\mu_1$ is the truth membership for (a, b, c) and $\mu_2$ is the negation of false membership for (a, b, c). For convenience, the tuple can also be abbreviated into $< [(a, b, c); \mu_1; \mu_2]>$, where $0 \leq \mu_1 \leq \mu_2 \leq 1$. The arithmetic operations of triangular vague sets $\breve{A}$ and $\breve{B}$, where

$\breve{A} - < [(a_1,b_1,c_1); \mu_1], [(a_1,b_1,c_1); \mu_2] > = < [(a_1,b_1,c_1); \mu_1; \mu_2] >$

$\breve{B} = < [(a_2,b_2,c_2); \mu_3], [(a_2,b_2,c_2); \mu_4] > = < [(a_2,b_2,c_2); \mu_3; \mu_4] >$,

$0 \leq \mu_3 \leq \mu_1 \leq \mu_4 \leq \mu_2 \leq 1$ are as follows [1]:

$$\breve{A} \otimes \breve{B} = < [(a_1, b_1, c_1); \mu_1; \mu_2] > \oplus < [(a_2, b_2, c_2); \mu_3; \mu_4] >$$
$$= < [(a_1 + a_2, b_1 + b_2, c_1 + c_2); \text{Min}(\mu_1, \mu_3); \text{Min}(\mu_2, \mu_4)]>$$

$$\breve{A} \otimes \breve{B} = < [(a_1, b_1, c_1); \mu_1; \mu_2] > \otimes < [(a_2, b_2, c_2); \mu_3; \mu_4] >$$
$$= < [(a_1 \text{ x } a_2, b_1 \text{ x } b_2, c_1 \text{ x } c_2); \text{Min}(\mu_1, \mu_3); \text{Min}(\mu_2, \mu_4)]>$$

Based on the above two operations, we now define the multiplication of two vague matrices (i.e., generalized fuzzy matrices). Let $\breve{P}$ and $\breve{Q}$ be two matrices whose entries are triangular vague set represented as $\breve{p}_{ij} =< [(p_{ij}^1, p_{ij}^2, p_{ij}^3); \mu_{ij}^1, \mu_{ij}^2] >$, where $\mu_{ij}^1$ is the truth membership for $(p_{ij}^1, p_{ij}^2, p_{ij}^3)$ and $\mu_{ij}^2$ is the negation of false membership of $(p_{ij}^1, p_{ij}^2, p_{ij}^3)$, $\breve{q}_{ij} =< [(q_{ij}^1, q_{ij}^2, q_{ij}^3); \mu_{ij}^3, \mu_{ij}^4] >$, where $\mu_{ij}^3$ is the truth membership of $(q_{ij}^1, q_{ij}^2, q_{ij}^3)$ and $\mu_{ij}^4$ is the negation of false membership of $(q_{ij}^1, q_{ij}^2, q_{ij}^3)$ respectively, given by

$$\breve{P} = (\breve{p}_{ij}) = \begin{pmatrix} \breve{p}_{11} & \breve{p}_{12} & \cdots & \breve{p}_{1n} \\ \breve{p}_{21} & \breve{p}_{22} & \cdots & \breve{p}_{2n} \\ \vdots & \vdots & & \vdots \\ \breve{p}_{n1} & \breve{p}_{n2} & \cdots & \breve{p}_{nn} \end{pmatrix} \qquad \breve{Q} = (\breve{q}_{ij}) = \begin{pmatrix} \breve{q}_{11} & \breve{q}_{12} & \cdots & \breve{q}_{1n} \\ \breve{q}_{21} & \breve{q}_{22} & \cdots & \breve{q}_{2n} \\ \vdots & \vdots & & \vdots \\ \breve{q}_{n1} & \breve{q}_{n2} & \cdots & \breve{q}_{nn} \end{pmatrix}$$

Then the multiplication of $\breve{P}$ and $\breve{Q}$ is defined by $\breve{P} \otimes \breve{Q} = \underset{k}{\oplus}\left(\breve{p}_{ik} \otimes \breve{q}_{kj}\right)$

## 4. GENERALIZED FUZZY MARKOV MODEL

A fuzzy Markov model is discrete time, discrete state space model similar to stochastic Markov model consisting of set of states and fuzzy transition possibility from one state to another state [4-6]. *GFMM* is also similar to stochastic Markov model $\{X_m, m \in T\}$ consisting of n states , whose transition between the states are vague transition possibility represented by $\breve{p}_{ij} =< [(p^1_{ij}, p^2_{ij}, p^3_{ij}); \mu^1_{ij}, \mu^2_{ij}] >$, where $\mu^1_{ij}$ is the truth membership for $(p^1_{ij}, p^2_{ij}, p^3_{ij})$ and $\mu^2_{ij}$ is the negation of false membership of $(p^1_{ij}, p^2_{ij}, p^3_{ij})$ satisfying the property $\breve{p}(X_m = j / X_{m-1} = i, X_{m-2} = i_{m-2}, ..., X_0 = i_0) = \breve{p}(X_m = j / X_{m-1} = i) = \breve{p}_{ij}$. These transition possibilities can be expressed in matrix form as

$$\breve{P} = (\breve{p}_{ij}) = \begin{array}{c} \\ S_1 \\ S_2 \\ \vdots \\ S_n \end{array} \begin{array}{c} \begin{matrix} S_1 & S_2 & \cdots & S_n \end{matrix} \\ \begin{pmatrix} \breve{p}_{11} & \breve{p}_{12} & \cdots & \breve{p}_{1n} \\ \breve{p}_{21} & \breve{p}_{22} & \cdots & \breve{p}_{2n} \\ \vdots & \vdots & & \vdots \\ \breve{p}_{n1} & \breve{p}_{n2} & \cdots & \breve{p}_{nn} \end{pmatrix} \end{array}$$

We now define the vague possibility of the system remaining in each state initially as a row vector $\breve{V}(0) = (\breve{v}_1(0), \breve{v}_2(0), ..., \breve{v}_n(0))$ called as initial state designator, where each $\breve{v}_j(0) =< [(v^1_j(0), v^2_j(0), v^3_j(0)); \mu^1_j; \mu^2_j] >$, $\mu^1_j$ is the truth membership of $(v^1_j(0), v^2_j(0), v^3_j(0))$ and $\mu^2_j$ is the negation of false membership of $(v^1_j(0), v^2_j(0), v^3_j(0))$. Similarly, we define the vague possibility of the system remaining in each state at m-steps is defined as a row vector $\breve{V}(m) = (\breve{v}_1(m), \breve{v}_2(m), ..., \breve{v}_n(m))$ called as m-step state designator, where each $\breve{v}_j(m) =< [(v^1_j(m), v^2_j(m), v^3_j(m)); \mu^1_{j_m}; \mu^2_{j_m}] >$, $\mu^1_{j_m}$ is the truth membership of

$\left(v_j^1(m), v_j^2(m), v_j^3(m)\right)$ and $\mu_{j_m}^2$ is the negation of false membership of $\left(v_j^1(m), v_j^2(m), v_j^3(m)\right)$.

## 5. STUDY ON STATE DESIGNATOR FOR GFMM

Consider a *GFMM* with state space consisting of n states.

We know that $\breve{p}_{ij}(m) = \breve{p}\left(X_{m+l} = j / X_l = i\right)$.

Consider $\breve{p}_{ij}(2) = \breve{P}(X_{l+2} = j / X_l = i)$

The state j can be reached from the state i in two steps through some intermediate step k. For a fixed value of k, we have

$$\begin{aligned}\breve{p}_{ij}(2) &= \breve{p}\left(X_{l+2} = j, X_{l+1} = k / X_l = i\right)\\ &= \breve{p}\left(X_{l+2} = j / X_{l+1} = k\right) \otimes \breve{p}\left(X_{l+1} = k / X_l = i\right)\\ &= \breve{p}_{kj}(1) \otimes \breve{p}_{ik}(1) = \breve{p}_{ik}(1) \otimes \breve{p}_{kj}(1)\end{aligned}$$

Since these intermediate steps can take values k=1, 2… we have

$$\breve{p}_{ij}(2) = \underset{k}{\oplus}(\breve{p}_{ik} \otimes \breve{p}_{kj})$$

By induction we have,

$$\begin{aligned}\breve{p}_{ij}(m+1) &= \breve{p}(X_{l+m+1} = j / X_l = i)\\ &= \underset{k}{\oplus}\left(\breve{p}\left(X_{l+m+1} = j / X_{l+m} = k\right) \otimes \breve{p}\left(X_{l+m} = k / X_l = i\right)\right)\\ &= \underset{k}{\oplus}\left(\breve{p}_{kj}(1) \otimes \breve{p}_{ik}(m)\right)\end{aligned}$$

$$\Rightarrow \breve{p}_{ij}(m+1) = \underset{k}{\oplus}\left(\breve{p}_{ik}(m) \otimes \breve{p}_{kj}(1)\right)$$

In general, $\breve{p}_{ij}(m+l) = \underset{k}{\oplus}\left(\breve{p}_{ik}(m) \otimes \breve{p}_{kj}(l)\right)$ (5.1)

The above equation gives the vague transition possibility from state i to state j at *(m+l)* steps. Let $\breve{P}(m)$ be the matrix of m-step transition possibilities whose entry $\breve{p}_{ij}(m)$ is obtained by replacing *l=1* and *m* replaced by *m-1* in equation (5.1), i.e. $\breve{p}_{ij}(m) = \underset{k}{\oplus}\left(\breve{p}_{ik}(m-1) \otimes \breve{p}_{kj}(1)\right)$.

This means that $\breve{P}(m) = \breve{P}(m-1) \otimes \breve{P}$

Using the same formula again, we get

$$\begin{aligned}\breve{P}(m) &= \breve{P}(m-1) \otimes \breve{P}\\ &= \breve{P}(m-2) \otimes \breve{P} \otimes \breve{P} = \breve{P}^m\end{aligned}$$

Thus the matrix of m-step transition possibilities is obtained by multiplying the matrix of 1-step transition possibility by itself m-1 times.

We can obtain the state designator after m-steps $\breve{V}(m)=(\breve{v}_1(m), \breve{v}_2(m),...,\breve{v}_n(m))$ from the m-step transition possibility and the initial state designator as follows:

$$\breve{v}_j(m)=\breve{p}\left(X_m=j\right)=\underset{i}{\oplus}\left[\breve{p}(X_0=i)\otimes\breve{p}\left(X_m=j/X_0=i\right)\right]$$
$$=\underset{i}{\oplus}\left[\breve{v}_i(0)\otimes\breve{p}_{ij}(m)\right]$$

i.e: $$\breve{V}(m)=\breve{V}(0)\otimes\breve{P}^m$$

This implies that m-step state designator is completely determined from one step vague transition possibility matrix $\breve{p}$ and initial state designator $\breve{V}(0)$, where each entry of m-step state designator is represented as triangular vague set. In the following we give the numerical example for finding state designators for *GFMM*.

## 6. EXAMPLE

Consider the GFMM consisting of 3 states, whose vague transition possibilities are represented as triangular vague sets. The vague transition possibility matrix is given by

$$\breve{P}=\begin{matrix} & S_1 & S_2 & S_3 \\ S_1 & <[(0,0.1,0.2);0.8;0.89]> & <[(0,0,0);1;1]> & <[(0.1,0.2,0.3);0.81;0.9]> \\ S_2 & <[(0.3,0.4,0.5);0.9;0.92]> & <[(0,0.2,0.3);0.9;0.91]> & <[(0,0,0);1;1]> \\ S_3 & <[(0,0,0.);1;1]> & <[(0.1,0.4,0.5);0.8;0.9]> & <[(0.2,0.4,0.5);0.9;0.95]> \end{matrix}$$

The initial state designator is taken as

$$\breve{V}(0)=(<[(1,1,1);0.8;0.9]> \quad <[(0,0,0);1;1]> \quad <[(0,0,0);1;1]>)$$

The state designator after step1 is given by $\breve{V}(1)=\breve{V}(0)\otimes\breve{P}$

$$\breve{V}(1)=(<[(0,0.1,0.2);0.8;0.89]> \quad <[(0,0,0);0.8;0.9]> \quad <[(0.1,0.2,0.3);0.8;0.9]>)$$

The vague transition possibility matrix after two steps is given by

$$\breve{P}^2=\begin{pmatrix} <[(0,0.01,0.04);0.8;0.89]> & <[(0.01,0.08,0.15);0.8;0.89]> & <[(0.02,0.1,0.21);0.8;0.89]> \\ <[(0,0.12,0.25);0.84;0.89]> & <[(0,0.04,0.09);0.8;0.9]> & <[(0.03,0.08,0.2);0.81;0.9]> \\ <[(0.03,0.16,0.25);0.8;0.89]> & <[(0.02,0.24,0.4);0.8;0.9]> & <[(0.04,0.16,0.25);0.8;0.9]> \end{pmatrix}$$

The state designator after two steps is given by $\breve{V}(2)=\breve{V}(0)\otimes\breve{P}^2$

$$\breve{V}(2)=(<[(0,0.01,0.04);0.8;0.89]> \quad <[(0.01,0.08,0.15);0.8;0.89]> \quad <[(0.02,0.1,0.21);0.8;0.89]>)$$

From the first entry of $\breve{V}(2)$, namely < [(0,0.01,0.04);0.8;0.89]> , we infer that 0.8 is the lower bound for the system to remain in the first state for the triplet

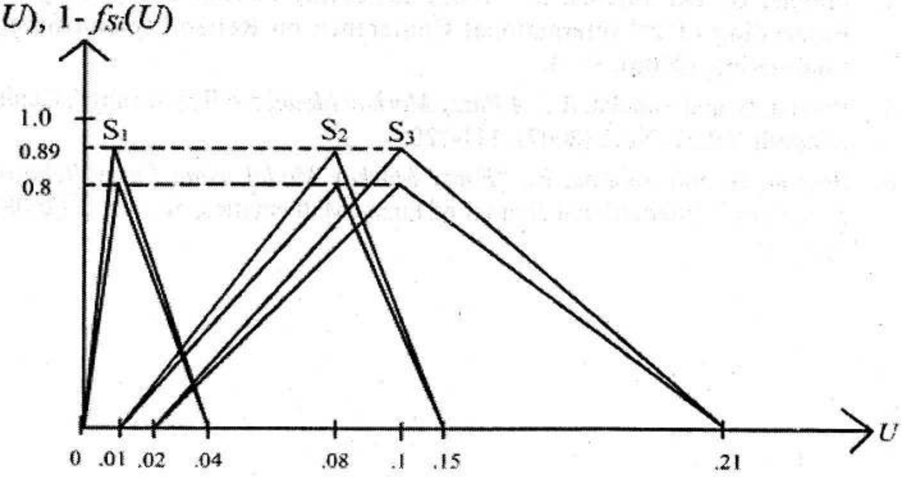


**Figure 2: Two step state designator**

(0,0.01,0.04) and 0.89 is the upper bound for the system to remain in the first state for the triplet (0,0.01,0.04). The state designator after two steps is depicted in figure 2.

## 7. CONCLUSION

In this paper, we have defined *GFMM* using vague set theory and derived the state designators for *GFMM.* Fuzzy set theory is an essential tool to capture vagueness, hence it is used in many real time applications. In this paper we made an attempt to capture the vagueness of fuzzy set using vague set theory.

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# APPLICATION OF FRACTIONAL DERIVATIVE OPERATOR IN THE DERIVATION OF BILATERAL EXPANSIONS CONCERNING CERTAIN SPECIAL FUNCTIONS

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## ABSTRACT

The object of the present paper is to establish a bilateral expansion pertaining to multivariable H-function by employing certain fractional derivative operator. Being unified and general in nature, our main result yields a large number of new and known results as its special cases. To illustrate, some special cases are mentioned briefly.

**Keywords:** Bilateral expansions, Multivariable H-function, General sequence of functions, General class of polynomials, Riemann-Liouville operator.

**Mathematics Subject Classification (2000):** 26A33, 33C60.

## 1. INTRODUCTION

The Riemann-Liouville operator $D_x^\sigma$ occurring in this paper is defined by Samko, Kilbas and Marichev [5]:

$$D_x^\sigma\{f(x)\} = \begin{cases} \dfrac{1}{\Gamma(-\sigma)}\int_0^x (x-\xi)^{-\sigma-1} f(\xi)\,d\xi\,, & \mathrm{Re}(\sigma) < 0, \\ \dfrac{d^m}{dx^m} D_x^{\sigma-m}\{f(x)\}, & m-1 \le \mathrm{Re}(\sigma) \le m,\ m \in N \end{cases} \quad \ldots (1.1)$$

provided that the integral exists and N is the set of positive integers.

Also, the Riemann-Liouville operator satisfies the generalized Leibnitz rule [5] as

$$D_x^\sigma\{f(x)\,g(x)\} = \sum_{n=-\infty}^{\infty} k \binom{\sigma}{\mu+kn} D_x^{\sigma-\mu-kn}\{f(x)\} D_x^{\mu+kn}\{g(x)\}, \quad \ldots (1.2)$$

where $\sigma$, $\mu$ are complex numbers and $0 \le k \le 1$.

The following special case of the fractional derivative is used in the present paper:

$$D_t^\gamma(t^\rho) = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\gamma+1)}\, t^{\rho-\gamma}, \quad \ldots (1.3)$$

for Re $(\gamma) > 0$, $t > 0$ and $\rho > -1$.

In the present study, we use the following special case of series representation for H-function of several complex variables given by Srivastava and Panda [7]:

$$H^{0,0:(1,v');\ldots;(1,v^{(r)})}_{A,C:(B',D'+1);\ldots;(B^{(r)},D^{(r)}+1)}\left[\begin{matrix}(a_j;\theta'_j,\ldots,\theta^{(r)}_j)_{1,A} : (b'_j;\phi'_j)_{1,B'} ;\ldots;(b^{(r)}_j;\phi^{(r)}_j)_{1,B^{(r)}}\\ (c_j;\psi'_j,\ldots,\psi^{(r)}_j)_{1,C}:(0,1),(d'_j;\delta'_j)_{1,D'};\ldots;(0,1),(d^{(r)}_j;\delta^{(r)}_j)_{1,D^{(r)}}\end{matrix}\middle| \begin{matrix} z_1\\ \vdots \\ z_r\end{matrix}\right]$$

$$= H^{0,0:(1,v');\ldots;(1,v^{(r)})}_{A,C:(B',D'+1);\ldots;(B^{(r)},D^{(r)}+1)}[z_1,\ldots,z_r]$$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} T(m_1,\ldots,m_r)\,R_1(m_1)\ldots R_r(m_r)\frac{(-z_1)^{m_1}}{m_1!}\ldots\frac{(-z_r)^{m_r}}{m_r!}, \quad \ldots (1.4)$$

where

$$T(m_1,\ldots,m_r) = \left\{\prod_{j=1}^{A}\Gamma\left(a_j-\sum_{i=1}^{r}\theta^{(i)}_j m_i\right)\prod_{j=1}^{C}\Gamma\left(1-c_j+\sum_{i=1}^{r}\psi^{(i)}_j m_i\right)\right\}^{-1},$$

$$R_i(m_i) = \frac{\prod_{j=1}^{v^{(i)}}\Gamma\left(1-b^{(i)}_j+\phi^{(i)}_j m_i\right)}{\prod_{j=1}^{D^{(i)}}\Gamma\left(1-d^{(i)}_j+\delta^{(i)}_j m_i\right)\prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma\left(b^{(i)}_j-\phi^{(i)}_j m_i\right)},$$

$i = 1,\ldots,r$.

The convergence conditions and other details of the multivariable H-function can be found in the works by Mathai, Saxena and Haubold [2] and Srivastava and Panda [7].

Also the series formula of the general sequence of functions appearing in this paper was established by Salim [4] as:

$$R^{\alpha,\beta}_n[x;E,F,g,h;p,q;\gamma;\delta;e^{-sx^r}] = \sum_{w,v,u,t,e,k_1,k_2}\phi(w,v,u,t,e,k_1,k_2)x^R, \quad \ldots (1.5)$$

where

$$\phi(w,v,u,,t,e,k_1,k_2) = \frac{(-1)^{t+w+k_2}(-v)_u(-t)_e(\alpha)_t\,\ell^n}{w!\,v!\,u!\,t!\,e!\,\ell'_n\,k_1!\,k_2!} \times\frac{s^{w+k_1}F^{\gamma n-t}}{(1-\alpha-t)_e}(-\alpha-\gamma n)_e(-\beta-\delta n)_v\,g^{v+k_2}h^{\delta n-v-k_2} \times(v-\delta n)_{k_2}E^t\left(\frac{pe+rw+\lambda+qu}{\ell}\right)_n, \quad \ldots (1.6)$$

$\{\ell'_n\}_{n=1}^{\infty}$, is a sequence of constants,

$$R = \ell n+qv+pt+rw+k_1r+k_2q, \quad \ldots (1.7)$$

$$\sum_{w,v,u,t,e,k_1,k_2} \equiv \sum_{w=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t=0}^{n}\sum_{e=0}^{t}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \quad \ldots (1.8)$$

and the infinite series on the right-hand side of (1.5) is absolutely convergent.

We shall also require the following series expansion of general class of polynomials defined by Srivastava [6] as:

$$S_V^U[x] = \sum_{\eta=0}^{[V/U]} \frac{(-V)_{U\eta} A_{V,\eta}}{\eta!} x^\eta, \qquad V = 0,1,\ldots \quad \ldots (1.9)$$

where U is an arbitrary positive integer and the coefficients $A_{V,\eta}$ $(V,\eta \geq 0)$ are arbitrary constants, real or complex.

## 2. MAIN RESULT

We state our main result in the form of the following

**THEOREM.** The following bilateral expansion

$$\sum_{\eta=0}^{[V/U]} \sum_{w,v,u,t,e,k_1,k_2} \phi(w,v,u,t,e,k_1,k_2) \frac{(-V)_{U\eta} A_{V,\eta}}{\eta!}$$

$$\times H^{0,1:(1,v');\ldots;(1,v^{(2r)})}_{A+M+1,C+N+1:(B',D'+1);\ldots;(B^{(2r)},D^{(2r)}+1)}\left[\begin{matrix}(2-\rho_1-\rho_2-\eta-R;q_1,\ldots,q_r,s_{r+1},\ldots,s_{2r}),(a_j;\theta'_j,\ldots,\theta_j^{(r)})_{1,A},\\ (c_j;\psi'_j,\ldots,\psi_j^{(r)})_{1,C},(c'_j;\psi_j^{(r+1)},\ldots,\psi_j^{(2r)})_{1,N},\end{matrix}\right.$$

$$\left.\begin{matrix}(a'_j;\theta_j^{(r+1)},\ldots,\theta_j^{(2r)})_{1,M}:(b'_j;\phi'_j)_{1,B'};\ldots;(b_j^{(2r)};\phi_j^{(2r)})_{1,B^{(2r)}}\\ (2-\rho_1-\rho_2-\eta-R+\sigma;q_1,\ldots,q_r,s_{r+1},\ldots,s_{2r}):(0,1),(d'_j;\delta'_j)_{1,D'};\ldots;(0,1),(d_j^{(2r)};\delta_j^{(2r)})_{1,D^{(2r)}}\end{matrix}\middle|\begin{matrix}y_1\\ \vdots\\ y_{2r}\end{matrix}\right]$$

$$= \sum_{m=-\infty}^{\infty}\sum_{\eta=0}^{[V/U]}\sum_{w,v,u,t,e,k_1,k_2} k\binom{\sigma}{\mu+km}\frac{(-V)_{U\eta}A_{V,\eta}}{\eta!}\phi(w,v,u,t,e,k_1,k_2)$$

$$\times H^{0,1:(1,v');\ldots;(1,v^{(r)})}_{A+1,C+1:(B',D'+1);\ldots;(B^{(r)},D^{(r)}+1)}\left[\begin{matrix}(1-\rho_1-\eta;,q_1,\ldots,q_r),(a_j;\theta'_j,\ldots,\theta_j^{(r)})_{1,A}:\\ (c_j;\psi'_j,\ldots,\psi_j^{(r)})_{1,C},(1-\rho_1-\eta+\sigma-\mu-km;q_1,\ldots,q_r):\end{matrix}\right.$$

$$\left.\begin{matrix}(b'_j;\phi'_j)_{1,B'};\ldots;(b_j^{(r)};\phi_j^{(r)})_{1,B^{(r)}}\\ (0,1),(d'_j,\delta'_j)_{1,D'};\ldots;(0,1),(d_j^{(r)};\delta_j^{(r)})_{1,D^{(r)}}\end{matrix}\middle|\begin{matrix}y_1\\ \vdots\\ y_r\end{matrix}\right]$$

$$\times H^{0,1:(1,v^{(r+1)});\ldots;(1,v^{(2r)})}_{M+1,N+1:(B^{(r+1)},D^{(r+1)}+1);\ldots;(B^{(2r)},D^{(2r)}+1)}\left[\begin{matrix}(1-\rho_2-R;s_{r+1},\ldots,s_{2r}),(a'_j;\theta_j^{(r+1)},\ldots,\theta_j^{(2r)})_{1,M}:\\ (c'_j;\psi_j^{(r+1)},\ldots,\psi_j^{(2r)})_{1,N},(1-\rho_2-R+\mu+km;s_{r+1},\ldots,s_{2r}):\end{matrix}\right.$$

$$\left.\begin{matrix}(b_j^{(r+1)};\phi_j^{(r+1)})_{1,B^{(r+1)}};\ldots;(b_j^{(2r)};\phi_j^{(2r)})_{1,B^{(2r)}}\\ (0,1),(d_j^{(r+1)},\delta_j^{(r+1)})_{1,D^{(r+1)}};\ldots;(0,1),(d_j^{(2r)};\delta_j^{(2r)})_{1,D^{(2r)}}\end{matrix}\middle|\begin{matrix}y_{r+1}\\ \vdots\\ y_{2r}\end{matrix}\right], \quad \ldots (2.1)$$

holds under the following conditions:

(i) $$\mathrm{Re}(\rho_1) + \sum_{i=1}^{r} q_i \left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,$$

(ii) $$\mathrm{Re}(\rho_2) + \sum_{i=r+1}^{2r} s_i \left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,$$

(iii) $\sigma$ and $\mu$ are complex numbers , $0 \le k \le 1$,

(iv) $$\sum_{j=1}^{A+M} \theta_j^{(i)} - \sum_{j=1}^{C+N} \psi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} \le 1, \ \forall\, i = 1,\ldots,2r.$$

**Proof:** Taking

$$f(x) = x^{\rho_1 - 1} S_V^U[x] \; H^{0,0:(1,v');\ldots;(1,v^{(r)})}_{A,C:(B',D'+1);\ldots;(B^{(r)},D^{(r)}+1)} [y_1 x^{q_1},\ldots,y_r x^{q_r}]$$

and

$$g(x) = x^{\rho_2 - 1} R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma,\delta;e^{-s x^r}]$$

$$\times H^{0,0:(1,v^{(r+1)});\ldots;(1,v^{(2r)})}_{M,N:(B^{(r+1)},D^{(r+1)}+1);\ldots;(B^{(2r)},D^{(2r)}+1)} [y_{r+1} x^{s_{r+1}},\ldots,y_{2r} x^{s_{2r}}]$$

in (1.2), and then using the series representations (1.4),(1.5) and (1.9), we get

$$\text{L.H.S.} = \sum_{\eta=0}^{[V/U]} \sum_{m_1,\ldots,m_r,m_{r+1},\ldots,m_{2r}=0} \sum_{w,v,u,t,e,k_1,k_2} (-V)_{U\eta} A_{V,\eta} \phi(w,v,u,t,e,k_1,k_2)$$

$$\times \frac{T_1(m_1,\ldots,m_r) T_2(m_{r+1},\ldots,m_{2r}) R_1(m_1)\ldots R_r(m_r) R_{r+1}(m_{r+1})\ldots R_{2r}(m_{2r})}{\eta!\, m_1!\ldots m_r!\, m_{r+1}!\ldots m_{2r}!}$$

$$\times (-y_1)^{m_1}\ldots(-y_r)^{m_r}(-y_{r+1})^{m_{r+1}}\ldots(-y_{2r})^{m_{2r}}$$

$$\times D_x^{\sigma}\left\{ x^{\rho_1+\rho_2+\eta+R+\sum_{i=1}^{r} q_i m_i + \sum_{i=r+1}^{2r} s_i m_i - 2} \right\}$$

Now, using the result (1.3) and making suitable changes in the parameters involved therein, straight forward simplifications imply the desired result.

## 3. SPECIAL CASES

(A) If we take $\rho_2 = 1$ and $y_{r+1} = \ldots = y_{2r} = 0$ in (2.1), then we get

**COROLLARY 1**. Under the hypothesis of the theorem

$$\sum_{\eta=0}^{[V/U]} \sum_{w,v,u,t,e,k_1,k_2} \frac{\phi(w,v,u,t,e,k_1,k_2)(-V)_{U\eta} A_{V,\eta}}{\eta!}$$

$$\times H^{0,1\,:(1,v');\ldots;(1,v^{(r)})}_{A+1,C+1:(B',D'+1);\ldots;(B^{(r)},D^{(r)}+1)}\left[\begin{matrix}(1-\rho_1-\eta-R;q_1,\ldots,q_r),(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,A}: \\ (c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,C},(1-\rho_1-\eta-R+\sigma;q_1,\ldots,q_r):\end{matrix}\right.$$

$$\left.\begin{matrix}(b_j';\phi_j')_{1,B'};\ldots;(b_j^{(r)};\phi_j^{(r)})_{1,B^{(r)}} \\ (0,1),(d_j';\delta_j')_{1,D'};\ldots;(0,1),(d_j^{(r)};\delta_j^{(r)})_{1,D^{(r)}}\end{matrix}\middle|\begin{matrix}y_1\\ \vdots \\ y_r\end{matrix}\right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{\eta=0}^{[V/U]} \sum_{w,v,u,t,e,k_1,k_2} k\binom{\sigma}{\mu+km}\frac{(-V)_{U\eta} A_{V,\eta}}{\eta!}\phi(w,v,u,t,e,k_1,k_2)$$

$$\times H^{0,1\,:(1,v');\ldots;(1,v^{(r)})}_{A+1,C+1:(B',D'+1);\ldots;(B^{(r)},D^{(r)}+1)}\left[\begin{matrix}(1-\rho_1-\eta;q_1,\ldots,q_r),(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,A} : \\ (c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,C},(1-\rho_1-\eta+\sigma-\mu-km;q_1,\ldots,q_r):\end{matrix}\right.$$

$$\left.\begin{matrix}(b_j';\phi_j')_{1,B'};\ldots;(b_j^{(r)};\phi_j^{(r)})_{1,B^{(r)}} \\ (0,1),(d_j';\delta_j')_{1,D'};\ldots;(0,1),(d_j^{(r)};\delta_j^{(r)})_{1,D^{(r)}}\end{matrix}\middle|\begin{matrix}y_1\\ \vdots \\ y_r\end{matrix}\right], \quad \ldots (3.1)$$

holds under the following conditions:

(i) $\mathrm{Re}(\rho_1) + \sum_{i=1}^{r} q_i \left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1$,

(ii) $\sigma$ and $\mu$ are complex numbers , $0 \le k \le 1$,

(iii) $\sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} \le 1, \ \forall\, i = 1,\ldots,r.$

(B) On setting $\rho_1 = 1$ and $y_1 = \ldots = y_r = 0$ in (2.1), we arrive at

**COROLLARY 2.** Under the hypothesis of the theorem

$$\sum_{\eta=0}^{[V/U]} \sum_{w,v,u,t,e,k_1,k_2} \phi(w,v,u,t,e,k_1,k_2)\frac{(-V)_{U\eta} A_{V,\eta}}{\eta!}$$

$$\times H^{0,1\,:(1,v^{(r+1)});\ldots;(1,v^{(2r)})}_{M+1,N+1:(B^{(r+1)},D^{(r+1)}+1);\ldots;(B^{(2r)},D^{(2r)}+1)}\left[\begin{matrix}(1-\rho_2-\eta-R;,s_{r+1},\ldots,s_{2r}),(a_j';\theta_j^{(r+1)},\ldots,\theta_j^{(2r)})_{1,M} : \\ (c_j';\psi_j^{(r+1)},\ldots,\psi_j^{(2r)})_{1,N},(1-\rho_2-\eta-R+\sigma;s_{r+1},\ldots,s_{2r}):\end{matrix}\right.$$

$$\left.\begin{matrix}(b_j^{(r+1)};\phi_j^{(r+1)})_{1,B^{(r+1)}};\ldots;(b_j^{(2r)};\phi_j^{(2r)})_{1,B^{(2r)}} \\ (0,1),(d_j^{(r+1)};\delta_j^{(r+1)})_{1,D^{(r+1)}};\ldots;(0,1),(d_j^{(2r)};\delta_j^{(2r)})_{1,D^{(2r)}}\end{matrix}\middle|\begin{matrix}y_{r+1}\\ \vdots \\ y_{2r}\end{matrix}\right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{\eta=0}^{[V/U]} \sum_{w,v,u,t,e,k_1,k_2} k\binom{\sigma}{\mu+km} \frac{(-V)_{U\eta} A_{V,\eta}}{\eta!} \phi(w,v,u,t,e,k_1,k_2)$$

$$\times H^{0,1\,:(1,v^{(r+1)});\ldots;(1,v^{(2r)})}_{M+1,N+1:(B^{(r+1)},D^{(r+1)}+1);\ldots;(B^{(2r)},D^{(2r)}+1)} \left[ \begin{matrix} (1-\rho_2-R;,s_{r+1},\ldots,s_{2r}),(a_j;\theta_j^{(r+1)},\ldots,\theta_j^{(2r)})_{1,M} : \\ (c_j';\psi_j^{(r+1)},\ldots,\psi_j^{(2r)})_{1,N},(1-\rho_2-R+\mu+km;s_{r+1},\ldots,s_{2r}): \end{matrix} \right.$$

$$\left. \begin{matrix} (b_j^{(r+1)};\phi_j^{(r+1)})_{1,B^{(r+1)}} ;\ldots;(b_j^{(2r)};\phi_j^{(2r)})_{1,B^{(2r)}} \\ (0,1),(d_j^{(r+1)},\delta_j^{(r+1)})_{1,D^{(r+1)}} ;\ldots;(0,1),(d_j^{(2r)};\delta_j^{(2r)})_{1,D^{(2r)}} \end{matrix} \middle| \begin{matrix} y_{r+1} \\ \vdots \\ y_{2r} \end{matrix} \right], \quad \ldots (3.2)$$

holds under the following conditions:

(i) $\quad \mathrm{Re}(\rho_2) + \sum_{i=r+1}^{2r} s_i \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > -1,$

(ii) $\quad \sigma$ and $\mu$ are complex numbers , $0 \le k \le 1,$

(iii) $\quad \sum_{j=1}^{M} \theta_j^{(i)} - \sum_{j=1}^{N} \psi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} \le 1, \ \forall\, i = r+1,\ldots,2r.$

(C) If we set $p = h = 1$, $-l_n' = 1$, $s = 0$ in (2.1) and replace $\beta$ by $\beta/\tau$ and g by $-\tau$ therein, $R_n^{\alpha,\beta}[x; E, F, g, h; p, q; \gamma, \delta; e^{-sx^r}]$ reduces to the sequence of functions $S_n^{\alpha,\beta,\tau}[x; q, \delta, \gamma, E, F, \lambda, \ell]$ studied by Raizada [3] and we arrive at

**COROLLARY 3.** Under the hypothesis of the theorem

$$\sum_{\eta=0}^{[V/U]} \sum_{v,u,t,e,k_2} \psi(v,u,t,e,k_2) \frac{(-V)_{U\eta} A_{V,\eta}}{\eta!}$$

$$\times H^{0,1\,:(1,v');\ldots;(1,v^{(2r)})}_{A+M+1,C+N+1:(B',D'+1);\ldots;(B^{(2r)},D^{(2r)}+1)} \left[ \begin{matrix} (2-\rho_1-\rho_2-\eta-R^*;q_1,\ldots,q_r,s_{r+1},\ldots,s_{2r}),(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,A}, \\ (c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,C},(c_j';\psi_j^{(r+1)},\ldots,\psi_j^{(2r)})_{1,N}, \end{matrix} \right.$$

$$\left. \begin{matrix} (a_j';\theta_j^{(r+1)},\ldots,\theta_j^{(2r)})_{1,M}:(b_j';\phi_j')_{1,B'};\ldots;(b_j^{(2r)};\phi_j^{(2r)})_{1,B^{(2r)}} \\ (2-\rho_1-\rho_2-\eta-R^*+\sigma;q_1,\ldots,q_r,s_{r+1},\ldots,s_{2r}):(0,1),(d_j';\delta_j')_{1,D'};\ldots;(0,1),(d_j^{(2r)};\delta_j^{(2r)})_{1,D^{(2r)}} \end{matrix} \middle| \begin{matrix} y_1 \\ \vdots \\ y_{2r} \end{matrix} \right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{\eta=0}^{[V/U]} \sum_{v,u,t,e,k_2} k\binom{\sigma}{\mu+km} \psi(v,u,t,e,k_2) \frac{(-V)_{U\eta} A_{V,\eta}}{\eta!}$$

$$\times H^{0,1\,:(1,v');\ldots;(1,v^{(r)})}_{A+1,C+1:(B',D'+1);\ldots;(B^{(r)},D^{(r)}+1)} \left[ \begin{matrix} (1-\rho_1-\eta;q_1,\ldots,q_r),(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,A} : \\ (c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,C},(1-\rho_1-\eta+\sigma-\mu-km;q_1,\ldots,q_r): \end{matrix} \right.$$

$$\left.\begin{matrix}(b'_j;\phi'_j)_{1,B'} & ;...;(b_j^{(r)};\phi_j^{(r)})_{1,B^{(r)}} \\ (0,1),(d'_j;\delta'_j)_{1,D'};...;(0,1),(d_j^{(r)};\delta_j^{(r)})_{1,D^{(r)}}\end{matrix}\right|\begin{bmatrix}y_1\\ \vdots \\ y_r\end{bmatrix}$$

$$\times H^{0,1\;:(1,v^{(r+1)})\;;...;(1,v^{(2r)})}_{M+1,N+1:(B^{(r+1)},D^{(r+1)}+1);...;(B^{(2r)},D^{(2r)}+1)}\left[\begin{matrix}(1-\rho_2-R^*;,s_{r+1},...,s_{2r}),(a_j;\theta_j^{(r+1)},...,\theta_j^{(2r)})_{1,M}: \\ (c_j;\psi_j^{(r+1)},...,\psi_j^{(2r)})_{1,N},(1-\rho_2-R^*+\mu+km;s_{r+1},...,s_{2r}):\end{matrix}\right.$$

$$\left.\begin{matrix}(b_j^{(r+1)};\phi_j^{(r+1)})_{1,B^{(r+1)}} & ;...;(b_j^{(2r)};\phi_j^{(2r)})_{1,B^{(2r)}} \\ (0,1),(d_j^{(r+1)};\delta_j^{(r+1)})_{1,D^{(r+1)}} & ;...;(0,1),(d_j^{(2r)};\delta_j^{(2r)})_{1,D^{(2r)}}\end{matrix}\right|\begin{bmatrix}y_{r+1}\\ \vdots \\ y_{2r}\end{bmatrix}, \quad \ldots (3.3)$$

holds under the following conditions:

(i) $$\mathrm{Re}(\rho_1)+\sum_{i=1}^{r} q_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1,$$

(ii) $$\mathrm{Re}(\rho_2)+\sum_{i=r+1}^{2r} s_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1,$$

(iii) $\sigma$ and $\mu$ are complex numbers , $0 \le k \le 1$,

(iv) $$\sum_{j=1}^{A+M}\theta_j^{(i)}-\sum_{j=1}^{C+N}\psi_j^{(i)}+\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}\le 1,\ \forall\, i=1,...,2r.$$

where

$$\psi(v,u,t,e,k_2)=\frac{(-1)^{t+k_2}(-\tau)^{v+k_2}(-v)_u(-t)_e(\alpha)_t(-\alpha-\gamma n)_e(v-\delta n)_{k_2}\ell^n E^t F^{\gamma n-t}}{v!\,u!\,t!\,e!\,k_2!(1-\alpha-t)_e}$$

$$\times\left(-\frac{\beta}{\tau}-\delta n\right)_v\left(\frac{e+\lambda+qu}{\ell}\right)_n \quad \ldots (3.4)$$

and

$$R^*=\ell n+qv+t+k_2 v. \quad \ldots (3.5)$$

(D) Further, setting $\tau\to 0$, $n=\delta=\lambda=F=0$, $-!=q=-1$ and $E=1$ in (3.3), the generalized polynomial set $S_n^{\alpha,\beta,\tau}(x;q,\delta,\gamma,E,F,\lambda,\ell]$ reduces to unity and also taking $\rho_2=1$, $y_{r+1}=\ldots=y_{2r}=0$, we get

**COROLLARY 4.** Under the hypothesis of the Theorem

$$\sum_{\eta=0}^{[V/U]}\frac{(-V)_{U\eta}A_{V,\eta}}{\eta!}H^{0,1\;:(1,v')\;;...;(1,v^{(r)})}_{A+1,C+1:(B',D'+1);...;(B^{(r)},D^{(r)}+1)}\left[\begin{matrix}(1-\rho_1-\eta;q_1,...,q_r),(a_j;\theta'_j,...,\theta_j^{(r)})_{1,A} & : \\ (c_j;\psi'_j,...,\psi_j^{(r)})_{1,C},(1-\rho_1-\eta+\sigma;q_1,...,q_r) & :\end{matrix}\right.$$

$$\left.\begin{matrix}(b'_j;\phi'_j)_{1,B'} & ;...;(b_j^{(r)};\phi_j^{(r)})_{1,B^{(r)}} \\ (0,1),(d'_j;\delta'_j)_{1,D'};...;(0,1),(d_j^{(r)};\delta_j^{(r)})_{1,D^{(r)}}\end{matrix}\right|\begin{bmatrix}y_1\\ \vdots \\ y_r\end{bmatrix}=\sum_{m=-\infty}^{\infty}\sum_{\eta=0}^{[V/U]}k\binom{\sigma}{\mu+km}\frac{(-V)_{U\eta}A_{V,\eta}}{\eta!}$$

$$\times H^{0,1\;:(1,v');\ldots;(1,v^{(r)})}_{A+1,C+1:(B',D'+1);\ldots;(B^{(r)},D^{(r)}+1)}\left[\begin{matrix}(1-\rho_1-\eta;q_1,\ldots,q_r),(a_j;\theta_j',\ldots,\theta_j^{(r)})_{1,A} & : \\ (c_j;\psi_j',\ldots,\psi_j^{(r)})_{1,C},(1-\rho_1-\eta+\sigma-\mu-km;q_1,\ldots,q_r) & : \\ (b_j';\phi_j')_{1,B'};\ldots;(b_j^{(r)};\phi_j^{(r)})_{1,B^{(r)}} & \\ (0,1),(d_j';\delta_j')_{1,D'};\ldots;(0,1),(d_j^{(r)};\delta_j^{(r)})_{1,D^{(r)}} & \end{matrix}\middle| \begin{matrix} y_1 \\ \vdots \\ y_r\end{matrix}\right] \quad \ldots (3.6)$$

holds under the following conditions:

(i) $\operatorname{Re}(\rho_1) + \sum_{i=1}^{r} q_i \left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,$

(ii) $\sigma$ and $\mu$ are complex numbers, $0 \le k \le 1,$

(iii) $\sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} \le 1,\ \forall\, i=1,\ldots,r.$

(E) Finally, on letting $V \to 0$, $\rho_2 = \tau + \sigma$, $q_{r+1} = \ldots = q_{2r} = 1$, $\rho_1 = 1$, $y_1 = \ldots = y_r = 0$ and replacing $(a') \to 1-(a')$; $(c') \to (1-c')$; $(b_j^{(i)}) \to (1-b_j^{(i)})$; $(d_j^{(i)}) \to (1-d_j^{(i)})$; $(y_i) \to (-y_i), (i = r+1,\ldots,2r)$ in (3.3), it reduces to the known result given by Kalia and Srivastava [1].

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# AN EPQ MODEL WITH DISCRETE TRANSPORTATION COST

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## ABSTRACT

In the present paper, an inventory model with discrete transportation cost is studied. The number of transport modes required would always be a whole number, irrespective of the lot-size to be transported. The shortages are allowed and backlogged. Here, we have discussed an alternative method, in which the number of steps required to solve the problem is smaller as compared to the method given in Gupta's [5] model. In this paper, we explore a lot-size model with discrete transportation costs and obtain a simpler algorithm for optimal solution.

**Key words:** Lot-size, Discrete transportation cost, Shortages.

## INTRODUCTION

The classical Harris-Wilson model [6] is an elementary model based upon many assumptions, which become almost invalid in real life. Still the model undoubtedly has its own importance. In addition, it has apprehended the attention of many researchers and enforced them to step into the field of inventory modelling with innovations. Models with more complexity and relaxations on the restrictive assumptions of instantaneous supply, no shortage and implicit transportation cost etc. have come to the forefront.

Most of the models studied earlier did not include the transportation cost explicitly. Many later generalizations followed to go well with the same conception. The transportation cost might have been included in the price of the unit. However, when the order of certain item is placed, the purchaser is obliged to pay for the transportation along with the other usual costs. Transportation and inventory cost interact and provide the background for the emergence of a joint transport-inventory model.

Many researchers have analysed the situation by taking into account the shipping cost while determining the optimal lot-size. Baumol and Vinod [1] studied an inventory theoretic model of freight transport considering per unit constant transportation cost. They minimized the total cost including the cost of transportation. That was the first serious effort in this perspective. Their work was followed by Das [4] with a different set of assumptions. Buffa and Reynolds [2] further extended this model and included explicitly the stock-out costs and freight discounts. Langley [7] considered different types of freight costs. Constable and Whybark [3] also incorporated the back-order costs and a number of transportation alternatives. Larson [8] considered a model for economic transport quantity with freight discount. Russell [9] developed a model in which per unit freight cost decreases as lot-size increases. Gupta [5] considered a realistic situation where a fixed cost for a transportation mode is incurred whether it is fully loaded or partially loaded. Transportation cost happens to be a discrete function of lot-size. He developed an algorithm to solve inventory problems in which transportation cost is considered explicitly along with ordering cost and carrying cost.

In this paper, we study a more realistic situation. The entire lot is not delivered instantly but there is a uniform rate of supply. Shortages are also allowed to occur which are backlogged. However, the nature of transportation cost is the same as considered by Gupta [5]. Inspite of generalization, which makes the problem complex, we have developed a simpler algorithm for finding optimal lot-size, which minimizes the total cost of inventory (holding, ordering, transportation and shortage costs). Gupta's [5] method involves a large number of iterative steps. This number of steps varies with the dimensions as well as the values of parameters involved in the problem, whereas in the method presented here we get the solution of the problem in a fixed number of steps irrespective of the size and the parametric values of the problem. This would save a lot of efforts, time and money, which is the objective of almost every business organization. Therefore, the significance of the algorithm discussed herein is self-evident.

## ASSUMPTIONS AND NOTATIONS

**Assumptions:**

- Demand rate is uniform and constant.
- Production rate is finite.
- Shortage are allowed and backlogged.
- Set-up cost is constant and does not include the transportation cost.
- Transportation cost is constant for a vehicle load, even though the quantity transported is less than its capacity.
- It is the liability of the buyer to pay for the transportation charges.

**Notations:**

- $D$ = Demand rate.
- $K$ = Production rate.
- $C_1$ = Carrying cost per unit per unit time.
- $C_2$ = Shortage cost per unit per unit time.
- $C_3$ = Transportation cost per vehicle.
- $C_4$ = Set-up cost per set-up.
- $N$ = Capacity of the vehicle (in units).
- % X % = Lowest integer greater than or equal to X.
- $q$ = Lot-size.
- $q^*$ = Optimal lot-size.

## MODEL DEVELOPMENT

Initially, the inventory level is zero. Production starts at $t = 0$ at the rate of K units per unit time. At the same time, supply also starts to satisfy market demand. At $t = t_1$, the stock level reaches P units, where production is stopped. Then the inventory level declines for a period $t_2$ until it reaches zero level. Now shortages start and accumulate to the level Q in time $t_3$. Production starts again and the backlog is cleared in time $t_4$. The cycle repeats itself after total time $T\left(=\sum_{i=1}^{4} t_i\right)$. The following figure describes the present inventory model.

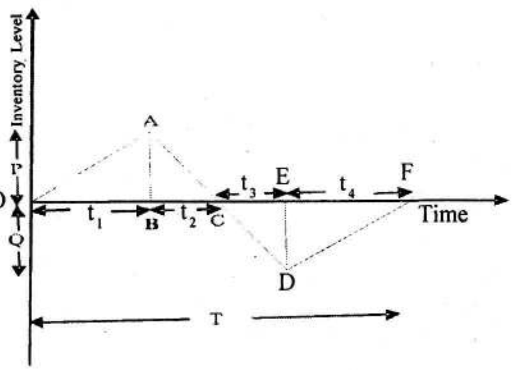



Figure-1

Carrying cost = $C_1P(t_1+t_2)/2$

Shortage cost = $C_2\,(t_3+t_4)/2$

Transportation cost for a load of M vehicles = $MC_3$

Thus, the average cost of the system per unit time is given by

$$C = \frac{1}{T}\left[\frac{1}{2}\{C_1P(t_1+t_2)+C_2Q(t_3+t_4)\}+MC_2+C_4\right] \qquad ...(1)$$

By Figure-1, we have

$$t_1 = \frac{P}{(K-D)} \qquad ...(2)$$

$$t_2 = \frac{P}{D} \qquad ...(3)$$

$$t_3 = \frac{Q}{D} \qquad ...(4)$$

$$t_4 = \frac{Q}{(K-D)} \qquad ...(5)$$

and

$$q = DT \qquad ...(6)$$

Using equations (2) to (5), equation (6) reduces to

$$q = \frac{K(P+Q)}{(K-D)} \text{ or} \quad ...(7)$$

$$Q = \frac{(K-D)q}{K} - P \quad ...(8)$$

Using above result in equation (1), we obtain

$$C = f(P, q) = \frac{K}{2q(K-D)}\left[C_1P^2 + C_2\left\{\frac{(K-D)q}{K} - P\right\}\right] + \frac{(MC_3+C_4)D}{q} \quad ...(9)$$

Applying the conditions of optimality, we get

$$P = \frac{C_2(1-D/K)q}{(C_1+C_2)} \quad ...(10)$$

$$Q = \frac{C_1(1-D/K)q}{(C_1+C_2)} \quad ...(11)$$

$$C = \frac{C_1C_2(1-D/K)q}{2(C_1+C_2)} + \frac{MC_3D}{q} + \frac{C_4D}{q} \quad ...(12)$$

$$= f_1(q) + f_2(q) + f_3(q)$$

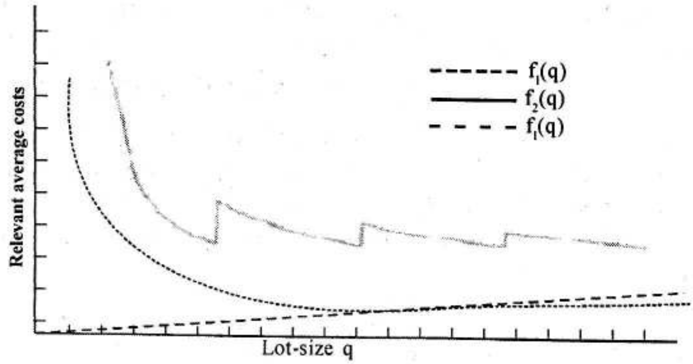
Also the optimal value of q is calculated as

$$q^* = \sqrt{\left[\frac{2(MC_3+C_4)D(C_1+C_2)}{C_1C_2(1-D/K)}\right]}$$

To obtain the optimal inventory cost, the analysis of transportation cost $\frac{MC_3D}{q}$ in relation with the set-up cost and carrying cost is important. Transportation cost $\frac{MC_3D}{q}$ varies with the value of q. We observe the following points, which are relevant to obtain optimum lot-size in this case.

- As shown in Figure-2 (curve for $f_2(q)$) cost $\frac{MC_3D}{q}$ suddenly jumps as q become slightly greater than MN. Then it decreases continuously till q becomes (M+1)N, and so on. Thus, $\frac{MC_3D}{q}$ is minimum at q = MN and this pattern is same for all M.
- We know that the cost $\frac{C_1C_2(1-D/K)q}{2(C_1+C_2)} + \frac{C_4D}{q}$ is minimum at $q = q_o = \sqrt{\left[\frac{2C_4D(C_1+C_2)}{C_1C_2(1-D/K)}\right]}$. As the value of q deviates from this optimal value, the cost continuously increases.

From the points noted above, we conclude that the average total cost f(q) increases for the values of q less than $q_{M^*-1}$ (= (M*-1)N) as well as for the values of q greater




**Variation of relevant average costs with respect to lot-size**

**Figure 2**

than $q_{M*}$ (= M*N), where $M^* = \lceil q_0/N \rceil$. Thus the minimum average total cost f(q) lies either between $q_{M*-1}$ and $q_{M*}$ or at one of them.

The value of q for minimum cost, between $q_{M*-1}$ and $q_{M*}$ can be calculated by the formula

$$q = \sqrt{\left[\frac{2(M^*C_3 + C_4)D(C_1 + C_2)}{C_1C_2(1 - D/K)}\right]}$$

If $q \leq M^*N$, we compare $f(q_{M*-1})$, $f(q_{M*})$ and f(q) and the least of these three values is the optimum cost and the corresponding value of lot-size is the optimum lot-size.

If $q > M^*N$, we compare $f(q_{M*-1})$ and $f(q_{M*})$. The lesser one is the optimum cost and the corresponding value of lot-size is the optimum lot-size.

These facts lead to the following algorithm.

**Algorithm**

**Step-1:** Calculate the optimum lot-size of inventory, when the transportation cost is not included, by using the formula:

$$q_o = \sqrt{\left[\frac{2C_4D(C_1 + C_2)}{C_1C_2(1 - D/K)}\right]}$$

**Step-2:** Calculate the number of vehicles M* to carry $q_o$, by using the formula

$$M^* = \lceil q_0/N \rceil$$

**Step-3:** Calculate

$$q = \sqrt{\left[\frac{2(M^*C_3 + C_4)D(C_1 + C_2)}{C_1C_2(1 - D/K)}\right]}$$

**Step-4:** Calculate $q_{M*} = M^*N$ and $q_{M*-1} = (M^*-1)N$

**Step-5(a):** If $q \leq M^*N$, we compare $f(q_{M*-1})$, $f(q_{M*})$ and f(q) and the least of these three values is the optimum cost and the corresponding value of lot-size is the optimum lot-size.

**Step-5(b):** If $q > q_{M^*}$, then compare $f(q_{M^*-1})$ and $f(q_{M^*})$. The lesser one is the optimum cost and the corresponding value of lot-size is the optimum lot-size.

## ILLUSTRATIVE EXAMPLES

**Example-1:** Let $C_1$ = \$ 2 per unit per unit time, $C_2$ = \$ 20 per unit per unit time, $C_3$ = \$ 1 up to one vehicle load, $C_4$ = \$ 15 per set-up, D = 90 units per unit time, K = 180 units per unit time and N = 5 units.

**Step-1:** Calculate the optimum lot-size of inventory, when the transportation cost is not included

$$q_o = \sqrt{\left[\frac{2C_4D(C_1+C_2)}{C_1C_2(1-D/K)}\right]}$$

$$= \sqrt{\left[\frac{2\times15\times90\times(2+20)}{2\times20\times(1-90/180)}\right]}$$

$$= 54.49771$$

**Step-2:** Calculate the number of vehicles M* to carry $q_o$

$$M^* = \lceil q_0/N \rceil$$

$$= 11$$

**Step-3:** Calculate q

$$q = \sqrt{\left[\frac{2(M^*C_3+C_4)D(C_1+C_2)}{C_1C_2(1-D/K)}\right]}$$

$$= \sqrt{\left[\frac{2\times(11\times1+15)\times90\times(2+20)}{2\times20\times(1-90/180)}\right]}$$

$$= 71.74956$$

**Step-4:** Calculate $q_{M^*}$ and $q_{M^*-1}$

$$q_{M^*} = M^*N$$

$$= 11\times5 = 55$$

$$q_{M^*-1} = (M^*-1)N$$

$$= (11-1)5 = 50.$$

Here q > M*N.

**Step-5(b):**

$$f(q_{M^*-1}) = \frac{C_1C_2(1-D/K)q_{M^*-1}}{2(C_1+C_2)} + \frac{C_4D}{q_{M^*-1}} + \frac{(M^*-1)C_3D}{q_{M^*-1}}$$

$$f(50) = \frac{2\times20\times(1-90/180)\times50}{2\times(2+20)} + \frac{15\times90}{50} + \frac{(11-1)\times1\times9c}{50}$$

$$= 67.72727$$

$$f(q_{M^*}) = \frac{C_1C_2(1-D/K)q_{M^*}}{2(C_1+C_2)} + \frac{C_4D}{q_{M^*}} + \frac{M^*C_3D}{q_{M^*}}$$

$$f(55) = \frac{2\times20\times(1-90/180)\times55}{2\times(2+20)} + \frac{15\times90}{55} + \frac{11\times1\times9C}{55} = 67.54545$$

Here f(55) < f(50), so f(55) = \$ 67.54545 is the optimum cost and the optimum lot-size is 55 units.

If this problem is solved by Gupta's [5] method, it requires 11 iterations in 44 steps involving quite lengthy calculations.

**Example-2:** Let $C_1$ = \$ 0.8 per unit per unit time, $C_2$ = \$ 8 per unit per unit time, $C_3$ = \$ 1 up to one vehicle load, $C_4$ = \$ 10 per set-up, D = 1000 units per unit time, R = 4000 units per unit time and N = 80 units.

**Step-1:** Calculate the optimum lot-size of inventory, when the transportation cost is not included

$$q_o = \sqrt{\left[\frac{2C_4D(C_1+C_2)}{C_1C_2(1-D/K)}\right]}$$

$$= \sqrt{\left[\frac{2\times10\times1000\times(0.8+8)}{0.8\times8\times(1-1000/4000)}\right]} = 191.48542$$

**Step-2:** Calculate the number of vehicles M* to carry $q_o$

$$M^* = \lceil q_0/N \rceil$$

$$= \lceil 191.4854/80 \rceil = 3$$

**Step-3:** Calculate q

$$q = \sqrt{\left[\frac{2(M^*C_3+C_4)D(C_1+C_2)}{C_1C_2(1-D/K)}\right]}$$

$$= \sqrt{\left[\frac{2\times(3\times1+10)\times1000\times(0.8+8)}{0.8\times8\times(1-1000/4000)}\right]} = 218.32697$$

**Step-4:** Calculate $q_{M^*}$ and $q_{M^*-1}$

$$q_{M^*} = M^*N = 3\times80 = 240.$$

$$q_{M^*-1} = (M^*-1)N$$

$$= (3-1)80 = 160.$$

Here q < M*N.

**Step-5(a):**

$$f(q_{M^*-1}) = \frac{C_1C_2(1-D/K)q_{M^*-1}}{2(C_1+C_2)} + \frac{C_4D}{q_{M^*-1}} + \frac{(M^*-1)C_3D}{q_{M^*-1}}$$

$$f(160) = \frac{0.8\times8\times(1-1000/4000)\times160}{2\times(0.8+8)} + \frac{10\times1000}{160} + \frac{(3-1)\times1\times1000}{160}$$

$$= 118.63636$$

$$f(q_{M^*}) = \frac{C_1C_2(1-D/K)q_{M^*}}{2(C_1+C_2)} + \frac{C_4D}{q_{M^*}} + \frac{M^*C_3D}{q_{M^*}}$$

$$f(240) = \frac{0.8\times8\times(1-1000/4000)\times240}{2\times(0.8+8)} + \frac{10\times1000}{240} + \frac{3\times1\times1000}{240}$$

$$= 119.62121$$

$$f(q) = \frac{C_1C_2(1-D/K)q}{2(C_1+C_2)} + \frac{C_4D}{q} + \frac{M^*C_3D}{q}$$

$$f(218.3269) = \frac{0.8\times 8\times(1-1000/4000)\times 218.3269}{2\times(0.8+8)} + \frac{10\times 1000}{218.3269} + \frac{3\times 1\times 1000}{218.3269}$$

$$= 119.08743$$

Here $f(q_{M^*-1})$ is the least, therefore 160 units is the optimum lot-size and 118.63636 is the optimal cost.

If this problem is solved by Gupta's [9] method, it requires 2 iterations in 8 steps.

## CONCLUSION

In many real-life situations transportation cost is fixed for a transport mode of finite capacity. A fixed cost is incurred when a vehicle is engaged whether its capacity is fully utilized or not. Most of the inventory models either consider transportation cost as fixed and include it in the set-up cost or consider it as variable and assume it as a part of the item cost. In the present paper, we have developed a transport-inventory model where transportation cost is considered explicitly. The cost function is a discrete function of lot-size. The algorithm proposed here is simple and saves much of time and efforts involved in calculations. Illustrative examples show how the lot-size and average inventory cost are influenced by the transportation cost considered explicitly.

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# BIANCHI TYPE-III COSMOLOGICAL MODELS WITH TIME DEPENDENT DISPLACEMENT VECTOR FOR DUST FILLED UNIVERSE IN LYRA GEOMETRY

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**ABSTRACT**

Bianchi type III cosmological models with time dependent displacement vector field for dust fluid distribution in Lyra geometry are investigated. To get the deterministic model of the universe, we have assumed that shear ($\sigma$) is proportional to the expansion ($\theta$) in the model. This leads to $B = C^n$ where B and C are metric potentials, $n$ is a constant and proportionality constant is taken as unity. The special models in terms of cosmic time $t$ are also investigated. The physical and geometrical aspects of the models with singularity also discussed.

**Key Words :** Bianchi III, displacement vector, dust, Lyra geometry.

**Mathematics Subject Classification (2000) :** 83 D05, 83 F05

## 1. INTRODUCTION

Bianchi Type I models are a very special subset of spatially homogeneous cosmological models. We, therefore, consider more general Bianchi Type III models in a similar investigation. On the large scale, the distribution of matter and radiation could be treated as a perfect fluid. The present day observations show that matter is dominant over radiation and it can be expressed as a distribution of pressure less dust. This picture might not have been valid in the early stages of the universe when radiation was dominant, Friedmann [2] was the first to obtain solution corresponding to pressure less dust distribution in a non-static universe in which there was no need to introduce the cosmological constant.

Einstein derived his field equations of general relativity by geometrizing gravitation. Inspired by the idea of geometrizing gravitation, Weyl [13] developed a theory to geometrize gravitation and electromagnetism both. But Weyl's theory was not given much importance due to non-integrability of length of vector under parallel displacement.

Later, Lyra [5] modified Riemannian geometry by introducing a gauge function into the structure less manifold. This step removed the main obstackle of Weyl's theory and made length of vector dependent upon parallel transport.

In continuation of investigation, Sen [10] formulated a new scalar tensor theory of gravitation and constructed an analogue of Einstein's field equations of General Relativity. Halford [3] pointed out that the constant displacement vector field (β) of Lyra geometry, plays the role of cosmological constant (Λ) in General Relativity. Soleng [12] has shown that the displacement vector includes a Creation field of Hoyle-Narlikar Creation field [4]. Singh and Singh [11] have reviewed Lyra geometry and investigated Bianchi Type I cosmological model with time dependent displacement vector field in Lyra geometry. Ram et al. [9] have obtained exact solutions of modified Einstein's field equation for Bianchi Type III and V space-time with stiff fluid distribution in Lyra geometry. Pradhan et al. [7] and Rahman et al. [8], have investigated cosmological models with constant and time dependent displacement vector field in Lyra geometry. Recently Bali and Chandnani [1] have investigated Bianchi Type I cosmological models with time dependent displacement vector field for perfect fluid distribution in Lyra geometry.

In this paper, we have investigated Bianchi Type III cosmological models with time dependent displacement vector for dust filled universe in Lyra geometry. To obtain the deterministic model of the universe, we have assumed that the shear (σ) is proportional to the expansion (θ) in the model. The physical and geometrical aspects of the model with singularities in the model are also discussed.

## 2. METRIC AND FIELD EQUATIONS

We consider Bianchi type III metric in the form

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 e^{-2x} dy^2 + C^2 dz^2 \qquad \text{.... (2.1)}$$

Where A, B, C are functions of t – alone.

Energy momentum tensor $(T_i^j)$ for perfect fluid distribution is given by

$$T_i^j = (\rho + p) v_i v^j + p g_i^j \qquad \text{...(2.2)}$$

Where $v_i = (0, 0, 0, -1)$; $v^i_- v_i = -1$ , $\phi_i = (0, 0, 0, \beta(t))$

$v_4 = -1$ and $v^4 = 1$. p is the isotropic pressure, ρ the energy density, $v^i$ the fluid flow vector and β the gauge function.

Einstein's modified field equation in normal gauge for Lyra's manifold obtained by Sen [10] is given by

$$R_i^j - \frac{1}{2} R g_i^j + \frac{3}{2}\phi_i \phi^j - \frac{3}{4}\theta_k \phi^k g_i^j = -T_i^j \qquad \text{... (2.3)}$$

(in geometrized units where $8 \pi G = 1$, $c = 1$) The field equation (2.3) for the metric (2.1) for dust filled universe leads to

$$\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{B_4C_4}{BC}+\frac{3}{4}\beta^2=0 \qquad \text{... (2.4)}$$

$$\frac{A_{44}}{A}+\frac{C_{44}}{C}+\frac{A_4C_4}{AC}+\frac{3}{4}\beta^2=0 \qquad \text{... (2.5)}$$

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}-\frac{\alpha^2}{A^2}+\frac{3}{4}\beta^2=0 \qquad \text{... (2.6)}$$

$$\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}+\frac{B_4C_4}{BC}-\frac{\alpha^2}{A^2}-\frac{3}{4}\beta^2=\rho \qquad \text{... (2.7)}$$

$$\left(\frac{A_4}{A}-\frac{B_4}{B}\right)=0 \qquad \text{...(2.8)}$$

The energy conservation equation $T_i^j;_j=0$ leads to

$$\rho_4+\rho\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)=0 \qquad \text{... (2.9)}$$

$(\because\ p=0)$

The conservation of left hand side of (2.3) leads to

$$(R_i^j-\frac{1}{2}R\,g_i^j);_j+\frac{3}{2}(\phi_i\phi^j);_j-\frac{3}{4}(\phi_k\phi^k g_i^j);_j=0 \qquad \text{...(2.10)}$$

which leads to

$$\beta\beta_4+\beta^2\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)=0 \qquad \text{...(2.11)}$$

## 3. SOLUTION OF FIELD EQUATIONS

For the complete determination of the model, we assume that shear ($\sigma$) is proportional to the expansion ($\theta$). This leads to

$$B=C^n \qquad \text{...(3.1)}$$

where $n$ is a constant. Equation (2.8) leads to

$$\frac{A_4}{A}=\frac{B_4}{B} \qquad \text{...(3.2)}$$

which leads to

$$A=lB \qquad \text{...(3.3)}$$

where $l$ is a constant. Equation (2.1) after using (3.1) and (3.2) leads to

$$\frac{\beta_4}{\beta}=-(2n+1)\frac{C_4}{C} \qquad \text{...(3.4)}$$

From equation (3.4) we have

$$\beta = \gamma C^{-(2n+1)} \qquad ...(3.5)$$

where $\gamma$ is constant of integration. From equation (2.4), (3.1) and (3.5), we have

$$2C_{44} + \frac{2n^2}{n+1}\frac{C_4^2}{C} = -\frac{3\gamma^2}{2(n+1)}C^{-4n-1} \qquad ...(3.6)$$

To find the solution of (3.6), we assume that $C_4 = f(C)$, thus equation (3.6) leads to

$$\frac{d}{dC}(f^2) + \frac{2n^2}{(n+1)}\frac{f^2}{C} = -\frac{3\gamma^2}{2(n+1)}C^{-4n-1} \qquad ...(3.7)$$

Equation (3.7) leads to

$$f^2.C^{\frac{2n^2}{n+1}} = \frac{-3\gamma^2}{2(n+1)}\int C^{-4n-1+\frac{2n^2}{n+1}}dC$$

$$= \frac{-3\gamma^2}{2(n+1)}\int C^{\frac{-4n^2-n-4n-1+2n^2}{n+1}} dC$$

$$= \frac{-3\gamma^2}{2(n+1)}\int C^{\frac{-2n^2-5n-1}{n+1}} dC$$

$$= \frac{-3\gamma^2}{2(n+1)}\frac{C^{\frac{-2n^2-5n-1+n+1}{n+1}}}{\frac{-2n^2-4n}{n+1}} + \alpha$$

$$= \frac{-3\gamma^2}{-4n(2+n)}C^{\frac{-2n^2-4n}{n+1}} + \alpha$$

$$= \frac{3\gamma^2}{4n(n+2)}C^{\frac{-2n^2-4n}{n+1}} + \alpha$$

$$f^2 = \frac{3\gamma^2}{4n(n+2)}.C^{\frac{-2n^2}{n+1}-\frac{4n}{n+1}-\frac{2n^2}{n+1}} + \alpha C^{-\frac{2n^2}{n+1}}$$

$$= \frac{3\gamma^2}{4n(n+2)} C^{\frac{-4n^2-4n}{n+1}} + \alpha C^{\frac{-2n^2}{n+1}}$$

$$= \frac{3\gamma^2}{4n(n+2)} C^{-4n} + \alpha C^{\frac{-2n^2}{n+1}}$$

$$\therefore \quad f^2 = KC^{-4n} + \alpha\ C^{\frac{-2n^2}{n+1}} \qquad ...(3.8)$$

where

$$K = \frac{3\gamma^2}{4n(n+2)} \qquad ...(3.9)$$

Equation (3.8) leads to

$$\left(\frac{dC}{dt}\right)^2 = K\,C^{-4n} + \alpha\,C^{\frac{-2n^2}{n+1}} \qquad ...(3.10)$$

Thus the metric (2.1) leads to

$$ds^2 = -\left(\frac{dt}{dC}\right)^2 dC^2 + l^2 C^{2n} dx^2 + C^{2n} e^{-2x} dy^2 + C^2 dz^2 \qquad ...(3.11)$$

Which after suitable transformation of coordinates leads to

$$ds^2 = -\frac{dT^2}{KT^{-4n} + \alpha T^{\frac{-2n^2}{n+1}}} + T^{2n} dX^2 + T^{2n} e^{\frac{-2X}{l}} dY^2 + T^2 dZ^2 \qquad ...(3.12)$$

where $C = T$, $lx = X$, $y = Y$, $z = Z$ and cosmic time $t$ is defined as

$$t = \int \frac{dT}{\sqrt{KT^{-4n} + \alpha T^{-2n^2/n+1}}} \qquad ...(3.13)$$

## 4. SPECIAL MODELS

To get the deterministic value of C in terms of cosmic time *t*, we assume $n = -\frac{1}{2}$, 1 thus

(i) For $n = -\frac{1}{2}$, equation (3.10)_ leads to

$$\frac{C^{1/2}dC}{\sqrt{KC^{2(3/2)}+\alpha}}=dt \qquad ...(4.1)$$

which leads to

$$C^{3/2}=N\sin h(at+b) \qquad ...(4.2)$$

where N is the constant of integration and

$$a=\frac{2\sqrt{K}}{3},\ N^2=\frac{\alpha}{K}, \text{ Thus}$$

$$B=C^{-1/2}=N^{-1/3}\sin h^{-1/3}(at+b) \qquad ...(4.3)$$

and

$$A=lB=lN^{-1/3}\sin h^{-1/3}(at+b) \qquad ...(4.4)$$

Therefore, the metric (2.1) leads to

$$ds^2=-dt^2+l^2N^{-2/3}\sin h^{-2/3}(at+b)\,dx^2+N^{-2/3}\sin h^{-2/3}$$

$$(at+b)e^{-2x}\,dy^2+N^{4/3}\sin h^{4/3}(at+b)\,dz^2 \qquad ...(4.5)$$

which after suitable transformation of coordinates leads to

$$ds^2=-\frac{dT^2}{a^2}+\sin h^{-2/3}T\,dX^2+\sin h^{-2/3}Te^{-\frac{2N^{1/3}x}{l}}dY^2+\sin h^{4/3}T\,dZ^2 \qquad ...(4.6)$$

where $at+b=T,\ l\,N^{-1/3}x=X,\ N^{-1/3}y=Y,\ N^{2/3}z=Z$

(ii) For $n=1$, equation (3.10) leads to

$$\frac{2C^2dC}{\sqrt{1+LC^3}}=N\,dt \qquad ...(4.7)$$

which leads to

$$C^3=\frac{(Qt+S)^2-1}{L} \qquad ...(4.8)$$

where Q, S and L are constants of integration. Thus

$$B=C=\left[\frac{(Qt+S)^2-1}{L}\right]^{1/3} \qquad ...(4.9)$$

and

$$A=lB=\frac{l}{L^{1/3}}\left[(Qt+S)^2-1\right]^{1/3} \qquad ...(4.10)$$

Thus the metric (2.1) reduces to the form

$$ds^2 = -\frac{d\tau^2}{Q^2} + (\tau^2-1)^{2/3}dX^2 + (\tau^2-1)^{2/3}e^{-\frac{2xl}{L^{1/3}}}dY^2 + (\tau^2-1)^{2/3}dZ^2 \qquad ...(4.11)$$

where

## 5. SOME PHYSICAL AND GEOMETRICAL FEATURES

The matter density $(\rho)$, the displacement vector $(\beta)$, the expansion $(\theta)$, spatial volume $(R^3)$ and the generalized mean Hubble parameter (H) for the model (3.12) are given by

$$\rho = \frac{b}{T^{2n+1}} \qquad ...(5.1)$$

$$\beta = \frac{\gamma}{T^{2n+1}} \qquad ...(5.2)$$

$$\theta = \frac{1}{T^{2n+1}}\sqrt{K + \alpha T^{\frac{2n(1+2n)}{1+n}}} \qquad ...(5.3)$$

$$R^3 = lT^{2n+1}$$

$$H = \frac{1}{3}(H_1 + H_2 + H_3) \qquad ...(5.4)$$

$$= \frac{2n+1}{T^{2n+1}}\sqrt{K + \alpha T^{\frac{2n(1+2n)}{1+n}}} \qquad ...(5.5)$$

For the model (4.5), the matter density $(\rho)$, the displacement vector $(\beta)$, the spatial volume $(R^3)$ are constants and there is no expansion in the model. Thus the model (4.5) represents a steady state.

The above mentioned quantities for the model (4.11) are given by

$$\rho = \frac{b}{\tau^2 - 1} \qquad ...(5.6)$$

$$\beta = \frac{\gamma}{\tau^2 - 1} \qquad ...(5.7)$$

$$\theta = \frac{2Q\tau}{\tau^2 - 1} \qquad ...(5.8)$$

$$R^3 = \tau^2 - 1 \qquad ...(5.9)$$

$$H = \frac{2Q\tau}{3(\tau^2 - 1)} \qquad ...(5.10)$$

## 6. CONCLUSIONS

The model (3.12) starts expanding with a big bang at $T = 0$ and the expansion in the model decreases as T increases where $2n+1>0$. The matter density $(\rho)$ and displacement vector $(\beta)$ are initially large but it decreases due to lapse of time. The spatial volume $(R^3)$ increases as T increases where $2n+1>0$

The mean generalized Hubble parameter is constant where $n=-1/2$. The model (3.12) is free from singularity.

The model (4.11) starts expanding with a big bang at $\tau = 1$ and the expansion in the model decreases as $\tau$ increases. The quantities $\rho, \beta, \theta, H$ are initially large but decreases as $\tau$ increases. The spatial volume $(R^3)$ increases as $\tau$ increases. The model (4.11) has Point type singularity at $\tau = 1$ (MacCallum [6]). Since shear $(\sigma)$ is zero for the model (4.11), hence the model (4.11) represents an isotropic universe.

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# COMPOSITION FORMULAE FOR THE MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS

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## ABSTRACT

*In the present paper we derive three new and interesting composition formulae of a class of multidimensional Fractional integral operators involving generalized multivariable polynomial and $\overline{H}$-function. On account of the general nature of the functions occurring as kernels here, the main findings of our paper are capable of yielding a number of corresponding results (new and known) involving simpler functions and polynomials (of one or more variables) as special cases of our formulae. We also give a two dimensional analogue of our second composition formulae. The results obtained by Erdélyi [2], Goyal and Jain [3], Goyal, Jain and Gaur [4], Raina [12], follow as simple cases of our composition formulae.*

**Keywords:** Fractional integral operator, - function, Mellin transform, Stieltjes transform, General class of multivariable polynomials.

**AMS Subject Classification Code:** 33C60, 44A35, 44A05.

## 1. INTRODUCTION

The subject of fractional calculus (that is, calculus of derivatives and integrals of an arbitrary order) has gained considerable importance and popularity during the past four decades or so, due mainly to its applications in numerous diverse fields of science and engineering. Fractional calculus is applicable in deriving the solutions of certain integral equations involving special functions of mathematical physics which possess a Mellin-Barnes type integral representation.

In recent years several authors (see, for example) Erdélyi [2], Nishimoto [10, 11], Gupta and Soni [7], Gupta et al. [6] have made significant contributions to the fractional calculus operators involving various functions and polynomials. Srivastava and Saxena [17] have presented a systemic account of Fractional calculus operators and their applications investigated by various authors. In the present paper we introduce and study a new pair of Fractional integral operators defined and represented in the following manner:

$$I_x\left[f(t_1,...,t_s)\right]=I_{x;U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left[f(t_1,...,t_s);x_1,...,x_s\right]$$

$$=\left(\prod_{j=1}^{s}x_j^{-\rho_j-\sigma_j}\right)\int_0^{x_1}...\int_0^{x_s}\left[\prod_{j=1}^{s}t_j^{\rho_j}(x_j-t_j)^{\sigma_j-1}\right]S_V^{U_1,...,U_s}\left[E_1\left(\frac{t_1}{x_1}\right)^{e_1}\left(1-\frac{t_1}{x_1}\right)^{f_1},...,E_s\left(\frac{t_s}{x_s}\right)^{e_s}\left(1-\frac{t_s}{x_s}\right)^{f_s}\right]$$

$$\times\overline{H}_{P,Q}^{M,N}\left[Z\prod_{j=1}^{s}\left(\frac{t_j}{x_j}\right)^{\eta_j}\left(1-\frac{t_j}{x_j}\right)^{\lambda_j}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]f(t_1,...,t_s)\,dt_1...dt_s \qquad ...(1.1)$$

$$J_x\left[f(t_1,...,t_s)\right]=J_{x;U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left[f(t_1,...,t_s);x_1,...,x_s\right]$$

$$=\left(\prod_{j=1}^{s}x_j^{\rho_j}\right)\int_{x_1}^{\infty}...\int_{x_s}^{\infty}\left[\prod_{j=1}^{s}t_j^{-\rho_j-\sigma_j}(t_j-x_j)^{\sigma_j-1}\right]S_V^{U_1,...,U_s}\left[E_1\left(\frac{x_1}{t_1}\right)^{e_1}\left(1-\frac{x_1}{t_1}\right)^{f_1},...,E_s\left(\frac{x_s}{t_s}\right)^{e_s}\left(1-\frac{x_s}{t_s}\right)^{f_s}\right]$$

$$\times\overline{H}_{P,Q}^{M,N}\left[Z\prod_{j=1}^{s}\left(\frac{x_j}{t_j}\right)^{\eta_j}\left(1-\frac{x_j}{t_j}\right)^{\lambda_j}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]f(t_1,...,t_s)\,dt_1...dt_s \qquad ...(1.2)$$

Throughout the paper, we assume that

$$f(t_1,...,t_s)=\begin{cases}O\prod_{j=1}^{s}\left(|t_j|^{U_j}\right) & \max\{|t_j|\}\to 0\\ O\prod_{j=1}^{s}\left(|t_j|^{-V_j}e^{-W_j|t_j|}\right) & \min\{|t_j|\}\to\infty\end{cases}\qquad j=1,...,s \qquad ...(1.3)$$

Such a class of function will be represented symbolically as $f(t_1,...,t_s)\in \mathrm{A}$.

We also assume that $\int...\int_{\Omega_s}|f(t_1,...,t_s)|dt_1...dt_s<\infty$ for every bounded s-dimensional region $\Omega_s$ excluding the origin. The operators defined by (1.1) and (1.2) exists if

(i) $\min \mathrm{Re}(e_j,f_j,\eta_j,\lambda_j)\geq 0 \quad (j=1,...,s)$ not all zero simultaneously;

(ii) $\min\limits_{1\leq k\leq M}\mathrm{Re}\left[1+\rho_j+U_j+\eta_j\frac{b_k}{\beta_k}\right]>0,\ \min\limits_{1\leq k\leq M}\mathrm{Re}\left[\sigma_j+\lambda_j\frac{b_k}{\beta_k}\right]>0$;

(iii) $\mathrm{Re}(W_j)=0,\ \min\limits_{1\leq k\leq M}\mathrm{Re}\left[\rho_j+V_j+\eta_j\frac{b_k}{\beta_k}\right]>0,\ \min\limits_{1\leq k\leq M}\mathrm{Re}\left[\sigma_j+\lambda_j\frac{b_k}{\beta_k}\right]>0$;

$$\text{or} \quad \mathrm{Re}\left(W_j\right) > 0, \min_{1 \le k \le M} \mathrm{Re}\left[\sigma_j + \lambda_j \frac{b_k}{\beta_k}\right] > 0 \qquad \text{...(1.4)}$$

The multivariable polynomial $S_V^{U_1,\dots,U_k}\left(x_1,\dots,x_k\right)$ introduced by Srivastava and Garg [16, p. 686, eq. (1.4)] is defined in the following manner:

$$S_V^{U_1,\dots,U_k}\left[x_1,\dots,x_k\right] = \sum_{\substack{R_1,\dots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i} A\left(V, R_1,\dots,R_k\right) \frac{x_i^{R_i}}{R_i!} \quad ,V = 0,1,2,\dots \qquad \text{...(1.5)}$$

where $U_1,\dots,U_k$ are arbitrary positive integers and the coefficients $A\left(V, R_1,\dots,R_k\right)$ are arbitrary constants (real or complex).

The $\overline{\mathrm{H}}$-Function occurring in the paper was introduced by Inayat Hussain [8, 9] and studied by Buschman and Srivastava [1] and others. It is represented in the following manner:

$$\overline{H}_{P,Q}^{M,N}\left[z\right] = \overline{H}_{P,Q}^{M,N}\left(z \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right)$$

$$= \frac{1}{2\pi i}\int_L \bar{\phi}(\xi) z^{\xi} d\xi, \qquad (z \ne 0) \qquad \text{... (1.6)}$$

$$\text{where} \quad \bar{\phi}(\xi) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{N} \{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q} \{\Gamma(1-b_j+\beta_j\xi)\}^{B_j} \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j \xi)} . \qquad \text{...(1.7)}$$

The following sufficient conditions for the absolute convergence of the defining integral for $\overline{\mathrm{H}}$-function given by (1.6) have been given by Buschman and Srivastava [1]

$$|\arg(z)| < \frac{1}{2}\pi\Omega , \qquad \text{...(1.8)}$$

$$\text{where} \quad \Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N} A_j \alpha_j - \sum_{j=M+1}^{Q} B_j \beta_j - \sum_{j=N+1}^{P} \alpha_j > 0 . \qquad \text{...(1.9)}$$

The following series representation of the $\overline{H}$-function was given by Rathie [14].

$$\overline{H}_{P,Q}^{M,N}\left(z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right)=\sum_{\upsilon=1}^{M}\sum_{P=0}^{\infty}\overline{\theta}\left(S_{P,\upsilon}\right)z^{S_{P,\upsilon}} \quad \ldots (1.10)$$

where

$$\overline{\theta}(S_{P,\upsilon})=\frac{\prod_{j=1,j\neq\upsilon}^{M}\Gamma(b_j-\beta_j S_{P,\upsilon})\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j S_{P,\upsilon})\}^{A_j}(-1)^P}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j S_{P,\upsilon})\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j S_{P,\upsilon})P!\beta_\upsilon},\quad S_{P,\upsilon}=\frac{b_\upsilon+P}{\beta_\upsilon} \quad \ldots (1.11)$$

In the sequel, we shall also make use of the following behavior of the $\overline{H}$-function for small and large value of z as recorded by Saxena and Gupta [15, p. 870, eqs. (2.3) and (2.4)].

$$\overline{H}_{P,Q}^{M,N}[z]=O\left[|z|^{\alpha}\right] \text{ for small z, where } \alpha=\min_{1\le j\le M}\left[\operatorname{Re}\left(\frac{b_j}{\beta_j}\right)\right] \quad \ldots (1.12)$$

$$\overline{H}_{P,Q}^{M,N}[z]=O\left[|z|^{\beta}\right] \text{ for large z, where } \beta=\max_{1\le j\le N}\left[\operatorname{Re}\left(\frac{a_j-1}{\alpha_j}\right)\right] \quad \ldots (1.13)$$

and the conditions (1.8) and (1.9) are satisfied.

## 2. COMPOSITION FORMULAE FOR THE MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS

***Result 1***

$$I_{x;U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left\{J_{y;U',V';Z'}^{\rho',\sigma';e',f';\eta',\lambda'}\left[f\left(t_1,\ldots,t_s\right)\right]\right\}$$

$$=\left(\prod_{j=1}^{s}x_j^{\rho_j-1}\right)\int_0^{x_1}\cdots\int_0^{x_s}\left(\prod_{j=1}^{s}t_j^{\rho_j}\right)G\left(\frac{t_1}{x_1},\ldots,\frac{t_s}{x_s}\right)f\left(t_1,\ldots,t_s\right)dt_1\ldots dt_s$$

$$+\left(\prod_{j=1}^{s}x_j^{\rho_j'}\right)\int_{x_1}^{\infty}\cdots\int_{x_s}^{\infty}\left(\prod_{j=1}^{s}t_j^{-\rho_j'-1}\right)G\left(\frac{x_1}{t_1},\ldots\frac{x_s}{t_s}\right)f\left(t_1,\ldots,t_s\right)dt_1\ldots dt_s\ , \quad \ldots (2.1)$$

where

$$G\left(t_1,\ldots,t_s\right)=\sum_{R_1,\ldots,R_s=0}^{\sum_{j=1}^{s}U_jR_j\le V}(-V)_{\sum_{j=1}^{s}U_jR_j}A\left(V,R_1,\ldots,R_s\right)\frac{E_1^{R_1}}{R_1!}\cdots\frac{E_s^{R_s}}{R_s!}\sum_{R_1',\ldots,R_s'=0}^{\sum_{j=1}^{s}U_j'R_j'\le V'}\left(-V'\right)_{\sum_{j=1}^{s}U_j'R_j'}A\left(V',R_1',\ldots,R_s'\right)$$

$$\frac{E_1^{'R_1'}}{R_1'!}\cdots\frac{E_s^{'R_s'}}{R_s'!}\sum_{\upsilon=1}^{M}\sum_{P'=0}^{\infty}\bar{\theta}\left(S_{P',\upsilon}\right)z^{S_{P',\upsilon}}\Gamma\left(\sigma_j'+f_j'R_j'+\lambda_j'S_{P',\upsilon}+l\right)t_j^{e_j'R_j'+l}\left(1-t_j\right)^{\sigma_j+\sigma_j'+f_jR_j+f_j'R_j'+\lambda_j'S_{P',\upsilon}-1}$$

$$\overline{H}_{P+2s,Q+2s}^{M,N+2s}\left[z\prod_{j=1}^{s}\left(t_j\right)^{\eta_j}\left(1-t_j\right)^{\lambda_j}\middle|\begin{matrix}A^*\\B^*\end{matrix}\right], \qquad \ldots(2.2)$$

where

$$A^*=(a_k,\alpha_k;A_k)_{1,N}\left(-\rho_j-\rho_j'-e_jR_j-e_j'R_j'-\eta_j'S_{P',\upsilon},\eta_j;1\right)$$
$$\left(-\rho_j-\rho_j'-\sigma_j-\sigma_j'-l-1-\left(e_j+f_j\right)R_j+\left(e_j'+f_j'\right)R_j'+\left(\eta_j'+\lambda_j'\right)S_{P',\upsilon},\eta_j\right)(a_k,\alpha_k;A_k)_{1,N}, \qquad \ldots(2.3)$$

$$B^*=(b_k,\beta_k)_{1,M}(b_k,\beta_k;B_k)_{M+1,Q}\left(-\rho_j-\rho_j'-\sigma_j'-l-e_jR_j-\left(e_j'+f_j'\right)R_j'-\left(\eta_j'+\lambda_j'\right)S_{P',\upsilon},\eta_j';1\right)$$
$$\left(-\rho_j-\rho_j'-\sigma_j-\sigma_j'-1-\left(e_j+f_j\right)R_j-\left(e_j'+f_j'\right)R_j'-\left(\eta_j'+\lambda_j'\right)S_{P',\upsilon},\left(\eta_j+\lambda_j\right);1\right), \qquad \ldots(2.4)$$

and $G'\left(t_1,\ldots,t_s\right)$ can be obtained from $G\left(t_1,\ldots,t_s\right)$ from (2.2) by interchanging the parameters with dashes with those without dashes, $f\left(t_1,\ldots,t_s\right)\in A$, the composite operator defined by the L.H.S. of (2.1) exists and the following conditions are satisfied:

$$\min_{1\le k\le M}\operatorname{Re}\left[1+\rho_j+\eta_j\frac{b_k}{\beta_k}\right]>0,\ \min_{1\le k\le M}\operatorname{Re}\left[1+\rho_j'+U_j+\eta_j'\frac{b_k}{\beta_k}\right]>0$$

$$\min_{1\le k\le M}\operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{\beta_k}\right]>0,\ \min_{1\le k\le M}\operatorname{Re}\left[\sigma_j'+\lambda_j'\frac{b_k'}{\beta_k'}\right]>0, \text{ and}$$

$$\operatorname{Re}\left(W_j\right)>0 \text{ or } \operatorname{Re}\left(W_j\right)=0,\ \min_{1\le k\le M}\operatorname{Re}\left[1+\rho_j+\sigma_j+V_j+\eta_j\frac{b_k}{\beta_k}\right]>0. \qquad \ldots(2.5)$$

**Result 2**

$$I_{x;U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left\{I_{y;U',V';Z'}^{\rho',\sigma';e',f';\eta',\lambda'}\left[f\left(t_1,\ldots,t_s\right)\right]\right\}$$

$$=\left(\prod_{j=1}^{s}x_j^{-\rho_j'-1}\right)\int_0^{x_1}\cdots\int_0^{x_s}\left(\prod_{j=1}^{s}t_j^{\rho_j'}\right)G\left(\frac{t_1}{x_1},\ldots,\frac{t_s}{x_s}\right)f\left(t_1,\ldots,t_s\right)dt_1\ldots dt_s, \qquad \ldots(2.6)$$

where

$$\left. \begin{matrix} (a_k,\alpha_k)_{N+1,P}\left(-\rho_j+\rho_j'+\sigma_j'-e_jR_j+\left(e_j'+f_j'\right)R_j'+\left(\eta_j'+\lambda_j'\right)S_{P',\upsilon},\eta_j\right) \\ (b_k,\beta_k;B_k)_{M+1,Q}\left(1-l-\sigma_j-\sigma_j-f_jR_j-f_j'R_j'-\lambda_j'S_{P',\upsilon},\lambda_j;1\right) \end{matrix} \right] , \quad \dots (2.7)$$

where $\bar{\theta}(S_{P,\upsilon})$ and $S_{P,\upsilon}$ are given by (1.13) and the following conditions are satisfied:

$$\min_{1\le k\le M} \mathrm{Re}\left[1+\rho_j+\eta_j\frac{b_k}{\beta_k}\right]>0, \ \min_{1\le k\le M'} \mathrm{Re}\left[1+\rho_j'+U_j+\eta_j'\frac{b_k}{\beta_k}\right]>0$$

$$, \min_{1\le k\le M} \mathrm{Re}\left[\sigma_j+\lambda_j\frac{b_k}{\beta_k}\right]>0, \ \min_{1\le k\le M'} \mathrm{Re}\left[\sigma_j'+\lambda_j'\frac{b_k'}{\beta_k'}\right]>0 \quad \dots (2.8)$$

**Result 3**

$$J^{\rho',\sigma';e',f';\eta',\lambda'}_{x;U',V';Z'}\left\{J^{\rho,\sigma;e,f;\eta,\lambda}_{y;U,V;Z}\left[f\left(t_1,\dots,t_s\right)\right]\right\}$$

$$=\left(\prod_{j=1}^{s}x_j^{\rho_j'}\right)\int_{x_1}^{\infty}\dots\int_{x_s}^{\infty}\left(\prod_{j=1}^{s}t_j^{-\rho_j'-1}\right)G\left(\frac{x_1}{t_1},\dots,\frac{x_s}{t_s}\right)f\left(t_1,\dots,t_s\right)dt_1\dots dt_s, \quad \dots (2.9)$$

where $G\left(t_1,\dots,t_s\right)$ is given by (2.7), $f\left(t_1,\dots,t_s\right)\in A$, the composite operator defined by the L.H.S. of (2.9) exists and the following conditions are satisfied:

$$\mathrm{Re}\left(W_j\right)>0 \text{ or } \mathrm{Re}\left(W_j\right)=0, \ \min_{1\le k\le M} \mathrm{Re}\left[1+\rho_j+\sigma_j+V_j+\eta_j\frac{b_k}{\beta_k}\right]>0,$$

$$\min_{1\le k\le M} \mathrm{Re}\left[\sigma_j+\lambda_j\frac{b_k}{\beta_k}\right]>0, \ \min_{1\le k\le M'} \mathrm{Re}\left[\sigma_j'+\lambda_j'\frac{b_k'}{\beta_k'}\right]>0 \quad \dots (2.10)$$

**Proof**: To prove result 1, we first express both the I- and J-operators involved in the L.H.S. of (2.1), in the integral form with the help of (1.1) and (1.2). Then we interchange the order of $t_j$ - and $y_j$ - integrals (which is permissible under the conditions stated) and get the following integral:

$$I^{\rho,\sigma;e,f;\eta,\lambda}_{x;U,V;Z}\left\{J^{\rho',\sigma';e',f';\eta',\lambda'}_{y;U',V';Z'}\left[f\left(t_1,\dots,t_s\right)\right]\right\}$$

$$= \int_0^{x_1}\cdots\int_0^{x_s}\left\{\int_0^{t_1}\cdots\int_0^{t_s}\Omega dy_1...dy_s\right\}f(t_1,...,t_s)dt_1...dt_s + \int_{x_1}^{\infty}\cdots\int_{x_s}^{\infty}\left\{\int_0^{x_1}\cdots\int_0^{x_s}\Omega dy_1...dy_s\right\}f(t_1,...,t_s)dt_1...dt_s$$

$$= \int_0^{x_1}\cdots\int_0^{x_s} I_1 f(t_1,...,t_s)dt_1...dt_s + \int_{x_1}^{\infty}\cdots\int_{x_s}^{\infty} I_2 f(t_1,...,t_s)dt_1...dt_s \quad , \qquad ...(2.11)$$

where

$$\Omega = \left(\prod_{j=1}^{s} x_j^{-\rho_j-\sigma_j} t_j^{-\rho_j'-\sigma_j'} y_j^{\rho_j+\rho_j'}\left(x_j-y_j\right)^{\sigma_j-1}\left(t_j-y_j\right)^{\sigma_j'-1}\right)$$

$$S_V^{U_1,...,U_s}\left[E_1\left(\frac{y_1}{x_1}\right)^{e_1}\left(1-\frac{y_1}{x_1}\right)^{f_1},...,E_s\left(\frac{y_s}{x_s}\right)^{e_s}\left(1-\frac{y_s}{x_s}\right)^{f_s}\right]S_{V'}^{U_1',...,U_s'}\left[E_1'\left(\frac{y_1}{t_1}\right)^{e_1'}\left(1-\frac{y_1}{t_1}\right)^{f_1'},...,E_s'\left(\frac{y_s}{t_s}\right)^{e_s'}\left(1-\frac{y_s}{t_s}\right)^{f_s'}\right]$$

$$\overline{H}_{P,Q}^{M,N}\left[z\prod_{j=1}^{s}\left(\frac{y_j}{x_j}\right)^{\eta_j}\left(1-\frac{y_j}{x_j}\right)^{\lambda_j}\right]\overline{H}_{P',Q'}^{M',N'}\left[z'\prod_{j=1}^{s}\left(\frac{y_j}{t_j}\right)^{\eta_j'}\left(1-\frac{y_j}{t_j}\right)^{\lambda_j'}\right]dy_1...dy_s$$

To evaluate $I_1$, involved in the first integral on the R.H.S. of (2.11), we express both the multivariable polynomials $S_V^{U_1,...,U_s}$ , $S_{V'}^{U_1',...,U_s'}$ and $\overline{H}_{P',Q'}^{M',N'}$ in terms of their respective series with the help of equations (1.5) and (1.12) respectively, $\overline{H}_{P,Q}^{M,N}$ is expressed in terms of the Mellin Barne's contour integral with the help of (1.6). Now interchanging the order of summations and Mellin Barne's contour integral with - integral and further, evaluating the -integral by setting in (2.12) and using the known result [13, p. 47, Th. 1.6], we get the following form after a little simplification:

$$I_1 = \sum_{R_1,...,R_s=0}^{\sum_{j=1}^{s}U_jR_j\le V}(-V)_{\sum_{j=1}^{s}U_jR_j}A(V,R_1,...,R_s)\frac{E_1^{R_1}}{R_1!}\cdots\frac{E_s^{R_s}}{R_s!}\sum_{R_1',...,R_s'=0}^{\sum_{j=1}^{s}U_j'R_j'\le V'}(-V')_{\sum_{j=1}^{s}U_j'R_j'}A(V',R_1',...,R_s')$$

$$\frac{E_1'^{R_1'}}{R_1'!}\cdots\frac{E_s'^{R_s'}}{R_s'!}\sum_{\upsilon=1}^{M'}\sum_{P'=0}^{\infty}\overline{\theta}\left(S_{P',\upsilon}\right)Z'^{S_{P\upsilon}}\frac{1}{2\pi i}\int_L\overline{\phi}(\xi)Z^{\xi}d\xi\, t_j^{\rho_j+e_jR_j+\eta_j\xi}x_j^{-\rho_j-e_jR_j-\eta_j\xi-1}$$

$$\frac{\Gamma\left(\rho_j+\rho_j'+e_jR_j+e_j'R_j'+\eta_j'S_{P',\upsilon}+\eta_j\xi+1\right)\Gamma\left(\sigma_j'+f_j'R_j'+\lambda_j'S_{P',\upsilon}\right)}{\Gamma\left(\rho_j+\rho_j'+\sigma_j'+e_jR_j+\left(e_j'+f_j'\right)R_j'+\left(\eta_j'+\lambda_j'\right)S_{P',\upsilon}+1\right)}$$

$${}_2F_1\left[\begin{matrix}1-\sigma_j+f_jR_j+\lambda_j\xi,1+\rho_j+\rho_j'+e_jR_j+e_j'R_j'+\eta_j'S_{P',\upsilon}+\eta_j\xi\\1+\rho_j+\rho_j'+\sigma_j'+e_jR_j+\left(e_j'+f_j'\right)R_j'+\left(\eta_j'+\lambda_j'\right)S_{P',\upsilon}+\eta_j\xi\end{matrix};\frac{t}{x}\right].$$

Now applying the transformation formula [13, p. 60, eq. (5)], expressing the thus obtained in the series form and re-interpreting the result in terms of $\overline{H}$ function we get the solution of $I_1$.

To calculate $I_2$, we proceed on similar lines, with the difference that the $\overline{H}_{P,Q}^{M,N}$ - function is now expressed in series and another one is expressed in terms of the Mellin Barne's contour integral.

Results 2 and 3 can be proved similarly by setting $\frac{x_j - y_j}{x_j - t_j} = u_j$ and using the known result [5, p. 287, eq. 3.197(8)].

## 3. SPECIAL CASES

We now presenting a two dimensional analogue of our second composition formula by taking s=2 after reducing the generalized class of polynomials to unity:

$$I_{x.y;Z}^{\rho,\sigma,\eta,\lambda}\left\{I_{s,t;Z'}^{\rho',\sigma';\eta',\lambda'}\left[f(u,v)\right]\right\} = I_{x.y;Z}^{\rho,\sigma,\eta,\lambda}\left\{s^{-\rho'-\sigma'}t^{-m'-n'}\int_0^s\int_0^t u^{\rho'}v^{m'}(s-u)^{\sigma'-1}(t-v)^{n'-1}\right.$$

$$\left. .\overline{H}_{P',Q'}^{M',N'}\left[z'\left(\frac{u}{s}\right)^{\eta'}\left(\frac{v}{t}\right)^{\delta'}\left(1-\frac{u}{s}\right)^{\lambda'}\left(1-\frac{v}{t}\right)^{\mu'}\right]f(u,v)\,du\,dv\right\}$$

$$=\sum_{\upsilon=1}^{M}\sum_{P'=0}^{\infty}\bar{\theta}\left(S_{P',\upsilon}\right)Z'^{S_{P,\upsilon}}\,x^{-\rho-\sigma-\rho'-\sigma'-k-\left(\eta'+\lambda'\right)S_{P',\upsilon}}\,y^{-m-n-m'-n'-l-\left(\delta'+\mu'\right)S_{P',\upsilon}}\left(\sigma'+\lambda'S_{P',\upsilon}\right)$$

$$\Gamma\left(\eta'+\mu'S_{P',\upsilon}\right)\int_0^x\int_0^y(x-u)^{\sigma+\sigma'+k+\lambda'S_{P',\upsilon}-1}(y-v)^{\eta+\eta'+l+\mu'S_{P',\upsilon}-1}\,u^{\rho'+\eta'S_{P',\upsilon}}\,v^{m'+\delta'S_{P',\upsilon}}$$

$$.\overline{H}_{P+4,Q+4}^{M+2,N+2}\left[z\left(1-\frac{u}{x}\right)^{\lambda}\left(1-\frac{v}{y}\right)^{\mu}\middle|\begin{matrix}A^{**}\\B^{**}\end{matrix}\right]f(u,v)\,du\,dv\quad, \qquad \ldots(3.1)$$

where

$$A^{*} = (a_j,\alpha_j;A_j)_{1,N}\left(1-\sigma-k,\lambda;1\right)\left(1-\eta-l,\mu;1\right)(a_j,\alpha_j)_{N+1,P}$$

$$\left(\rho'+\sigma'+\left(\eta'+\lambda'\right)S_{P',\upsilon},\eta\right)\left(m'+n'+\left(\delta'+\mu'\right)S_{P',\upsilon},\delta\right)$$

$$B^{*} = (b_j,\beta_j)_{1,M}\left(\rho^{'}+\sigma^{'}+\left(\eta^{'}+\lambda^{'}\right)S_{P^{'},\upsilon}+k,\eta\right)\left(m^{'}+n^{'}+\left(\delta^{'}+\mu^{'}\right)S_{P^{'},\upsilon}+l,\delta\right)$$

$$(b_j,\beta_j;B_j)_{M+1,Q}\left(1-\sigma-\sigma^{'}-k-\lambda^{'}S_{P^{'},\upsilon},\lambda;1\right)\left(1-\eta-\eta^{'}-l-\mu^{'}S_{P^{'},\upsilon},\mu;1\right),$$

and the following conditions are satisfied:

$$\min_{1\le k\le M}\operatorname{Re}\left[1+\rho+m+(\eta+\delta)\left(\frac{b_k}{\beta_k}\right)\right]>0,\ \min_{1\le k\le M}\operatorname{Re}\left[\sigma+\eta+(\lambda+\mu)\left(\frac{b_k}{\beta_k}\right)\right]>0,$$

$$\min_{1\le k\le M^{'}}\operatorname{Re}\left[1+\rho^{'}+m^{'}+U_1+U_2+\left(\eta^{'}+\delta^{'}\right)\left(\frac{b_k^{'}}{\beta_k^{'}}\right)\right]>0,\ \min_{1\le k\le M^{'}}\operatorname{Re}\left[\sigma^{'}+\eta^{'}+\left(\lambda^{'}+\mu^{'}\right)\left(\frac{b_k^{'}}{\beta_k^{'}}\right)\right]>0.$$

Similar two dimensional formulae can be easily obtained from results 1 and 3. These formulae can be further reduced to results given by Raina [12, pp. 511-513, eqs. (2.8), (2.9) and (2.15)] by taking $\overline{H}$ function to unity.

If in these composition formula we reduce both the generalized class of polynomials, function to unity, we arrive at the multidimensional analogue of the results given by Erdélyi [2, p. 166, eq. (6.2); p. 167, eq. (6.3)]. Again reducing the generalized class of polynomials to unity and the function to the generalized hypergeometric function, we arrive at the corresponding result given by Goyal and Jain [3, p. 253, eq. (2.4); p. 254, eq. (2.7); p.255, eq. (2.12)] after a little simplification. Further, if we reduce generalized class of polynomials to polynomials we arrive at the result which are in essence the same as those obtained by Goyal, Jain and Gaur [4, pp. 404-405, eq. (2.1); p. 406, eq. (2.7); pp. 407-403, eq. (2.12)].

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# UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATION IN RAYLEIGH DISTRIBUTION

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## ABSTRACT

*In this paper, the uniformly minimum variance unbiased estimator(umvue) of the probability density function(pdf) of Rayleigh distribution is obtained .This umvue is used to obtain the umvue of the moments and reliability function of the Rayleigh distribution.*

**Keywords:** Uniformly minimum variance unbiased estimator(umvue), reliability function,complete sufficient statistic ,moments.

**Mathematics Subject Classification (2000) : 62F10**

## 1. INTRODUCTION

The Rayleigh distribution is very useful distribution in reliability and life testing experiments. In actuarial science this distribution is used as a loss distribution. Siddiqui [8] discussed the origin and properties of Rayleigh distribution. Polvoko[6] and Dyer and whisenand [4] discussed the importance of this distribution in electro vaccum devices and communication engineering.

The umvue of the probability densities has been discussed by many author.Singh [7] has obtained the umvue of some unvariate probability densities. Asrabadi [1] has considered the umvue of the pdf of the pareto distribution. Ghosh and Dutta [5] have obtained the umvue of the probability densities of serveral discrete distribution whose support depends on the parameter.

In this paper,we obtain the umvue of the pdf of the Rayleigh distribution. In section 2,we discuss the method and obtain the umvue of the pdf of the Rayleigh distribution. Finally in section 3. using this umvue, we obtain the umvue of the moments and reliability function of the Rayleigh distribution.

## 2. THE METHOD

Let $X_{1,}X_{2,}..............X_n$ be a random sample from a probability distribution with pdf $f(x,\theta)$, $\theta \in \Omega$ (parameter space) and $x \in S$ (sample space). For every fixed , can be considered as a function of the parameter and our problem is to obtain the umvue of for every fixed . To this end we proceed as below:

Let be complete sufficient statistic for the parameter . The conditional pdf of given , denoted by , does not depend upon . Hence is a statistic and can be used to estimate . Note that

$$E_T[g(x\,|\,t)] = \int_t g(x\,|\,t)\ h(t,\theta)dt$$

$$= \int_t k(x,t,\theta)dt$$

$$= f(x,\theta). \tag{2.1}$$

Where $k(x,t,\theta)$ and $h(t,\theta)$ denote respectively the joint pdf of $X_1$ and $T(X)$ and the marginal pdf of $T(X)$. Since $g(x\,|\,t)$ is a function of and is unbiased for , hence is the umvue of by Rao-Blackwell and Lehmann-scheffe theorem.

### 2.1 THE UMVUE

The Rayleigh distribution considered here has the pdf

$$f(x,\theta) = \begin{cases} \dfrac{x}{\theta^2}\exp(-\dfrac{x^2}{2\theta^2}) & if \quad x > 0, \theta > 0 \\ 0 & Otherwiswe \end{cases} \tag{2.2}$$

If $X_{1,}X_{2,}..............X_n$ is a random sample from $f(x,\theta)$ in (2.2), then $T(X) = \sum_{i=1}^{n} X_i^{\,2}$ is complete sufficient statistic for $\theta$. The pdf $h(t,\theta)$ of $T(X) =$ is given by

$$h(t,\theta) = \begin{cases} \dfrac{1}{\Gamma n\ (2\theta^2)^n}\ e^{-\frac{t}{2\theta^2}}\, t^{n-1} & if \quad t > 0, \quad \theta > 0 \\ 0 & otherwiswe \end{cases} \tag{2.3}$$

Where is $\Gamma.$ gamma function.

The conditional pdf g(x|t) of $X_1$ given T(X) = t is given by

$$g(x \mid t) = \frac{k(x,t,\theta)}{h(t,\theta)} \tag{2.4}$$

where $k(x,t,\theta)$ and $h(t,\theta)$ are respectively the joint pdf of $X_1$ at x and T(X)= $\sum_{i=1}^{n} X_i^{\,2}$ at t and marginal pdf of $T(X)$ at $t$.Thus.

$$g(x|t) = \frac{p(t \mid x) f(x,\theta)}{h(t,\theta)} \tag{2.5}$$

where $p(t \mid x)$ is the conditional pdf of $T(X) = \sum X_i^{\,2}$ at t given at x.

Suppose, then

$$\begin{aligned} T(X) &= \sum_{i=1}^{n} X_i^{\,2} \\ &= Y + X_1^{\,2} \end{aligned} \tag{2.6}$$

Since Y is independent of $X_1$,the conditional pdf p(t|x) is the unconditional pdf of Y evaluated at $(t-x^2)$. We also Note that Y follows a gamma distribution with parameter (n-1) and $(2\theta^2)$. Now we can show that

$$g(x \mid t) = \begin{cases} 2(n-1)\dfrac{x}{t}(1-\dfrac{x^2}{t})^{n-2} & if \quad 0 < x < \sqrt{t} \\ 0 & otherwiswe \end{cases}$$

Thus we have the following

Theorem 2.1 The umvue of the pdf in (2.2) is given by

$$g(x \mid t) = \begin{cases} 2(n-1)\dfrac{x}{t}(1-\dfrac{x^2}{t})^{n-2} & if \quad 0 < x < \sqrt{t} \\ 0 & otherwiswe \end{cases}$$

## 3. THE UMVUE OF MOMENTS AND RELIABLITY FUNCTION

Note that the $r^{th}$ moment of the Rayleigh distribution with pdf $f(x,\theta)$ in (2.2) is given by

$$E[X^r] = \int_0^\infty x^r f(x,\theta)\,dx$$

$$= \int_0^\infty x^r \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) dx$$

$$= \Gamma\left(\frac{r}{2}+1\right) (2\theta^2)^{\frac{r}{2}} \tag{3.1}$$

TTo obtain the umvue of $E[X^r]$, we compute,

$$E[X^r \mid T(x)=t] = \int_0^{\sqrt{t}} x^r g(x \mid t)\,dx$$

$$= 2(n-1) \int_0^{\sqrt{t}} x^r \frac{x}{t} \left(1-\frac{x^2}{t}\right)^{n-2} dx$$

Using the transformation

$\frac{x^2}{t} = u$ ,we have

$$E[X^r \mid T(X)=t] = (n-1)\; t^{+\frac{r}{2}} B\left(\frac{r}{2}+1, n-1\right) \tag{3.2}$$

Where

$$B(m,n) = \frac{\Gamma m \; \Gamma n}{\Gamma m+n}$$

We also note that

$$E[E\{X^r \mid T(X)=t\}] = E[X^r]$$

Since $E[E\{X^r \mid T(X)=t\}]$ is a function of $T(X)$ and is unbiased for $E[X^r]$,hence $E[X^r \mid T(X)=t]$ in (3.2) is the the unique umvue of $E[X^r]$.

Further, the reliability function of the Rayleigh distribution is given by

$$R(m) = P[X \geq m]\,, \qquad (m \geq 0)$$

$$= \int_m^\infty f(x,\theta)dx$$

$$= \int_m^\infty \frac{x}{\theta^2} \exp(-\frac{x^2}{2\theta^2})dx$$

$$= \exp(-\frac{m^2}{2\theta^2}) \tag{3.3}$$

The umvue of $R(m)$ is given by

$$\hat{R}(m) = E[U(X_1) \mid T(X) = t] \tag{3.4}$$

where $U(X_1) = \begin{cases} 1 & if \quad X_1 \geq m \\ 0 & Otherwiswe \end{cases}$

Thus

$$\hat{R}(m) = P[X_1 \geq m \mid T(X) = t]$$

$$= \int_m^{\sqrt{t}} g(x \mid t)dx$$

$$= 2(n-1)\int_m^{\sqrt{t}} \frac{x}{t}(1-\frac{x^2}{t})^{n-2} dx$$

$$= \begin{cases} (1 - \frac{m^2}{t})^{n-1} & if \quad 0 < m < \sqrt{t} \\ 0 & otherwiswe \end{cases}$$

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