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TRUNCATED UNILATERAL HYPERGEOMETRIC SERIES INVOLVING NEGATIVE UNIT ARGUMENT

M. I. QURESHI[1] and KALEEM A. QURAISHI[2]

[1]Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia(A Central University), New Delhi-110025(India)
[2]Mathematics Section, Mewat Engineering College (Wakf), Palla, Nuh, Mewat-122107, Haryana(India)
E-mails: miqureshi_delhi@yahoo.co.in; kaleemspn@yahoo.co.in

ABSTRACT

In this paper we obtain some summation theorems for truncated unilateral generalized hypergeometric series associated with negative unit argument given by

$${}_{B+1}F_B[c_0,(g_B);1+(h_B);-1]_{2M-\varepsilon},\ {}_{B+2}F_{B+1}[c_0,(g_B),1-\nu;1+(h_B),-\nu;-1]_{2M-\varepsilon},$$
$${}_{B+3}F_{B+2}[c_0,(g_B),1-\mu,1-\varsigma;1+(h_B),-\mu,-\varsigma;-1]_{2M-\varepsilon},$$
$${}_{B+4}F_{B+3}[c_0,(g_B),1-\sigma,1-\omega,1-\xi;1+(h_B),-\sigma,-\omega,-\xi;-1]_{2M-\varepsilon},$$
$${}_{B+K+1}F_{B+K}[c_0,(g_B),1-(\delta_K);1+(h_B),-(\delta_K);-1]_{2M-\varepsilon}$$

and
$${}_{2B+1}F_{2B}[c_0,(g_B),1-(\rho_B);1+(h_B),-(\rho_B);-1]_{2M-\varepsilon},$$

using series iteration techniques; where $\nu,\mu,\varsigma,\sigma,\omega,\xi,\delta_K$ and ρ_B are the functions of parameters $c_0, g_1, g_2,\dots, g_B, h_1, h_2,\dots, h_B$. Applying Rainville's limit formula for certain infinite products, some non terminating hypergeometric summation theorems with negative unit argument are also deduced, in terms of Gamma functions subject to certain conditions. The results presented here are presumably new.

Keywords and Phrases: Pochhammer symbol; Gaussian ordinary hypergeometric function; Gamma function; Rainville's limit formula; Truncated unilateral and non terminating series

2010 AMS Subject Classifications: 33-Special Functions, Primary 33C99; Secondary 33C20

1.0 INTRODUCTION

Truncated Unilateral Generalized Hypergeometric Series

$$\text{to } (N+1)\text{ terms} = {}_AF_B\left[\begin{matrix}(a_A);\\(b_B);\end{matrix}\ z\right]_N = \sum_{k=0}^{N}\frac{\prod_{j=1}^{A}(a_j)_k\, z^k}{\prod_{j=1}^{B}(b_j)_k\, k!} \quad \dots(1.1)$$

where numerator and denominator parameters are neither zero nor negative integers and A, B are non-negative integers. When $N\to\infty$ then (1.1) reduces to non-terminating generalized hypergeometric series and Pochhammer's symbol $(c)_k$ is given by $(c)_k=\prod_{j=0}^{k-1}(c+j)$.

Rainville's Limit Formula for Certain Infinite Products

If
$$U_n=\frac{(n+a_1)(n+a_2)\cdots(n+a_k)}{(n+b_1)(n+b_2)\cdots(n+b_\ell)}$$

then product $\prod U_n$ can only converge if $k = \ell$ and $\sum a_i = \sum b_i$. When these conditions are satisfied, we can express the infinite product in terms of Gamma functions [2,p.115(Q.No.11)]. Now limit formula for certain infinite products can be written in the following form, if

$$(1+a_1)+(1+a_2)+(1+a_3)+\cdots+(1+a_s)=(1+b_1)+(1+b_2)+(1+b_3)+\cdots+(1+b_s) \qquad \ldots(1.2)$$

and no a_s or b_s is a negative integer, then without any loss of absolute convergence, we have the following theorem [2,p.128(Q.No.1);3,pp.6-7(1.3.8);5,pp.14-15(Th.5)].

$$\prod_{n=1}^{\infty}\left\{\frac{(n+a_1)(n+a_2)\cdots(n+a_s)}{(n+b_1)(n+b_2)\cdots(n+b_s)}\right\}=\lim_{k\to\infty}\left\{\frac{(1+a_1)_k(1+a_2)_k\cdots(1+a_s)_k}{(1+b_1)_k(1+b_2)_k\cdots(1+b_s)_k}\right\} \qquad \ldots(1.3)$$

$$=\frac{\Gamma(1+b_1)\Gamma(1+b_2)\cdots\Gamma(1+b_s)}{\Gamma(1+a_1)\Gamma(1+a_2)\cdots\Gamma(1+a_s)} \qquad \ldots(1.4)$$

If condition (1.2) is not true, then product in (1.3) diverges.

In our analysis, the symbol $S_r(g_1, g_2, \ldots, g_B)$ represents the sum of all possible combinations of the products of parameters taken "r" at a time from the set of "B" parameters $\{g_1, g_2, \ldots, g_B\}$.

We shall discuss the applications of summation theorems of Slater, Verma, Qureshi and Quraishi with positive unit argument, for truncated unilateral hypergeometric series involving negative unit argument in next sections.

Since Pochhammer's symbol is associated with Gamma function and Gamma function is undefined for zero and negative integers therefore numerator and denominator parameters are adjusted in such a way that each term of following results is completely well defined and meaningful then without any loss of convergence, we have the following theorems.

2.0 COMPANION OF SLATER-THEOREM

$${}_{B+1}F_B\left[\begin{matrix} c_0, (g_B) ; \\ 1+(h_B) ; \end{matrix} \; -1\right]_{2M-\varepsilon} = T_1 - \frac{c_0\prod_{j=1}^{B}(g_j)}{\prod_{j=1}^{B}(1+h_j)}T_2 \qquad \ldots(2.1)$$

where

$$T_1=\frac{\left(\frac{2+c_0}{2}\right)_{M-\varepsilon}\left(\frac{3+c_0}{2}\right)_{M-\varepsilon}\prod_{j=1}^{B}\left\{\left(\frac{2+g_j}{2}\right)_{M-\varepsilon}\left(\frac{3+g_j}{2}\right)_{M-\varepsilon}\right\}}{(M-\varepsilon)!\left(\frac{1}{2}\right)_{M-\varepsilon}\prod_{j=1}^{B}\left\{\left(\frac{1+h_j}{2}\right)_{M-\varepsilon}\left(\frac{2+h_j}{2}\right)_{M-\varepsilon}\right\}} \qquad \ldots(2.2)$$

$$T_2=\frac{\left(\frac{3+c_0}{2}\right)_{M-1}\left(\frac{4+c_0}{2}\right)_{M-1}\prod_{j=1}^{B}\left\{\left(\frac{3+g_j}{2}\right)_{M-1}\left(\frac{4+g_j}{2}\right)_{M-1}\right\}}{(M-1)!\left(\frac{3}{2}\right)_{M-1}\prod_{j=1}^{B}\left\{\left(\frac{2+h_j}{2}\right)_{M-1}\left(\frac{3+h_j}{2}\right)_{M-1}\right\}} \qquad \ldots(2.3)$$

subject to the following conditions, given by

$$S_r(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) = S_r(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B) \quad \ldots(2.4)$$

$$S_{2B+2}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) \neq 0 \quad \ldots(2.5)$$

when $r = 1,2,3,\ldots,(2B+1)$; $\varepsilon \in \{0,1\}, B \in \{1,2,3,\ldots\}$ and $M \in \{2,3,4,\ldots\}$. ...(2.6)

Proof of (2.1)

Consider the following series identities:

$$\sum_{i=0}^{2M} \Phi(i) = \sum_{i=0}^{M} \Phi(2i) + \sum_{i=0}^{M-1} \Phi(2i+1) \quad \ldots(2.7)$$

$$\sum_{i=0}^{2M-1} \Phi(i) = \sum_{i=0}^{M-1} \Phi(2i) + \sum_{i=0}^{M-1} \Phi(2i+1) \quad \ldots(2.8)$$

where $M \in \{2,3,4,\ldots\}$.

The finite series identities (2.7) and (2.8) can be unified in the following form

$$\sum_{i=0}^{2M-\varepsilon} \Phi(i) = \sum_{i=0}^{M-\varepsilon} \Phi(2i) + \sum_{i=0}^{M-1} \Phi(2i+1) \quad \ldots(2.9)$$

where $\varepsilon \in \{0,1\}$ and $M \in \{2,3,4,\ldots\}$.

Suppose left hand side of (2.1) is denoted by S, then

$$S = \sum_{i=0}^{2M-\varepsilon} \frac{(c_0)_i (g_1)_i (g_2)_i \cdots (g_B)_i (-1)^i}{i!\,(1+h_1)_i (1+h_2)_i \cdots (1+h_B)_i} \quad \ldots(2.10)$$

Applying the finite series identity (2.9) in the right hand side of (2.10), we get

$$S = \sum_{i=0}^{M-\varepsilon} \frac{(c_0)_{2i} (g_1)_{2i} (g_2)_{2i} \cdots (g_B)_{2i} (-1)^{2i}}{(2i)!\,(1+h_1)_{2i} (1+h_2)_{2i} \cdots (1+h_B)_{2i}} + \sum_{i=0}^{M-1} \frac{(c_0)_{2i+1} (g_1)_{2i+1} (g_2)_{2i+1} \cdots (g_B)_{2i+1} (-1)^{2i+1}}{(2i+1)!\,(1+h_1)_{2i+1} (1+h_2)_{2i+1} \cdots (1+h_B)_{2i+1}} \quad \ldots(2.11)$$

Now write finite power series of (2.11) in truncated hypergeometric notation, we have

$$S = {}_{2B+2}F_{2B+1}\left[\begin{matrix} \frac{c_0}{2}, \frac{1+c_0}{2}, \frac{(g_B)}{2}, \frac{1+(g_B)}{2} ; \\ \frac{1}{2}, \frac{1+(h_B)}{2}, \frac{2+(h_B)}{2} ; \end{matrix} \; 1 \right]_{M-\varepsilon} -$$

$$- \frac{c_0 \prod_{j=1}^{B} (g_j)}{\prod_{j=1}^{B} (1+h_j)} \, {}_{2B+2}F_{2B+1}\left[\begin{matrix} \frac{1+c_0}{2}, \frac{2+c_0}{2}, \frac{1+(g_B)}{2}, \frac{2+(g_B)}{2} ; \\ \frac{3}{2}, \frac{2+(h_B)}{2}, \frac{3+(h_B)}{2} ; \end{matrix} \; 1 \right]_{M-1} \quad \ldots(2.12)$$

In (2.12) apply Slater's theorem[1,p.18(4.10,4.11);6,pp.83-84(2.6.1.1, 2.6.1.7);7,p.233(3.1); see also 4,equations (3.5.1)-(3.5.3)], we get the right hand side of (2.1).

Deduction of (2.1)

Since in the right hand side of (2.1), (1.2) type condition associated with (1.3) type products in (2.2) and (2.3) are satisfied, hence we can take the limit $M \to \infty$ in (2.1)

$$ {}_{B+1}F_B\left[\begin{matrix} c_0,\ (g_B)\ ; \\ 1+(h_B)\ ; \end{matrix}\ -1\right] = T_3 - \frac{c_0 \prod_{j=1}^{B}(g_j)}{\prod_{j=1}^{B}(1+h_j)} T_4 \qquad ...(2.13) $$

where

$$ T_3 = \frac{\sqrt{\pi}\ \prod_{j=1}^{B}\left\{\Gamma\left(\frac{1+h_j}{2}\right)\Gamma\left(\frac{2+h_j}{2}\right)\right\}}{\Gamma\left(\frac{2+c_0}{2}\right)\Gamma\left(\frac{3+c_0}{2}\right)\prod_{j=1}^{B}\left\{\Gamma\left(\frac{2+g_j}{2}\right)\Gamma\left(\frac{3+g_j}{2}\right)\right\}} \qquad ...(2.14) $$

$$ T_4 = \frac{\sqrt{\pi}\ \prod_{j=1}^{B}\left\{\Gamma\left(\frac{2+h_j}{2}\right)\Gamma\left(\frac{3+h_j}{2}\right)\right\}}{2\Gamma\left(\frac{3+c_0}{2}\right)\Gamma\left(\frac{4+c_0}{2}\right)\prod_{j=1}^{B}\left\{\Gamma\left(\frac{3+g_j}{2}\right)\Gamma\left(\frac{4+g_j}{2}\right)\right\}} \qquad ...(2.15) $$

subject to the conditions (2.4)-(2.6)

3.0 COMPANION OF VERMA-THEOREM

If we proceed on the same parallel lines of preceding section and apply Verma theorem [7,p.233(3.3); 1,p.19(4.12); see also 4,equations (3.2.1)-(3.2.4)], we obtain

$$ {}_{B+2}F_{B+1}\left[\begin{matrix} c_0,\ (g_B), 1-\nu\ ; \\ 1+(h_B), -\nu\ \ ; \end{matrix}\ -1\right]_{2M-\varepsilon} = T_1 - \frac{(\nu-1)c_0 \prod_{j=1}^{B}(g_j)}{\nu \prod_{j=1}^{B}(1+h_j)} T_2 \qquad ...(3.1) $$

subject to the following conditions, given by

$$ S_r(c_0; 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) = S_r(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B) \qquad ...(3.2) $$

$$ S_{2B+1}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) \neq S_{2B+1}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B) \qquad ...(3.3) $$

$$ S_{2B+2}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) \neq 0 \qquad ...(3.4) $$

$$ \nu = \frac{-S_{2B+2}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B)}{\{S_{2B+1}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{2B+1}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}} \qquad ...(3.5) $$

where $r = 1, 2, 3, \ldots, (2B)$; $\varepsilon \in \{0, 1\}, B \in \{1, 2, 3, \ldots\}$ and $M \in \{2, 3, 4, \ldots\}$. ...(3.6)

and T_1, T_2 are given by (2.2), (2.3) respectively.

Deduction of (3.1)

Since in the right hand side of (3.1), (1.2) type condition associated with (1.3) type products in (2.2) and (2.3) are satisfied, hence we can take the limit $M \to \infty$ in (3.1)

$$ {}_{B+2}F_{B+1}\left[\begin{matrix} c_0,\,(g_B),1-\nu\ ; \\ 1+(h_B),-\nu\ \ ; \end{matrix}\ -1\right] = T_3 - \frac{(\nu-1)c_0\prod_{j=1}^{B}(g_j)}{\nu\prod_{j=1}^{B}(1+h_j)}T_4 \qquad ...(3.7) $$

subject to the conditions (3.2)-(3.6), where T_3 and T_4 are given by (2.14) and (2.15) respectively.

4.0 COMPANION OF FIRST THEOREM OF QURESHI AND QURAISHI

If we proceed on the same parallel lines of preceding sections and apply first theorem of authors [4,equations (3.3.1)-(3.3.6)], we obtain

$$ {}_{B+3}F_{B+2}\left[\begin{matrix} c_0,\,(g_B),1-\mu,1-\varsigma\ ; \\ 1+(h_B),-\mu,-\varsigma\ \ ; \end{matrix}\ -1\right]_{2M-\varepsilon} = T_1 - \frac{(\mu-1)(\varsigma-1)c_0\prod_{j=1}^{B}(g_j)}{\mu\varsigma\prod_{j=1}^{B}(1+h_j)}T_2 \qquad ...(4.1) $$

subject to the following conditions, given by

$$ S_r(c_0,\,1+c_0,g_1,1+g_1,\ldots,g_B,1+g_B) = S_r(-1,-1+h_1,h_1,\ldots,-1+h_B,h_B) \qquad ...(4.2) $$

$$ S_{2B}(c_0,\,1+c_0,g_1,1+g_1,\ldots,g_B,1+g_B) \neq S_{2B}(-1,-1+h_1,h_1,\ldots,-1+h_B,h_B) \qquad ...(4.3) $$

$$ S_{2B+2}(c_0,\,1+c_0,g_1,1+g_1,\ldots,g_B,1+g_B) \neq 0 \qquad ...(4.4) $$

$$ \mu = \frac{-S_{2B+1}(c_0,\,1+c_0,g_1,1+g_1,\ldots,g_B,1+g_B)+S_{2B+1}(-1,-1+h_1,h_1,\ldots,-1+h_B,h_B)+\sqrt{D}}{2\{S_{2B}(c_0,\,1+c_0,g_1,1+g_1,\ldots,g_B,1+g_B)-S_{2B}(-1,-1+h_1,h_1,\ldots,-1+h_B,h_B)\}} \qquad ...(4.5) $$

$$ \varsigma = \frac{-S_{2B+1}(c_0,\,1+c_0,g_1,1+g_1,\ldots,g_B,1+g_B)+S_{2B+1}(-1,-1+h_1,h_1,\ldots,-1+h_B,h_B)-\sqrt{D}}{2\{S_{2B}(c_0,\,1+c_0,g_1,1+g_1,\ldots,g_B,1+g_B)-S_{2B}(-1,-1+h_1,h_1,\ldots,-1+h_B,h_B)\}} \qquad ...(4.6) $$

$$ \begin{aligned} D = {} & \{S_{2B+1}(c_0,\,1+c_0,g_1,1+g_1,\ldots,g_B,1+g_B)-S_{2B+1}(-1,-1+h_1,h_1,\ldots,-1+h_B,h_B)\}^2 - \\ & -4\{S_{2B}(c_0,\,1+c_0,g_1,1+g_1,\ldots,g_B,1+g_B)-S_{2B}(-1,-1+h_1,h_1,\ldots,-1+h_B,h_B)\}\times \\ & \times\{S_{2B+2}(c_0,\,1+c_0,g_1,1+g_1,\ldots,g_B,1+g_B)\} \end{aligned} \qquad ...(4.7) $$

where $r = 1,2,3,\ldots,(2B-1);\ \varepsilon\in\{0,1\}, B\in\{1,2,3,\ldots\},\ M\in\{2,3,4,\ldots\}.$ ...(4.8)

and T_1, T_2 are given by (2.2), (2.3) respectively.

Deduction of (4.1)

When $M\to\infty$ in (4.1), we get

$$ {}_{B+3}F_{B+2}\left[\begin{matrix} c_0,\,(g_B),1-\mu,1-\varsigma\ ; \\ 1+(h_B),-\mu,-\varsigma\ \ ; \end{matrix}\ -1\right] = T_3 - \frac{(\mu-1)(\varsigma-1)c_0\prod_{j=1}^{B}(g_j)}{\mu\varsigma\prod_{j=1}^{B}(1+h_j)}T_4 \qquad (4.9) $$

subject to the conditions (4.2)-(4.8), where T_3 and T_4 are given by (2.14) and (2.15) respectively.

5.0 COMPANION OF SECOND THEOREM OF QURESHI AND QURAISHI

If we proceed on the same parallel lines of preceding sections and apply second theorem of authors[4,equations (3.4.1)-(3.4.8)], we can obtain

$$ {}_{B+4}F_{B+3}\left[\begin{matrix} c_0, (g_B), 1-\sigma, 1-\omega, 1-\xi \; ; \\ 1+(h_B), -\sigma, -\omega, -\xi \quad ; \end{matrix} \; -1\right]_{2M-\varepsilon} = T_1 - \frac{(\sigma-1)(\omega-1)(\xi-1)c_0\prod_{j=1}^{B}(g_j)}{\sigma\omega\xi\prod_{j=1}^{B}(1+h_j)}T_2 \qquad ...(5.1) $$

subject to the following conditions, given by

$$ S_r(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) = S_r(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B) \qquad ...(5.2) $$

$$ S_{2B-1}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) \neq S_{2B-1}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B) \qquad ...(5.3) $$

$$ S_{2B+2}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) \neq 0 \qquad ...(5.4) $$

where $\frac{\sigma}{2}, \frac{\omega}{2}, \frac{\xi}{2}$ are the roots of the following cubic equation

$$ \begin{aligned} &[\{S_{2B-1}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{2B-1}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m)^3 + \\ &+ \{S_{2B}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{2B}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m)^2 + \\ &+ \{S_{2B+1}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{2B+1}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m) + \\ &\qquad + \{S_{2B+2}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B)\}] = 0 \end{aligned} \qquad ...(5.5) $$

when $r = 1, 2, 3, \ldots, (2B-2); \varepsilon \in \{0,1\}, B, M \in \{2, 3, 4, \ldots\}$ ...(5.6)

and T_1, T_2 are given by (2.2), (2.3) respectively.

Deduction of (5.1)

When $M \to \infty$ in (5.1), we get

$$ {}_{B+4}F_{B+3}\left[\begin{matrix} c_0, (g_B), 1-\sigma, 1-\omega, 1-\xi \; ; \\ 1+(h_B), -\sigma, -\omega, -\xi \quad ; \end{matrix} \; -1\right] = T_3 - \frac{(\sigma-1)(\omega-1)(\xi-1)c_0\prod_{j=1}^{B}(g_j)}{\sigma\omega\xi\prod_{j=1}^{B}(1+h_j)}T_4 \qquad ...(5.7) $$

subject to the conditions (5.2)-(5.6), where T_3 and T_4 are given by (2.14) and (2.15) respectively.

6.0 COMPANION OF THIRD THEOREM OF QURESHI AND QURAISHI

If we apply third theorem of authors[4,equations (3.6.1)-(3.6.4)] and proceed on the same parallel lines of preceding sections, we can obtain

$$ {}_{B+K+1}F_{B+K}\left[\begin{matrix} c_0, (g_B), 1-(\delta_K) \; ; \\ 1+(h_B), -(\delta_K) \quad ; \end{matrix} \; -1\right]_{2M-\varepsilon} = T_1 - \frac{c_0\prod_{j=1}^{K}(\delta_j-1)\prod_{j=1}^{B}(g_j)}{\prod_{j=1}^{K}(\delta_j)\prod_{j=1}^{B}(1+h_j)}T_2 \qquad ...(6.1) $$

subject to the following conditions, given by

$$ S_r(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) = S_r(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B) \qquad ...(6.2) $$

$$ S_{2B-K+2}(c_0, 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) \neq S_{2B-K+2}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B) \qquad ...(6.3) $$

$$S_{2B+2}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) \neq 0 \qquad \ldots(6.4)$$

where $\frac{\delta_1}{2}, \frac{\delta_2}{2}, \cdots, \frac{\delta_K}{2}$ are the roots of the following equation

$$\begin{aligned}&[\{S_{2B-K+2}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{2B-K+2}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m)^K + \\ &+ \{S_{2B-K+3}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{2B-K+3}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m)^{K-1} + \\ &+ \cdots + \{S_{2B}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{2B}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m)^2 + \\ &+ \{S_{2B+1}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{2B+1}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m) + \\ &\qquad + \{S_{2B+2}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B)\}] = 0 \qquad \ldots(6.5)\end{aligned}$$

when $r = 1, 2, 3, \ldots, (2B-K+1)$; $\varepsilon \in \{0,1\}, B \in \{1,2,3,\ldots\}, M \in \{2,3,4,\ldots\}$, $K < 2B+1$ ...(6.6)
and T_1, T_2 are given by (2.2), (2.3) respectively.

Deduction of (6.1)

When $M \to \infty$ in (6.1), we get

$$_{B+K+1}F_{B+K}\left[\begin{matrix} c_0,\ (g_B), 1-(\delta_K)\ ; \\ 1+(h_B),\ -(\delta_K)\ \ ; \end{matrix}\ -1\right] = T_3 - \frac{c_0 \prod_{j=1}^{K}(\delta_j - 1) \prod_{j=1}^{B}(g_j)}{\prod_{j=1}^{K}(\delta_j) \prod_{j=1}^{B}(1+h_j)} T_4 \qquad \ldots(6.7)$$

subject to the conditions (6.2)-(6.6), where T_3 and T_4 are given by (2.14) and (2.15) respectively.

7.0 COMPANION OF FOURTH THEOREM OF QURESHI AND QURAISHI

If we apply fourth theorem of authors[4,equations (3.1.1),(3.1.5),(3.1.6),(3.6.1)-(3.6.4)] and proceed on the same parallel lines of preceding sections, we have

$$_{2B+1}F_{2B}\left[\begin{matrix} c_0,\ (g_B), 1-(\rho_B)\ ; \\ 1+(h_B),\ -(\rho_B)\ \ ; \end{matrix}\ -1\right]_{2M-\varepsilon} = T_1 - \frac{c_0 \prod_{j=1}^{B}\{(\rho_j - 1)(g_j)\}}{\prod_{j=1}^{B}\{(\rho_j)(1+h_j)\}} T_2 \qquad \ldots(7.1)$$

subject to the following conditions, given by

$$S_r(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) = S_r(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B) \qquad \ldots(7.2)$$

$$S_{B+2}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) \neq S_{B+2}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B) \qquad \ldots(7.3)$$

$$S_{2B+2}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) \neq 0 \qquad \ldots(7.4)$$

where $\frac{\rho_1}{2}, \frac{\rho_2}{2}, \cdots, \frac{\rho_B}{2}$ are the roots of the following equation

$$\begin{aligned}&[\{S_{B+2}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{B+2}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m)^B + \\ &+ \{S_{B+3}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{B+3}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m)^{B-1} + \\ &+ \cdots + \{S_{2B}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{2B}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m)^2 + \\ &+ \{S_{2B+1}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B) - S_{2B+1}(-1, -1+h_1, h_1, \ldots, -1+h_B, h_B)\}(2m) + \\ &\qquad + \{S_{2B+2}(c_0,\ 1+c_0, g_1, 1+g_1, \ldots, g_B, 1+g_B)\}] = 0 \qquad \ldots(7.5)\end{aligned}$$

when $r=1,2,3,\ldots,(B+1);\ \varepsilon\in\{0,1\}, B\in\{1,2,3,\ldots\}, M\in\{2,3,4,\ldots\}$ ...(7.6)
and T_1, T_2 are given by (2.2), (2.3) respectively.

Deduction of (7.1)

When $M\to\infty$ in (7.1), we get

$$ {}_{2B+1}F_{2B}\left[\begin{matrix} c_0,(g_B),1-(\rho_B)\ ; \\ 1+(h_B),-(\rho_B)\ ; \end{matrix}\ -1\right]=T_3-\frac{c_0\prod_{j=1}^{B}\{(\rho_j-1)(g_j)\}}{\prod_{j=1}^{B}\{(\rho_j)(1+h_j)\}}T_4 \qquad \ldots(7.7) $$

subject to the conditions (7.2)-(7.6), where T_3 and T_4 are given by (2.14) and (2.15) respectively.

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MUNICIPAL SOLID WASTE HANDLING MODEL

ARCHNA GUPTA[1] and D. C. SHARMA[2]

[1]*Department of Mathematics,Rajdhani Engineering College, Rajasthan*
[2]*School of Mathematics, Statistics & Computational Sciences*
Department of Mathematics, Central University of Rajasthan
E –mail: archanaguptamath@gmail.com ,dcsharmacuraj@gmail.com

ABSTRACT

The increasing level of solid waste is now a days, a serious problem in the urban areas of the world. In general all Indian cities face similar problems with their solid waste management. Amount and content of generated solid waste may differ among different cities but problems related to collection, transport and disposal are about the same. A mathematical model is presented for the best utilization of resources to minimize the cost involve in it.The rising popularity of incineration of municipal solid waste calls for detailed mathematical modeling and understanding of the incineration process. In this paper a municipal solid waste management system, including one SOM plant for treatment of organic material, one RDF plant for production of refuse derive fuel, one recycling plant, one landfill, and two incinerators with energy recovery. Here first incinerator work at low humidity and moderate temperature, while second incinerator work at very high temperature. The objective function in the model describes total investment and maintenance costs, transportation cost. The benefits from refuse derive fuel; energy generation, compost, and recycling are also incorporated in the objective function. The models can be used an important tool for planners, in municipal solid waste management in urban environment.

Keywords: MSW- municipal solid waste,RDF- refuse derived fuel;SOM- stabilized organic material,

1.0 INTRODUCTION

India is still considered to be a so-called developing country and an enormous gap exists between the rich elite and the poor masses. All local bodies lack in technical, managerial, administrative financial resources, adequate institutional arrangement and the technical know how to managing urban solid waste. It is therefore very essential to provide proper guidance and trainings to the personnel in the urban local bodies to make them efficient in managing the solid waste generated in their respective areas/cities/towns.This is a hard task since it is necessary to take into account economic, technical, normative aspects, paying particular attention to environmental problems. Municipal solid waste(MSW) management involves the collection of waste from its sources and the transportation of waste to processing plants where it can either be converted into fuel(RDF), electrical energy, compost(SOM) or recycled for reuse. The unrecoverable waste can either be transported directly from the waste source to landfills or from treatment plant to landfills. A careful planning is required in order to execute these activities in an optimal way.

Among others the following methodologies have been proposed. Badran and EL-Haggar (1) present a mixed integer linear programming model whose objective covers collection cost from collection stations, transportation cost from collection stations to either composting plant to landfills. The model of Chang and Chang (2) minimized overall cost through the solution of a nonlinear programming problem.Costi et al (3) have presented a comprehensive mixed integer non linear programming problem, whose planning horizon in a year.One similarity between our model and that of costi et.al. (3) is that collection cost from waste sources to collection points are not part of themodel.Fiorucci et al (4) can be derived from that of costiet.al (3) by ignoring environmental constraintsR. Minciardi et.al. (5) presents a multiobjective approach for solid waste management. Michael (6) present mathematical models in municipal solid waste management.The difference between our models and that of Michael (6) is that some of the variables in his model measure the number of replacement trucks. In our model,we consider only the number of trucks (excluding replacement trucks) used per day. In his model he considers only one incinerator plant.In this model we introduce a new concept of two incinerators. So the waste from the waste source transport to first incinerator and then from first incinerator to second incinerator and then to landfill. Part of the waste from incinerator one to landfill will also be their. Energy recovers by first and second incinerator goesto market. The aim of this work is to maximize the benefit from incinerator's plant and this will minimize the total cost in objective function. These will also minimizing the amount of waste and filling time to the sanitary landfill.

2.0 FORMULATION OF THE MODEL

The model has been formulated as an integer linear programming problem. The aim of this work is to present the structure and the application of a decision support system (DSS) designed to help decision makers (DMs) of a municipality in the development integrated program for solid waste management.

A detailed representation of the model is shown in figure (1).

The variable x,y,v along the arcs gives the waste flow amounts in terms of number of trucks.The total daily waste production enters the source where it is separated then sent to the plants. From the source metals are taken to recycling, organic material is taken for compost (SOM)production, part of the waste with low humidity and high heating value is sent to incinerator's for energy generation, or sent for RDF production, or disposed of in a sanitary landfill. Recycling is considered for paper, glass, plastic, wood, organic materials, and textiles. The fuel from RDF producing plant is sold in the market, while the scraps are sent to an incinerator orlandfill. The SOM joins the market while the scraps are taken to an incinerator or landfill. Scraps from recycling, RDF producing plant and SOM producing plant to incinerators will not be incorporated in the model. Part of the waste with low humidity and at moderate temperature sent to incinerator at J.the set of incinerator at J can produce energy under certain conditions. They have their limitations. Therefore another set of incinerator has been introduced in this paper so as to create very high temperature in these incinerators. Though the wastes remain at incinerator J will be approximate 20% of the waste sent to J′ but still at J′ remaining waste will be approximate 50% to be sent to landfill. Therefore we are creating energy at J′ as well.

2.1 INDICES

$i = 1,2,\ldots..I$: location of waste sources (collection points).

$j = 1,2\ldots\ldots J$:location of first incinerators.

$j' = 1,2.:\ldots.J'$: location of second incinerator.

$k = 1,2\ldots\ldots K$: location of sanitary landfills.

$m = 1,2\ldots\ldots M$: location of (RDF) plants.

$h = 1,2\ldots\ldots.H$: location of Composing (SOM) plants.

$s = 1,2\ldots\ldots\ldots S$: location of recycling plants.

$l = 1,2\ldots\ldots\ldots.L$: truck type.

$g = 1,2\ldots\ldots\ldots.G$: waste type.

2.2 VARIABLES

X_{ij}, X_{im}, X_{ih} ,X_{is} ,X_{ik}:-respectivelytotal number of trips made by trucks from waste source i to an incinerator at j, an RDF plant at m, an SOM plant at h, a recycling plant at s, and a landfill at k

x_{ij}, x_{im}, x_{ih}, x_{is}, x_{ik} :- respectively total number of trucks used everyday to carry waste from source i to anincinerator at j, an RDF plant at m, an SOM plant at h, a recycling plant at s, and a landfill at k.

Y_{jk}, $Y_{j'k}$,Y_{mk}, Y_{hk}, Y_{sk}:- respectively total number of trips made by trucks from an incinerators at j and j′, an RDF plant at m, an SOM plant at h, a recycling plant at s to a landfill at k.

$Y_{jk}, y_{j'k}, y_{mk}, y_{hk}, y_{sk}$:- respectively total number of trucks used everyday from an incinerator at j and j′, an RDF plant at m, an SOM plant at h, a recycling plant at s to a landfill at k.

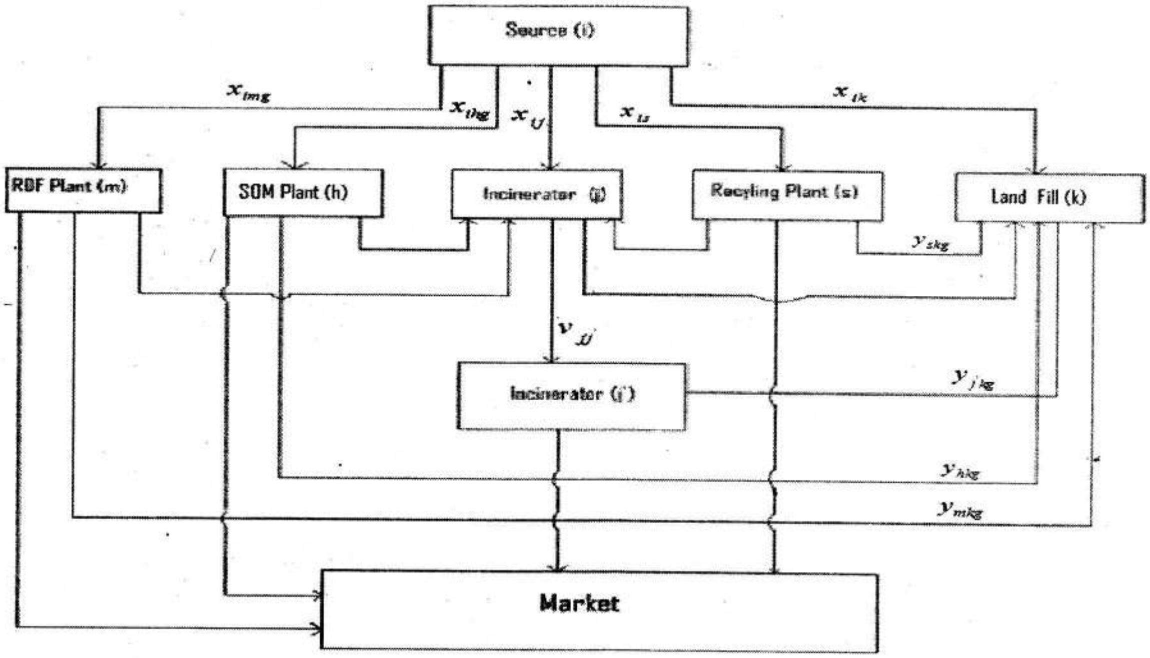
$z_j, z_{j'}, z_m, z_h, z_s, z_k$:- 0-1 variables indicating respectively , the presence of an incinerator's at j and j′ , RDF plant at m, an SOM plant at h , a recycling plant at s and a landfill at k.

$w_j, w_m, w_h, w_s, t_k, w_{j'}$:- amount of waste transported everyday respectively ,to an incinerator at j, an RDF plant at m, an SOM plant at h, a recycling plant at s and a landfill at k while $w_{j'}$ is the amount of waste transported from first incinerator at j to an second incinerator at j′.

T :- total number of trucks used everyday.

$V_{jj'}$:- total number of trips made by trucks from an incinerator at j to an incinerator at j′.

$v_{jj'}$:- numbers of trucks used to carry waste from an incinerator at j to an incinerator at j′.



(Figure 1) A simple mathematical model

2.3 INPUT DATA PARAMETERS

$a_{ij}, a_{im}, a_{ih}, a_{is}, a_{ik}$:- expected number of trips made by trucks per day between waste source at i and an incinerator at j, an RDF plant at m, an SOM plant at h, a recycling plant at s and a landfill at k.

$b_{jk}, b_{j'k}, b_{mk}, b_{hk}, b_{sk}$:- expected number of trips a truck of type l can make respectively per day between an incinerator at j and j′, an RDF plant at m, an SOM plant at h, a recycling plant at s and a landfill at k.

$p_{jj'}$:-expected number of trips a truck of type l can make per day between an incinerator at j and an incinerator atj′.

α :- Capacity (in tones) of a truck of type l.

$c_{ij}, c_{im}, c_{ih}, c_{is}, c_{ik}$:- respectively transportation cost per unit of waste from a waste source at i to j, m, h, s,k.

d_{jk}, $d_{j'k}$, d_{mk}, d_{hk}, $d_{sk}d_{jj'}$:- respectively transportation cost per unit of waste carried by a truck from an incinerator at j and j′, an RDF plant at m, an SOM plant at h, a recycling plant at s to a landfill at k, while $d_{jj'}$ is the transportation cost per unit of waste carried from an incinerator at j to an Incinerator j′.

c_j, $c_{j'}$, c_m, c_h, c_s:-revenue respectively per unit of waste from an incinerator at j and j′, an RDF plant at m, an SOM plant at h, and a recycling plant at s.

f_l:- cost of buying a new truck of type l. (l=1, 2, 3, ...L)

d_i:- amount of waste at source i.

ρ_j, $\rho_{j'}$, ρ_m, ρ_h, ρ_s, ρ_{1j}:- Fraction (%) of unrecovered waste respectively at an incinerator at j and j′, an RDF plant at m, an SOM plant at h, and a recycling plant at s, that requires disposal to a landfill while ρ_{1j} is the unrecovered waste when sent it from j to j′.

Q_j, $Q_{j'}$, Q_m, Q_h, Q_s, Q_k :- Capacity per day respectively for an incinerators at j and j′, an RDF plant at m, an SOM plant at h, and a recycling plant at s, and a landfill at K.

δ_j, $\delta_{j'}$, δ_m, δ_h, δ_s, δ_k:- respectively fixed cost incurred in opening an incinerators at j and j′, an RDF plant at m, an SOM plant at h, and a recycling plant at s, and a landfill at K.

γ_j, $\gamma_{j'}$, γ_m, γ_h, γ_s, γ_k:- :- respectively variable cost incurred in handling an incinerators at j and j′, an RDF plant at m, an SOM plant at h, and a recycling plant at s, and a landfill at k.

3.0 OBJECTIVE FUNCTION

The objective function represents the over all daily waste management cost. The first component (F1) gives the investment and waste handling expenses as well as transportation cost. The second component (F2) gives the total cost for buying all trucks required in the daily management of waste. The third component (B) gives the benefits at the plants owing to the production of electric energy, compost, refuses derived fuel, and recycled material.

$$F_1(Z,w,X,Y,V)=\Big[\sum_j(\delta_j Z_j+\gamma_j w_j)+\sum_{j'}(\delta_{j'}Z_{j'}+\gamma_{j'}w_{j'})+\sum_m(\delta_m Z_m+\gamma_m w_m)+\sum_h(\delta_h Z_h+\gamma_h w_h)$$

$$+\sum_s(\delta_s Z_s+\gamma_s w_s)+\sum_k(\delta_k Z_k+\gamma_k w_k)\Big]$$

$$+\Big[\sum_{ij}C_{ij}\alpha X_{ij}+\sum_{im}C_{im}\alpha X_{im}+\sum_{ih}C_{ih}\alpha X_{ih}+\sum_{is}C_{is}\alpha X_{is}+\sum_{ik}C_{ik}\alpha X_{ik}\Big]$$

$$+\Big[\sum_{jk}d_{jk}\alpha Y_{jk}+\sum_{j'k}d_{j'k}\alpha Y_{j'k}+\sum_{mk}d_{mk}\alpha Y_{mk}+\sum_{hk}d_{hk}\alpha Y_{hk}+\sum_{sk}d_{sk}\alpha Y_{sk}\Big]$$

$$+\sum_{jj'}d_{jj'}\alpha V_{jj'} \qquad [3.1]$$

$$F_2(x,y,v)=\sum f_l(T) \qquad \text{...[3.2]}$$

$$B(w)=\sum_j C_j(1-\rho_j)w_j+\sum_{j'} C_{j'}(1-\rho_{j'})w_{j'}+\sum_m C_m(1-\rho_m)w_m + \sum_h C_h(1-\rho_h)w_h+\sum_s C_s(1-\rho_s)w_s \qquad \text{...[3.3]}$$

So the objective function F to be minimized is

$$\mathbf{F = F_1 + F_2 - B} \qquad \text{...[3.4]}$$

Constraints:-

Mass Balance Constraints:

Total waste moved from each waste collection point *i* should at least be equal to the amount of waste found at that point

$$\sum_{glj} \alpha_l X_{ijg}+\sum_{glm} \alpha_l X_{img}+\sum_{glh} \alpha_l X_{ihg}+\sum_{gls} \alpha_l X_{isg}+\sum_{glk} \alpha_l X_{ikg} \geq d_i \quad \text{i=1,....}l \qquad \text{...[3.5]}$$

In Constraint (6-9)Amount of waste carried away from every plant to landfill should at least be equal to amount of waste found at that point

$$\rho_j w_j \leq \sum \alpha Y_{jkg} \qquad \text{j=1...J} \qquad \text{...[3.6]}$$

$$\rho_{j'} w_{j'} \leq \sum \alpha Y_{j'kg} \text{ j'=1..J} \qquad \text{...[3.7]}$$

$$\rho_m w_m \leq \sum \alpha Y_{mkg} \qquad \text{m=1..M} \qquad \text{...[3.8]}$$

$$\rho_h w_h \leq \sum \alpha Y_{hkg} \text{ h=1..H} \qquad \text{...[3.9]}$$

$$\rho_s w_s \leq \sum \alpha Y_{skg} \text{ s=1..S} \qquad \text{...[3.10]}$$

Amount of waste carried away from j to j' should at least be equal to amount of waste found at j.

$$\rho_{1j} w_j \leq \sum \alpha V_{jj'g} \text{ [3.11]}$$

Capacity limitation constraints:-

In constraints (12)-(16) the maximum capacities for processing plants are accounted. Means amount of waste taken to different plants should not exceed the plant capacities.In constraint (17) same thing done for sanitary landfill.

$w_{j'} \leq Q_{j'}Z_{j'}$ j'=1,…J'[3.12]

$w_j \leq Q_jZ_j$ j=1,…J ...[3.13]

$w_m \leq Q_mZ_m$ m=1,…M[3.14]

$w_h \leq Q_hZ_h$ h=1,…H ...[3.15]

$w_s \leq Q_s Z_s$ s=1,…S   …[3.16]

$t_k \leq Q_k Z_k$ k=1,…K   …[3.17]

Technical constraint:-

Constraints (18)-(27) means that, once the flow to either plant or sanitary landfill is positive, that plant or landfill must actually exist. In constraint (28) same thing done for j′.

$\alpha_l X_{ijg} \leq Q_j Z_j$, l =1,…,L , i, (j) =1,…,I,(J), g =1,…,G   …[3.18]

$\alpha_l X_{img} \leq Q_m Z_m$, l =1,…,L , i, (m) =1,…,I (M),g =1,…,G   …[3.19]

$\alpha_l X_{ihg} \leq Q_h Z_h$, l =1,…,L , i, (h) =1,…,I(H), g =1,…,G   …[3.20]

$\alpha_l X_{isg} \leq Q_s Z_s$, l =1,…,L , i, (s) =1,…,I,(S), g =1,…,G   …[3.21]

$\alpha_l X_{ikg} \leq Q_k Z_k$, l =1,…,L , i, (k) =1,…,I(K), g =1,…,G   …[3.22]

$\alpha_l Y_{jkg} \leq Q_k Z_k$, l =1,…,L , j, (k) =1,…,J, (K), g =1,…,G[3.23]

$\alpha_l Y_{j'kg} \leq Q_k Z_k$, l =1,…,L , j′, (k) =1,…,J′,(K), g =1,…,G   …[3.24]

$\alpha_l Y_{mkg} \leq Q_k Z_k$, l =1,…,L , m, (k) =1,…,M(K), g =1,…,G[3.25]

$\alpha_l Y_{hkg} \leq Q_k Z_k$, l =1,…,L , h, (k) =1,…,H,(K), g =1,…,G   …[3.26]

$\alpha_l Y_{skg} \leq Q_k Z_k$, l =1,…,L , s, (k) =1,…,S, (K), g =1,…,G   …[3.27]

$\alpha_l V_{jj'g} \leq$, $Q_{j'} Z_{j'}$ l =1,…,L , j, (j′) =1,…,J, (J′), g =1,…,G   …[3.28]

Variable conditions:-

X_{ijg}integer≥ 0 , l =1,…,L , i, (j) =1,….,I, (J), g =1,…,G   …[3.29]

X_{img}integer≥ 0 , l =1,…,L , i, (m) =1,…,I, (M), g =1,…,G   …[3.30]

X_{ihg}integer≥ 0 , l =1,…,L , i, (h) =1,…,I, (H), g =1,…,G   …[3.31]

X_{isg}integer≥ 0 , l =1,…,L , i, (s) =1,…,I, (S), g =1,…,G[3.32]

X_{ikg}integer≥ 0 , l =1,…,L , i, (k) =1,…,I, (K), g =1,…,G   …[3.33]

Y_{jkg}integer≥ 0 , l =1,…,L , j, (k) =1,…,J, (K), g =1,…,G   … [3.34]

$Y_{j'kg}$integer≥ 0 , l =1,…,L , j′, (k) =1,…,J′, (K), g =1,…,G   …[3.35]

Y_{mkg}integer≥ 0 , l =1,…,L , m, (k) =1,…,M, (K), g =1,…,G   …[3.36]

Y_{hkg}integer≥ 0 , l =1,…,L , h, (k) =1,….H, (K), g =1,…,G   …[3.37]

Y_{skg}integer≥ 0 , l =1,…,L , s, (k) =1,…,S, (K), g =1,…,G   …[3.38]

$V_{jj'g}$integer≥ 0 , l =1,…,L , j, (j′) =1,…,J, (J′), g =1,…,G   …[3.39]

Variables in (40)-(45) are defined as Boolean. These are used to determine the existence of either a plant or a landfill.

$Z_j \in \{0, 1\} J = 1,\ldots J$ ...[3.40]

$Z_{j'} \in \{0, 1\}$ $j' = 1,\ldots J'$[3.41]

$Z_m \in \{0, 1\}$ $m = 1,\ldots M$ ...[3.42]

$Z_h \in \{0, 1\}$ $h = 1,\ldots H$ ...[3.43]

$Z_s \in \{0, 1\}$ $s = 1,\ldots S$ ...[3.44]

$Z_k \in \{0, 1\}$ $k = 1,\ldots K$ ...[3.45]

Definitions:-

In definitions (46)-(55) which were already mentioned in the beginning, gives the expected number of trips made per day by trucks of type l from waste sources to plants, waste sources to landfills, and plants to landfills are given. While (56) gives the same from j to j'.

$X_{ijg} = a_{ij} x_{ijg},\ l = 1,\ldots,L,\ i, (j) = 1,\ldots,I, (J),\ g = 1,\ldots,G$ ...[3.46]

$X_{img} = a_{im} x_{img},\ l = 1,\ldots,L, i, (m) = 1,\ldots,I, (M), g = 1,\ldots,G$ ...[3.47]

$X_{ihg} = a_{ih} x_{ihg},\ l = 1,\ldots,L, i, (h) = 1,\ldots,I, (H), g = 1,\ldots,G$ ...[3.48]

$X_{isg} = a_{is} x_{isg},\ l = 1,\ldots,L, i, (s) = 1,\ldots,I, (S), g = 1,\ldots,G$[3.49]

$X_{ikg} = a_{ik} x_{ikg},\ l = 1,\ldots,L, i, (k) = 1,\ldots,I, (K), g = 1,\ldots,G$ ...[3.50]

$Y_{jkg} = b_{ij} y_{jkg},\ l = 1,\ldots,L,\ j, (k) = 1,\ldots,J, (K), g = 1,\ldots,G$ ...[3.51]

$Y_{j'kg} = b_{j'k} y_{j'kg},\ l = 1,\ldots,L, j', (k) = 1,\ldots,J', (K), g = 1,\ldots,G$ ...[3.52]

$Y_{mkg} = b_{mk} y_{mkg},\ l = 1,\ldots,L, m, (k) = 1,\ldots,M, (K), g = 1,\ldots,G$ ...[3.53]

$Y_{hkg} = b_{hk} y_{hkg},\ l = 1,\ldots,L, h, (k) = 1,\ldots,H, (K), g = 1,\ldots,G$ ...[3.54]

$Y_{skg} = b_{sk} x_{skg},\ l = 1,\ldots,L, s, (k) = 1,\ldots,S, (K), g = 1,\ldots,G$ ...[3.55]

$V_{jj'g} = p_{jj'} v_{jj'g},\ l = 1,\ldots,L, j, (j') = 1,\ldots J, (J'), g = 1,\ldots,G$ [3.56]

Definitions (3.57 – 3.61) indicate the amount of waste transported from source (i) to processing plants. While (62) indicates amount of waste from incinerator (j) to (j'). Definition (63) gives amount of waste from all waste sources to a landfill K.

$$w_j = \sum_{gli} \alpha_l X_{ijg} \quad J = 1------J \qquad ...[3.57]$$

$$w_m = \sum_{gli} \alpha_l X_{img} \quad M = 1,2----M \qquad ...[3.58]$$

$$w_h = \sum_{gli} \alpha_l X_{ihg} \quad h = 1,2----H \qquad ...[3.59]$$

$$w_s = \sum_{gli} \alpha_l X_{isg} \quad s = 1,2----s \qquad ...[3.60]$$

$$w_k = \sum_{gli} \alpha_l X_{ikg} \quad k = 1,2----K \qquad ...[3.61]$$

$$w_{j'} = \sum_{glj} \alpha_l V_{jj'g} \quad j' = 1,2----J' \qquad ...[3.62]$$

$$tk = wk + \sum_{glj} \alpha_l Y_{jkg} + \sum_{glj'} \alpha_l Y_{j'kg} + \sum_{glm} \alpha_l Y_{mkg} + \sum_{glh} \alpha_l Y_{hkg} + \sum_{gls} \alpha_l Y_{skg} \quad k = 1, 2...K \qquad [3.63] ...$$

Equation (64) gives total amount of waste collected from all waste sources per day. (This excludes waste generated by plants):

$$W = \sum_{j} w_j + \sum_{j'} w_{j'} + \sum_{m} w_m + \sum_{h} w_h + \sum_{s} w_s + \sum_{k} w_k \qquad ...[3.64]$$

Equation (65) gives total number of trucks used in the model.

$$T = \sum_{gij} x_{ijg} + \sum_{gim} x_{img} + \sum_{gih} x_{ihg} + \sum_{gis} x_{isg} + \sum_{gik} x_{ikg} + \sum_{gjk} y_{jkg} + \sum_{gj'k} Y_{j'kg}$$

$$+ \sum_{gmk} Y_{mkg} + \sum_{ghk} Y_{hkg} + \sum_{gsk} Y_{skg} + \sum_{gjj'} v_{jj'} \qquad ...[3.65]$$

An integer Linear model example illustrating how the above model problems can be solved:-

Let

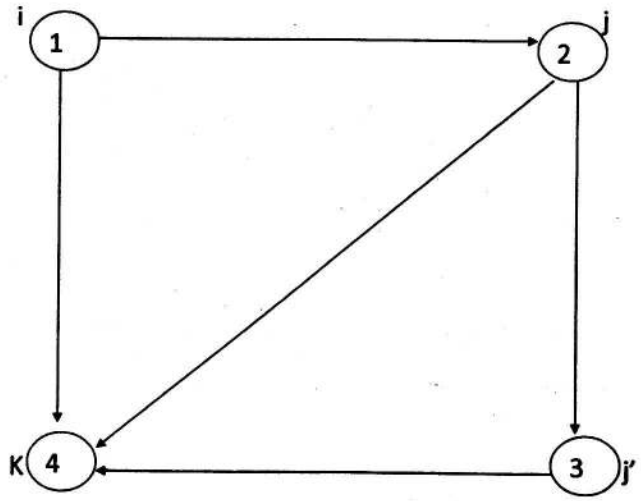
1: denote a waste source (ie collection point)

2: denote a first incinerator

3: denote a second incinerator

4: denote a landfill

Figure (2) illustrate a simple model, where the waste source i , the incinerators' j and j' , the landfill k, are all known. Here all trucks are of the same capacity.



Figure(2) A simple model representation

Variables:-

U_{12},U_{14} :-respectively represent the amount of waste (in tons) collected everyday by trucks of capacity 5 tons from a source at node1 to an incinerator at node 2, and a landfill at node 4.

V_{23}, V_{24},V_{34}:- respectively represent the amount of waste (in tons) collected everyday by trucks of capacity 5 tons from a incinerator at node 2 to an incinerator at node 3, and a landfill at node 4 while V_{34}representfrom an incinerator at node 3 to alandfill atnode 4.

x_{12}, x_{14} :- respectively represent the number of trucks of capacity 5 tons used everyday to carry waste from a waste source at node1 to an incinerator at node 2, and a landfill at node 4.

y_{23}, y_{24}, y_{34}:- respectively represent the number of trucks of capacity 5 tons used everyday to carry waste from an incinerator at node 2 to incinerator at node 3 and a landfill at node 4 and also from incinerator at 3to a landfill at node 4.

t_4- represent the amount of waste(in ton) transported everyday to an incinerator at node 2,and a landfill at node 4.

Input data/parameters:-

18, 12:- :- respectively represent the number of trips a truck of capacity 5 tons can make everyday to carry waste from a waste source at node1 to an incinerator at node 2, and a landfill at node 4.

4,1 :- respectively represent the number of trips a truck of capacity 5 tons can make everyday to carry waste from a incinerator at node 2 to an incinerator at node 3, and a landfill at node 4.

2:- represent the number of trips a truck of capacity 5 tons can make everyday to carry waste from a incinerator at node 3 to a landfill at node 4.

Rs. 400,450:- respectively are the transportation costs per ton of waste transported from a waste source at 1 to an incinerator at 2, and a landfill at 4.

Rs.100 :- the transportation cost per ton of waste transported from a incinerator at 2 to an incinerator at 3,from an incinerator at 2 to a landfill at 4 also from an incinerator at 3 to a landfill at 4.

Rs.3000:- is the revenue per unit of waste from an incinerator at 2 and also from an incinerator at 3.

150:- is the amount of waste in (tons) at a waste source at 1.

o.1:-is the fraction (%) of unrecovered waste at an incinerator at 2.

150, 50, 1000:- are the respective capacities for incinerators at node 2, node 3, and a landfill at node 4.

Rs.600, 600, 200:- are the respective costs of handling a ton of waste at an incinerator at 2, an incinerator at 3 and a landfill at 4.

Rs.15lacks:-cost of buying a truck.

The model:-

This model is an integer programming model and we seek to minimize the cost

$F_1 + F_2 - B$, where

$$F_1 = [(400 *5 *18*x_{12}) + (450* 5*12* x_{14}) + (100 *5 *1 * y_{24}) + (100*5* 4*y_{23}) + (100*5* 2*y_{34})] + [(600*5*18*x_{12}) + (600*5* 4*y_{23}) + (200* t_4)] \quad [3.66]$$

$$F_2 = 15,00,000*(T) \quad [3.67]$$

$$B = (3000*5*18* x_{12}) + (3000*5*4* y_{23}) \quad [3.68]$$

Constraints:-

$$5 *18*x_{12} + 5*12* x_{14} \geq 150 \quad ...[3.69]$$

$$0.01*5 *18*x_{12} \leq 5 *1 * y_{24} \quad [3.70]$$

$$5 *4 * y_{23} \geq 5* 2*y_{34} \quad [3.71]$$

The restriction on the waste material goes from node 3 to node 4 is given by,

$$5* 2*y_{34} \leq 9 \quad [3.72]$$

Capacity limitation constraints are,

$$5 *18*x_{12} \leq 150 \quad ...[3.73]$$

$$5* 4*y_{23} \leq 50 \quad [3.74]$$

$$t_4 \leq 1000 \quad [3.75]$$

Variable conditions:-

$x_{12}, x_{14}, y_{23}, y_{24}, y_{34}$ integer ≥ 0 ...[3.76]

Definitions:-

$t_4 = 5*12*x_{14} + 5*1*x_{24} + 5*2*y_{34}$ ...[3.77]

$T = x_{12} + x_{14} + y_{23} + y_{24} + y_{34}$ ...[3.78]

The solution:-

We begin by generating a feasible solution by carrying all the waste from node 1 to node 2, since there are benefits at node 2 and node 3.

There fore$x_{14} = 0$

From inequalities (69),(70),(71), (72),(78) respectively we get $x_{12} = 2, y_{24} = 1, y_{23} = 1, y_{34} = 1$, and $T = 5$

After putting the values of variables in equations (66) - (68), we get

F_1=19, 65,00 F_2= 75, 00,000 B =60, 00,00

$F_1 + F_2 - B$ = 70, 96,500

4.0 CONCLUSION

The model developed in the paper is general in nature which may be suitable to almost all urban areas. We have used second incinerator to gain more energy and also to minimize the quantity of waste which is sent to landfill. The paper may be a useful tool in planning and management of municipal solid waste transportation, recycling, composting, and disposal program. It can also be helpful to design plants and landfill.

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FRACTIONAL DIFFERENTIATION OF GENERALIZED HYPERGEOMETRIC FUNCTION

R.K. KUMBHAT AND (MRS.) SHANU SHARMA

ABSTRACT

The present paper is all about the fractional differentiation of the generalized hypergeometric function ${}_2R_1{}^{\tau}(.)$.The fractional differential operator is $D^n_{k,\alpha,x}$ .Our result provides interesting unification and extension of a number of new and known results. Some special cases and generalization of the hypergeometric function have also been worked out.

Keywords: Fractional differentiation, Fractional calculus, generalized hypergeometric function.

1.0 INTRODUCTION

The fractional differential operator is worked out by many mathematicians like Garg and Arora [2] and Nigam and Garg [3].The fractional differential operator is defined as :

$$D^n_{k,\alpha,x}(x^{\mu}) = \prod_{r=0}^{n-1} \frac{\Gamma(\mu+rk+1)}{\Gamma(\mu+rk-\alpha+1)} x^{\mu+nk} \qquad \text{...(1.1)}$$

where $\alpha \neq \mu+1$, α and k are not necessarily integers.

Virchenko Kalla and Al-Zamel [5] gave the generalized hypergeometric function ${}_2R_1(\lambda,\beta;\gamma;\tau;z)$ which is defined in the following form :

$${}_2R_1^{\tau}(z) = {}_2R_1(\lambda,\beta;\gamma;\tau;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)} \sum_{r=0}^{\infty} \frac{(\lambda)_r \Gamma(\beta+\tau r)\, z^r}{\Gamma(\gamma+\tau r)\; r!} \qquad \text{...(1.2)}$$

where $\tau \in R,\ \tau > 0,\ |z| < 1$,

and its integral representation is given

$${}_2R_1^{\tau}(z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-z\,t^{\tau})^{-\lambda}\, dt \qquad \text{...(1.3)}$$

where $R(\beta,\gamma) > 0, R(\beta-\gamma), \tau > 0, |z| < 1$

Recently Saxena, Chena Ram and Naresh [4] extended the generalized hypergeometric ${}_2R_1$ as ${}_3R_2(\cdot)$ as follows :

$${}_3R_2^{\tau}(z) = {}_3R_2(\lambda,\beta,\gamma;\delta,\mu;\tau;z) = \frac{\Gamma(\delta)\,\Gamma(\mu)}{\Gamma(\beta)\,\Gamma(\gamma)} \sum_{r=0}^{\infty} (\lambda)_r \frac{\Gamma(\beta+kr)\,\Gamma(\gamma+kr)}{\Gamma(\delta+kr)\,\Gamma(\mu+kr)} \frac{z^r}{r!} \qquad \text{...(1.4)}$$

where $\tau \in R,\ \tau > 0,\ |z| < 1$

and its integral representation is :

$$ {}_3R_2^{\tau}(z) = \frac{\Gamma(\mu)}{\Gamma(\beta)\,\Gamma(\mu-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\mu-\beta-1}\,{}_2R_1(\lambda,\gamma\,;\delta\,;k\,;z\,t^{\tau})dt \qquad ...(1.5)$$

where $R(\beta,\gamma,\delta,\mu) > 0\,,\tau > 0, |z| < 1$

The generalized hypergeometric function ${}_2R_1(\lambda,\,\beta\,;\gamma\,;\tau\,;z)$ holds the following relation when $\tau = n$ where n is a positive integer as

$$ {}_2R_1(\lambda,\,\beta\,;\gamma\,;n\,;z) = \mathrm{A}\,{}_{2n+1}F_{2n}\left(\lambda, \frac{\beta}{n}, \frac{\beta+1}{n}, ..., \frac{\beta+n-1}{n}; \frac{\gamma}{n}, \frac{\gamma+1}{n}, ..., \frac{\gamma+n-1}{n}; z\right) \qquad ...(1.6)$$

where

$$ A = n^{\beta-\gamma}\,\frac{\Gamma(\gamma)\Gamma(\frac{\beta}{n})\Gamma(\frac{\beta+1}{n}).......\Gamma(\frac{\beta+n-1}{n})}{\Gamma(\beta)\Gamma(\frac{\gamma}{n})\Gamma(\frac{\gamma+1}{n}).......\Gamma(\frac{\gamma+n-1}{n})} \qquad ...(1.7)$$

Particular Case: When $\tau = 1$, (1.2) and (1.4) reduce to hypergeometric function ${}_2F_1(.)$ and ${}_3F_2(.)$ respectively. Also for $\gamma = \mu$, (1.4) reduces to generalized hypergeometric function ${}_2R_1^{\ \tau}(.)$ studied by Virchenko et.al.[5].

2.0 MAIN RESULTS

The fractional derivative of ${}_2R_1^{\ \tau}(.)$ is :

$$ D^n_{k,\alpha,x}[x^{\mu}\,{}_2R_1(\lambda,\beta;\gamma;\tau;x^{\tau})] = \prod_{p=0}^{n-1} \frac{\Gamma(\mu + pk + 1)}{\Gamma(\mu + pk - \alpha + 1)}$$

$$ {}_3R_2(\lambda,\beta,\mu+pk+1;\gamma,\mu+pk-\alpha+1;\tau;x^{\tau}).x^{\mu+pk} \qquad ...(2.1)$$

Proof: Using the definition of the generalized hypergeometric function

${}_2R_1(\alpha,\beta;\gamma;\tau;x^{\tau})$, the left hand side of (2.1) can be written as :

$$ D^n_{k,\alpha,x}\Big[\ x^{\mu}\,\frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{r=0}^{\infty}\frac{(\lambda)_r\Gamma(\beta+\tau r)}{\Gamma(\gamma+\tau r)r!}x^{\tau_r}\ \Big] = \frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{r=0}^{\infty}\frac{(\lambda)_r\Gamma(\beta+\tau r)}{\Gamma(\gamma+\tau r)r!}D^n_{k,\alpha,x}(x^{\mu+\tau r})$$

Using (1.1), we get

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{r=0}^{\infty}\frac{(\lambda)_r\Gamma(\beta+\tau r)}{\Gamma(\gamma+\tau r)r!}\prod_{p=0}^{n-1}\frac{\Gamma(\mu+pk+1+\tau r)}{\Gamma(\mu+pk-\alpha+1+\tau r)}.x^{\mu+\tau r+pk}$$

$$\Rightarrow\prod_{p=0}^{n-1}\frac{\Gamma(\mu+pk+1)}{\Gamma(\mu+pk-\alpha+1)}{}_3R_2(\lambda,\beta,\mu+pk+1;\gamma,\mu+pk-\alpha+1;\tau;x^{\tau})x^{\mu+pk}$$

Generalization of the hypergeometric function ${}_pR_q(\cdot)$:

$$D^n_{k,\alpha,x}[x^{\mu}\,{}_{q+1}R_q(\lambda,\alpha_1,\alpha_2,....,\alpha_q;\beta_1,\beta_2,......,\beta_q;\tau;x^{\tau})]$$

$$=\prod_{p=0}^{n-1}\frac{\Gamma(\mu+pk+1)}{\Gamma(\mu+pk-\alpha+1)}{}_{q+2}R_{q+1}(\lambda,\alpha_1,\alpha_2,....,\alpha_q,\mu+pk+1;\beta_1,\beta_2,......,\beta_q,$$

$$\mu+pk-\alpha+1;\tau;x^{\tau})x^{\mu+pk} \qquad ...(3.1)$$

where

$$\tau\in R,\ \tau>0,\ |x|<1,\ R(\mu+pk-\lambda+1)>0$$

3.0 SPECIAL CASES:

Our main result provides unification and extension of various known and new results on fractional differential operator

(i). When we take $\tau=1$ in the main result, it reduces to

$$D^n_{k,\alpha,x}\left[x^{\mu}\,{}_2F_1(\lambda,\beta;\gamma;x)\right]$$

$$=\prod_{p=0}^{n-1}\frac{\Gamma(\mu+pk+1)}{\Gamma(\mu+pk-\alpha+1)}{}_3F_2(\lambda,\beta,\mu+pk+1;\gamma,\mu+pk-\alpha+1;x)x^{\mu+pk} \qquad ...(4.1)$$

(ii). When $\tau=1$ and $\gamma=\beta$,the main result (2.1) reduce to

$$D^n_{k,\alpha,x}[x^{\mu}(1-x)^{-\alpha}]$$

$$=\prod_{p=0}^{n-1}\frac{\Gamma(\mu+pk+1)}{\Gamma(\mu+pk-\alpha+1)}{}_2F_1(\lambda,\mu+pk+1;\mu+pk-\alpha+1;x)x^{\mu+pk} \qquad ...(4.2)$$

(iii). On replacing x by $\frac{x}{\lambda}$ and taking $\lambda\to 0$ in (4.1), we get

$$D^n_{k,\alpha,x}[x^{\mu}\,{}_1F_1(\beta;\gamma;x)]$$

$$=\prod_{p=0}^{n-1}\frac{\Gamma(\mu+pk+1)}{\Gamma(\mu+pk-\alpha+1)}{}_2F_2(\beta,\mu+pk+1;\gamma,\mu+pk-\alpha+1;x).x^{\mu+pk} \qquad ...(4.3)$$

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Corresponding author's address:	Deptt. Of Mathematics & Statistics,Jai Narayan Vyas University,Jodhpur-342002
Corresponding author's e-mail address:	drrkkumbhat@yahoo.com, shanu_19802006@yahoo.com

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A REDUCTION FORMULA FOR THE KAMPÉ DE FÉRIET FUNCTION - II

SHOUKAT ALI

Department of Mathematics, Govt. Engineering College Bikaner, Bikaner - 334 001
Rajasthan State, India
E-mail: dr.alishoukat@rediffmail.com

ABSTRACT

This paper is in continuation of the earlier paper by the author in which we have obtained an interesting reduction formula for the Kampé de Fériet function contiguous to that of obtained by Pathan, Qureshi and Khan.

The aim of this research paper is to obtain one more interesting contiguous result to that of Pathan, Qureshi and Khan.

Key Words: Kampé de Fériet function, Dixon's theorem.

Mathematics Subject Classification (2000): Primary 33C20, 33C60; Secondary 33C65, 33C70.

1.0 INTRODUCTION

We recall the definition of generalized Kampé de Fériet's function as follows [5]

$$F^{p:q;k}_{\ell:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k)\\(\alpha_\ell):(\beta_m);(\gamma_n)\end{matrix}\ \middle|\ \begin{matrix}x\\y\end{matrix}\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s\,x^r y^s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s\,r!s!}\qquad ...(1.1)$$

Where for convergence

(i) $p+q<\ell+m+1,\ p+k<\ell+n+1,\ |x|<\infty,\ |y|<\infty$

or

(ii) $p+q<\ell+m+1,\ p+k<\ell+n+1$ and

$$\begin{cases}|x|^{\frac{1}{p-\ell}}+|y|^{\frac{1}{p-\ell}}<1, \text{if } p>\ell\\ \max\{|x|,|y|\}<1, \text{if } p\le\ell\end{cases}$$

Although the double hypergeometric series defined by (1.1) reduces to the Kampé de Fériet function in the special case:

$q=k$ and $m=n$

yet it is usually referred to in the literature as the Kampé de Fériet series.

The following are the cases in which the Kampé de Fériet function defined in (1.1) can be expressed in terms of generalized hypergeometric series.

$$F^{p:0;0}_{q:0;0}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix}\ \Big|\ \begin{matrix}x\\ y\end{matrix}\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix}\ \Big|\ x+y\right] \quad \ldots(1.2)$$

$$F^{0:p;r}_{0:q;s}\left[\begin{matrix}-;\alpha_1,\ldots,\alpha_p;\gamma_1,\ldots,\gamma_r\\ -;\beta_1,\ldots,\beta_q;\delta_1,\ldots,\delta_s\end{matrix}\ \Big|\ \begin{matrix}x\\ y\end{matrix}\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix}\ \Big|\ x\right]\ {}_rF_s\left[\begin{matrix}\gamma_1,\ldots,\gamma_r\\ \delta_1,\ldots,\delta_s\end{matrix}\ \Big|\ y\right] \quad \ldots(1.3)$$

$$F^{p:1;1}_{q:0;0}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\nu;\sigma\\ \beta_1,\ldots,\beta_q;-;-\end{matrix}\ \Big|\ \begin{matrix}x\\ x\end{matrix}\right] = {}_{p+1}F_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\nu+\sigma\\ \beta_1,\ldots,\beta_q\end{matrix}\ \Big|\ x\right] \quad \ldots(1.4)$$

$$F^{p:0;0}_{q:1;1}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;-;-\\ \beta_1,\ldots,\beta_q;\nu;\sigma\end{matrix}\ \Big|\ \begin{matrix}x\\ x\end{matrix}\right] = {}_{p+2}F_{q+3}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,\Delta(2;\nu+\sigma-1)\\ \beta_1,\ldots,\beta_q,\nu,\sigma,\nu+\sigma-1\end{matrix}\ \Big|\ 4x\right] \quad \ldots(1.5)$$

where, and in what follows, $\Delta(\ell;\lambda)$ abbreviates the array of ℓ parameters

$$\frac{\lambda}{\ell},\frac{(\lambda+1)}{\ell},\ldots,\frac{(\lambda+\ell-1)}{\ell},\quad \ell=1,2,3,\ldots$$

For more detail see [5, pp. 28-32].

Very recently, the author [1] has obtained an interesting case of reducibility of Kampé de Fériet Function closely related to the result (1.6) by employing contiguous Dixon's theorem obtained earlier by Lavoie, Grondin, Rathie and Arora [3]. In this paper we have obtained one more interesting result for the reducibility of Kampé de Fériet Function.

In 1985, Pathan, Qureshi and Khan [4] obtained the following result for the Kampé de Fériet Function [2].

$$F^{p:1;1}_{q:1;1}\left[\begin{matrix}(a_p):d-2e+1;d\\ (b_q):2-2e;2e\end{matrix}\ \Big|\ x,-x\right]$$

$$= {}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p),\frac{1}{2}(a_p+1);(e-d+\frac{1}{2}),(\frac{1}{2}+d-e)\\ \frac{1}{2}(b_q),\frac{1}{2}(b_q+1);\frac{1}{2},\frac{1}{2}(1+2e),\frac{1}{2}(3-2e)\end{matrix}\ \Big|\ 4^{p-q-1}x^2\right] - \frac{x(e-d)(2e-1)\prod_{i=1}^{p}(a_i)}{2e(1-e)\prod_{i=1}^{q}(b_i)}$$

$${}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p+1),\frac{1}{2}(a_p)+1;(1+e-d),(1-e+d)\\ \frac{1}{2}(b_q+1),\frac{1}{2}(b_q)+1;\frac{3}{2},(2-e),(e+1)\end{matrix}\ \Big|\ 4^{p-q-1}x^2\right] \quad \ldots(1.6)$$

They have obtained the result with the help of classical Dixon's theorem on the sum of a $_3F_2$ viz.

$$ {}_3F_2\left[\begin{matrix} a,b,c \\ 1+a-b,1+a-c \end{matrix} \mid 1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)} \cdot \frac{\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a-b-c)} \quad ...(1.7) $$

provided $\Re e\,(a-2b-2c) > -2$.

In 1994, Lavoie et al. [3] have obtained a large number of summation formulae closely related to (1.7) of which one is given below:

$$ {}_3F_2\left[\begin{matrix} a,b,c \\ a-b,1+a-c \end{matrix} \mid 1\right] = \frac{\Gamma(\frac{1}{2}a)\Gamma(a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(a)\Gamma(\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(a-b-c+1)} + \frac{\Gamma(a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}a-b-c+\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a)}{2\Gamma(a)\Gamma(\frac{1}{2}+\frac{1}{2}a-c)\Gamma(a-b-c+1)\Gamma(\frac{1}{2}a-b+\frac{1}{2})} \quad ...(1.8) $$

provided $\Re e\,(a-2b-2c) > -1$

The aim of this research note is to obtain one result closely related to (1.6) by employing the summation formula (1.8).

2.0 RESULTS REQUIRED

The following results will be required in our present investigations.

$$ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A(n,m) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} A(n,m-n) \quad ...(2.1) $$

$$ (\alpha)_{m-n} = \frac{(-1)^n\Gamma(\alpha+m)}{\Gamma(\alpha)(1-\alpha-m)_n} \quad ...(2.2) $$

$$ (m-n)! = \frac{(-1)^n m!}{(-m)_n} \quad ...(2.3) $$

$$(\alpha)_{2m} = 2^{2m}\left(\frac{1}{2}\alpha\right)_m\left(\frac{1}{2}\alpha+\frac{1}{2}\right)_m \qquad ...(2.4)$$

$$\Gamma(\alpha-2m) = \frac{(-1)^{2m}\Gamma(\alpha)}{(1-\alpha)_{2m}} \qquad ...(2.5)$$

$$\Gamma(\alpha-m) = \frac{(-1)^{m}\Gamma(\alpha)}{(1-\alpha)_{m}} \qquad ...(2.6)$$

$$(2m)! = 2^{2m}\left(\frac{1}{2}\right)_m m! \qquad ...(2.7)$$

$$(2m+1)! = 2^{2m}\left(\frac{3}{2}\right)_m m! \qquad ...(2.8)$$

$$(\alpha)_{2m+1} = \alpha 2^{2m}\left(\frac{1}{2}\alpha+1\right)_m\left(\frac{1}{2}\alpha+\frac{1}{2}\right)_m \qquad ...(2.9)$$

3.0 MAIN RESULT

The following result for reducibility of Kampé de Fériet Function will be established in this section.

$$F^{p:1;1}_{q:1;1}\left[\begin{matrix}(a_p):d-2e+1;d \\ (b_q):2-2e;2e-1\end{matrix} \;\middle|\; x,-x\right]$$

$$= {}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p),\frac{1}{2}(a_p+1);\frac{1}{2}(2e-2d+1),\frac{1}{2}(1-2e+2d) \\ \frac{1}{2}(b_q),\frac{1}{2}(b_q+1);\frac{1}{2},\frac{1}{2}(2e-1),\frac{1}{2}(3-2e)\end{matrix} \;\middle|\; 4^{p-q-1}x^2\right]$$

$$+\frac{1}{2}\,{}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p),\frac{1}{2}(a_p+1);(e-d),(1-e+d)\\ \frac{1}{2}(b_q),\frac{1}{2}(b_q+1);(1-e),\frac{1}{2},e\end{matrix}\;\middle|\;4^{p-q-1}x^2\right]-\frac{x(d-2e+1)\prod_{i=1}^{p}(a_i)}{(2e-d-1)\prod_{i=1}^{q}(b_i)}$$

$$\left\{\frac{(e-d)}{(e-1)}\,{}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p+1),\frac{1}{2}(a_p+2);(e-d+1),(1-e+d)\\ \frac{1}{2}(b_q+1),\frac{1}{2}(b_q+2);\frac{3}{2},(2-e),e\end{matrix}\;\middle|\;4^{p-q-1}x^2\right]-\frac{(e-d-\frac{1}{2})}{2(e-\frac{1}{2})}\right.$$

$$\left.{}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p+1),\frac{1}{2}(a_p+2);\frac{1}{2}(2e-2d+1),\frac{1}{2}(3-2e+2d)\\ \frac{1}{2}(b_q+1),\frac{1}{2}(b_q+2);\frac{1}{2}(3-2e),\frac{3}{2},\frac{1}{2}(2e+1)\end{matrix}\;\middle|\;4^{p-q-1}x^2\right]\right\}$$

...(3.1)

4.0 DERIVATION

To prove (3.1), we proceed as follows:

Let

$$S=F^{p:1;1}_{q:1;1}\left[\begin{matrix}(a_p):d-2e+1;d\\ (b_q):2-2e;2e-1\end{matrix}\;\middle|\;x,-x\right]$$

It can be written in power series form as

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_p)_{m+n}(d-2e+1)_m(d)_n(-1)^n x^{m+n}}{(b_q)_{m+n}(2-2e)_m(2e-1)_n\,m!n!}$$

which on using (2.1) reduces to

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(a_p)_m(d-2e+1)_{m-n}(d)_n(-1)^n x^m}{(b_q)_m(2-2e)_{m-n}(2e-1)_n(m-n)!n!}$$

By virtue of relations (2.2) and (2.3), we have

$$S=\sum_{m=0}^{\infty}\frac{(a_p)_m(d-2e+1)_m x^m}{(b_q)_m(2-2e)_m\,m!}\,{}_3F_2\left[\begin{matrix}2e-1-m,-m,d\\ 2e-1,2e-d-m\end{matrix}\;\middle|\;1\right]$$

...(4.1)

On using (1.8) in (4.1), we get

$$ {}_3F_2\left[\begin{matrix}2e-1-m,-m,d\\ 2e-1,2e-d-m\end{matrix}\ \Big|\ 1\right] = \frac{(2e-1-m)_m(e-d+\frac{1}{2}-\frac{1}{2}m)_m}{(e-\frac{1}{2}-\frac{1}{2}m)_m(2e-d-m)_m} + \frac{(2e-1-m)_m(e-d-\frac{1}{2}m)_m}{2(2e-d-m)_m(e-\frac{1}{2}m)_m} $$

...(4.2)

Substituting the values from (4.2) to (4.1), we get

$$ S=\sum_{m=0}^{\infty}\frac{(a_p)_m(d-2e+1)_m x^m(2e-1-m)_m(e-d+\frac{1}{2}-\frac{1}{2}m)_m}{(b_q)_m(2-2e)_m m!(2e-d-m)_m(e-\frac{1}{2}-\frac{1}{2}m)_m} $$

$$ +\sum_{m=0}^{\infty}\frac{(a_p)_m(d-2e+1)_m x^m(2e-1-m)_m(e-d-\frac{1}{2}m)_m}{(b_q)_m(2-2e)_m m!2(2e-d-m)_m(e-\frac{1}{2}m)_m(2m+1)!} $$

$$ S=\sum_{m=0}^{\infty} A(m)\ \text{(Let)} $$

We know that

$$ \sum_{m=0}^{\infty} A(m)=\sum_{m=0}^{\infty} A(2m)+\sum_{m=0}^{\infty} A(2m+1) $$

...(4.3)

Now,

$$ \sum_{m=0}^{\infty} A(2m)=\sum_{m=0}^{\infty}\frac{(a_p)_{2m}(d-2e+1)_{2m}x^{2m}(2e-1-2m)_{2m}(e-d+\frac{1}{2}-m)_{2m}}{(b_q)_{2m}(2-2e)_{2m}2m!(2e-d-2m)_{2m}(e-\frac{1}{2}-m)_{2m}} $$

$$ +\sum_{m=0}^{\infty}\frac{(a_p)_{2m}(d-2e+1)_{2m}x^{2m}(2e-1-2m)_{2m}(e-d-m)_{2m}}{(b_q)_{2m}(2-2e)_{2m}2m!2(2e-d-2m)_{2m}(e-m)_{2m}} $$

using (2.4), (2.5), (2.6), and (2.7) in the above result, we get

$$ \sum_{m=0}^{\infty} A(2m)=\sum_{m=0}^{\infty}\frac{2^{2(p-q-1)m}(\frac{1}{2}a_p)_m(\frac{1}{2}a_p+\frac{1}{2})_m(e-d+\frac{1}{2})_m(\frac{1}{2}-e+d)_m x^{2m}}{(\frac{1}{2}b_q)_m(\frac{1}{2}b_q+\frac{1}{2})_m m!(\frac{1}{2})_m(e-\frac{1}{2})_m(\frac{3}{2}-e)_m} $$

$$+\sum_{m=0}^{\infty}\frac{2^{2(p-q-1)m}(\frac{1}{2}a_p)_m(\frac{1}{2}a_p+\frac{1}{2})_m(e-d)_m(1-e+d)_m x^{2m}}{2(\frac{1}{2}b_q)_m(\frac{1}{2}b_q+\frac{1}{2})_m m!(\frac{1}{2})_m(1-e)_m(e)_m}$$

Summing up the series, we finally have

$$\sum_{m=0}^{\infty}A(2m) = {}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p),\frac{1}{2}(a_p+1);\frac{1}{2}(2e-2d+1),\frac{1}{2}(1-2e+2d)\\ \frac{1}{2}(b_q),\frac{1}{2}(b_q+1);\frac{1}{2},\frac{1}{2}(2e-1),\frac{1}{2}(3-2e)\end{matrix}\ \middle|\ 4^{p-q-1}x^2\right]$$
$$+\frac{1}{2}\,{}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p),\frac{1}{2}(a_p+1);(e-d),(1-e+d)\\ \frac{1}{2}(b_q),\frac{1}{2}(b_q+1);(1-e),\frac{1}{2},e\end{matrix}\ \middle|\ 4^{p-q-1}x^2\right] \quad ...(4.4)$$

Also,

$$\sum_{m=0}^{\infty}A(2m+1) = \sum_{m=0}^{\infty}\frac{(a_p)_{2m+1}(d-2e+1)_{2m+1}x^{2m+1}(2e-2-2m)_{2m+1}(e-d-m)_{2m+1}}{(b_q)_{2m+1}(2-2e)_{2m+1}(2m+1)!(2e-d-1-2m)_{2m+1}(e-1-m)_{2m+1}}$$

$$+\sum_{m=0}^{\infty}\frac{(a_p)_{2m+1}(d-2e+1)_{2m+1}x^{2m+1}(2e-2-2m)_{2m+1}(e-d-\frac{1}{2}-m)_{2m+1}}{(b_q)_{2m+1}(2-2e)_{2m+1}(2m+1)!2(2e-d-1-2m)_{2m+1}(e-\frac{1}{2}-m)_{2m+1}}$$

Using (2.8) and (2.9), we get and after summing up the series finally we have

$$\sum_{m=0}^{\infty}A(2m+1) = -\frac{x(d-2e+1)\prod_{i=1}^{p}(a_i)}{(2e-d-1)\prod_{i=1}^{q}(b_i)}$$
$$\left\{\frac{(e-d)}{(e-1)}\,{}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p+1),\frac{1}{2}(a_p+2);(e-d+1),(1-e+d)\\ \frac{1}{2}(b_q+1),\frac{1}{2}(b_q+2);\frac{3}{2},(2-e),e\end{matrix}\ \middle|\ 4^{p-q-1}x^2\right]-\frac{(e-d-\frac{1}{2})}{2(e-\frac{1}{2})}\right.$$
$$\left.{}_{2p+2}F_{2q+3}\left[\begin{matrix}\frac{1}{2}(a_p+1),\frac{1}{2}(a_p+2);\frac{1}{2}(2e-2d+1),\frac{1}{2}(3-2e+2d)\\ \frac{1}{2}(b_q+1),\frac{1}{2}(b_q+2);\frac{1}{2}(3-2e),\frac{3}{2},\frac{1}{2}(2e+1)\end{matrix}\ \middle|\ 4^{p-q-1}x^2\right]\right\} \quad ...(4.5)$$

Substituting the values from (4.4) and (4.5) in (4.3), finally we get the desired result (3.1).

Clearly our main result (3.1) is closely related to (1.6).

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A TRANSFORMATION FORMULA FOR THE KAMPÉ DE FÉRIET FUNCTION - II

SHOUKAT ALI

Department of Mathematics, Govt. Engineering College Bikaner, Bikaner - 334 001

Rajasthan State, India

E-mail: dr.alishoukat@rediffmail.com

ABSTRACT

This paper is in continuation of the earlier paper by the author in which we have obtained an interesting transformation formula for the Kampé de Fériet function contiguous to that of obtained by Exton.

The aim of this research paper is to obtain one more interesting contiguous result to that of Exton.

Key Words: Kampé de Fériet function, Watson's theorem.

Mathematics Subject Classification (2000): Primary 33C20, 33C60; Secondary 33C65, 33C70.

1.0 INTRODUCTION

We recall the definition of generalized Kampé de Fériet's function as follows [5]

$$F^{p:q;k}_{\ell:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k)\\(\alpha_\ell):(\beta_m);(\gamma_n)\end{matrix}\;\middle|\;\begin{matrix}x\\y\end{matrix}\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s\,x^r y^s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s\,r!s!} \quad ...(1.1)$$

Where for convergence

(i) $p+q<\ell+m+1,\ p+k<\ell+n+1,\ |x|<\infty,\ |y|<\infty$

or

(ii) $p+q<\ell+m+1,\ p+k<\ell+n+1$ and

$$\begin{cases}|x|^{\frac{1}{p-\ell}}+|y|^{\frac{1}{p-\ell}}<1, \text{if } p>\ell\\ \max\{|x|,|y|\}<1, \text{if } p\le\ell\end{cases}$$

Although the double hypergeometric series defined by (1.1) reduces to the Kampé de Fériet function in the special case: $q=k$ and $m=n$

yet it is usually referred to in the literature as the Kampé de Fériet series.

The following are the cases in which the Kampé de Fériet function defined in (1.1) can be expressed in terms of generalized hypergeometric series.

$$F^{p:0;0}_{q:0;0}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix} \;\middle|\; \begin{matrix}x\\ y\end{matrix}\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix} \;\middle|\; x+y\right] \quad \ldots(1.2)$$

$$F^{0:p;r}_{0:q;s}\left[\begin{matrix}-;\alpha_1,\ldots,\alpha_p;\gamma_1,\ldots,\gamma_r\\ -;\beta_1,\ldots,\beta_q;\delta_1,\ldots,\delta_s\end{matrix} \;\middle|\; \begin{matrix}x\\ y\end{matrix}\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix} \;\middle|\; x\right] {}_rF_s\left[\begin{matrix}\gamma_1,\ldots,\gamma_r\\ \delta_1,\ldots,\delta_s\end{matrix} \;\middle|\; y\right] \quad \ldots(1.3)$$

$$F^{p:1;1}_{q:0;0}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\nu;\sigma\\ \beta_1,\ldots,\beta_q;-;-\end{matrix} \;\middle|\; \begin{matrix}x\\ x\end{matrix}\right] = {}_{p+1}F_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\nu+\sigma\\ \beta_1,\ldots,\beta_q\end{matrix} \;\middle|\; x\right] \quad \ldots(1.4)$$

$$F^{p:0;0}_{q:1;1}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;-;-\\ \beta_1,\ldots,\beta_q;\nu;\sigma\end{matrix} \;\middle|\; \begin{matrix}x\\ x\end{matrix}\right] = {}_{p+2}F_{q+3}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,\Delta(2;\nu+\sigma-1)\\ \beta_1,\ldots,\beta_q,\nu,\sigma,\nu+\sigma-1\end{matrix} \;\middle|\; 4x\right] \quad \ldots(1.5)$$

where, and in what follows, $\Delta(\ell;\lambda)$ abbreviates the array of ℓ parameters

$$\frac{\lambda}{\ell},\frac{(\lambda+1)}{\ell},\ldots,\frac{(\lambda+\ell-1)}{\ell},\quad \ell=1,2,3,\ldots$$

For more detail see [5, pp. 28-32].

Very recently, the author [1] has obtained an interesting case of transformation of Kampé de Fériet Function closely related to the result (1.7) by employing contiguous Watson's theorem obtained earlier by Lavoie, Grondin and Rathie [4]. In this paper we have obtained one more interesting result for the transformation of Kampé de Fériet Function.

In 1997, Exton [3] obtained the following results for the Kampé de Fériet Function [2].

$$(1-x)^{-1-z_1-z_2}\; F^{1:c+2}_{1:\rho}\left[\begin{matrix}d;c_1,\frac{1}{2}z_1,\frac{1}{2}z_1+\frac{1}{2};c_2,\frac{1}{2}z_2,\frac{1}{2}z_2+\frac{1}{2}\\ d-1;\rho_1;\rho_2\end{matrix} \;\middle|\; \frac{4xy_1}{(1-x)^2},\frac{4xy_2}{(1-x)^2}\right]$$

$$-\frac{(d-z_1-z_2-1)}{(d-1)}x(1-x)^{-1-z_1-z_2}$$

$$F^{1:c+2}_{1:\rho}\left[\begin{matrix}z_1+z_2-d+2;c_1,\frac{1}{2}z_1,\frac{1}{2}z_1+\frac{1}{2};c_2,\frac{1}{2}z_2,\frac{1}{2}z_2+\frac{1}{2}\\ z_1+z_2-d+1;\rho_1;\rho_2\end{matrix} \;\middle|\; \frac{4xy_1}{(1-x)^2},\frac{4xy_2}{(1-x)^2}\right]$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(d)_{m_1+m_2} (z_1)_{m_1} (z_2)_{m_2} x^{m_1+m_2}}{(d-1)_{m_1+m_2} m_1! m_2!} \; {}_{c+2}F_{\rho}\left[\begin{matrix} c_1, z_1+m_1, -m_1 \\ \rho_1 \end{matrix} \mid -y_1\right] \; {}_{c+2}F_{\rho}\left[\begin{matrix} c_2, z_2+m_2, -m_2 \\ \rho_2 \end{matrix} \mid -y_2\right] \quad ...(1.6)$$

and

$$F_{1:1}^{1:2}\left[\begin{matrix} d; c_1, \frac{1}{2}z_1; c_2, \frac{1}{2}z_2 \\ d-1; 2c_1; 2c_2 \end{matrix} \mid \frac{-4x}{(1-x)^2}, \frac{-4x}{(1-x)^2}\right] - \frac{(d-z_1-z_2-1)}{(d-1)} x \; F_{1:1}^{1:2}\left[\begin{matrix} z_1+z_2-d+2; c_1, \frac{1}{2}z_1; c_2, \frac{1}{2}z_2 \\ z_1+z_2-d+1; 2c_1; 2c_2 \end{matrix} \mid \frac{-4x}{(1-x)^2}, \frac{-4x}{(1-x)^2}\right]$$

$$= (1-x)^{1+z_1+z_2} \; F_{1:1}^{1:2}\left[\begin{matrix} \frac{1}{2}d+\frac{1}{2}; \frac{1}{2}z_1, \frac{1}{2}z_1+\frac{1}{2}-c_1; \frac{1}{2}z_2, \frac{1}{2}z_2+\frac{1}{2}-c_2 \\ \frac{1}{2}d-\frac{1}{2}; \frac{1}{2}+c_1; \frac{1}{2}+c_2 \end{matrix} \mid x^2, x^2\right] \quad ..(1.7)$$

Exton has obtained the result with the help of classical Watson's theorem on the sum of a ${}_3F_2$ viz.

$${}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(1+a+b), 2c \end{matrix} \mid 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a+c)} \cdot \frac{\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma(\frac{1}{2}-\frac{1}{2}b+c)} \quad ...(1.8)$$

provided $\Re e\,(2c-a-b) > -1$

In 1992, Lavoie, Grondin and Rathie [4] have obtained a large number of summation formulae closely related to (1.8) of which one is given below:

$${}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(1+a+b), 2c+1 \end{matrix} \mid 1\right] = \frac{2^{a+b-2}\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \left[\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})} - \frac{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma(c-\frac{1}{2}a+1)\Gamma(c-\frac{1}{2}b+1)}\right] \quad ...(1.9)$$

provided $\Re e\,(2c - a - b) > -1$

The aim of this research note is to obtain one result closely related to (1.7) by employing the summation formula (1.9).

2.0 RESULTS REQUIRED

The following results will be required in our present investigations.

$$(2m)\,! = 2^{2m}\left(\frac{1}{2}\right)_m m\,! \qquad ...(2.1)$$

$$(\alpha)_{2m} = 2^{2m}\left(\frac{1}{2}\alpha\right)_m\left(\frac{1}{2}\alpha+\frac{1}{2}\right)_m \qquad ...(2.2)$$

$$\Gamma(\alpha - m) = \frac{(-1)^m\,\Gamma(\alpha)}{(1-\alpha)_m} \qquad ...(2.3)$$

3.0 MAIN RESULT

The following result for transformation of Kampé de Fériet Function will be established in this section.

$$F^{1:2}_{1:1}\left[\begin{matrix} d; c_1, \frac{1}{2}z_1; c_2, \frac{1}{2}z_2 \\ d-1; 2c_1+1; 2c_2 \end{matrix} \;\middle|\; \frac{-4x}{(1-x)^2}, \frac{-4x}{(1-x)^2}\right] - \frac{(d - z_1 - z_2 - 1)}{(d-1)}\, x$$

$$F^{1:2}_{1:1}\left[\begin{matrix} z_1+z_2-d+2; c_1, \frac{1}{2}z_1; c_2, \frac{1}{2}z_2 \\ z_1+z_2-d+1; 2c_1+1; 2c_2 \end{matrix} \;\middle|\; \frac{-4x}{(1-x)^2}, \frac{-4x}{(1-x)^2}\right]$$

$$= (1-x)^{1+z_1+z_2}\, F^{1:2}_{1:1}\left[\begin{matrix} \frac{1}{2}d+\frac{1}{2}; \frac{1}{2}z_1, \frac{1}{2}z_1+\frac{1}{2}-c_1; \frac{1}{2}z_2, \frac{1}{2}z_2+\frac{1}{2}-c_2 \\ \frac{1}{2}d-\frac{1}{2}; \frac{1}{2}+c_1; \frac{1}{2}+c_2 \end{matrix} \;\middle|\; x^2, x^2\right]$$

$$+ \frac{(1-x)^{1+z_1+z_2}}{(2c_1+1)}\, F^{1:3:2}_{1:2:1}\left[\begin{matrix} \frac{1}{2}d+\frac{1}{2}; \frac{1}{2}z_1, \frac{3}{2}, \frac{1}{2}z_1+\frac{1}{2}-c_1; \frac{1}{2}z_2, \frac{1}{2}z_2+\frac{1}{2}-c_2 \\ \frac{1}{2}d-\frac{1}{2}; \frac{1}{2}, \frac{3}{2}+c_1; \frac{1}{2}+c_2 \end{matrix} \;\middle|\; x^2, x^2\right] \qquad ...(3.1)$$

4.0 DERIVATION

To prove (3.1), we proceed as follows:

If we set $c = 1$, $\rho = 2$, $y_1 = y_2 = -1$ in (1.6), we get

$$F^{1:3}_{1:2}\left[\begin{matrix} d; c_1, \frac{1}{2}z_1, \frac{1}{2}z_1+\frac{1}{2}; c_2, \frac{1}{2}z_2, \frac{1}{2}z_2+\frac{1}{2} \\ d-1; \rho_{1,1}, \rho_{2,1}; \rho_{1,2}, \rho_{2,2} \end{matrix} \;\middle|\; \frac{-4x}{(1-x)^2}, \frac{-4x}{(1-x)^2}\right] - \frac{(d-z_1-z_2-1)}{(d-1)}x$$

$$F^{1:3}_{1:2}\left[\begin{matrix} z_1+z_2-d+2; c_1, \frac{1}{2}z_1, \frac{1}{2}z_1+\frac{1}{2}; c_2, \frac{1}{2}z_2, \frac{1}{2}z_2+\frac{1}{2} \\ z_1+z_2-d+1; \rho_{1,1}, \rho_{2,1}; \rho_{1,2}, \rho_{2,2} \end{matrix} \;\middle|\; \frac{-4x}{(1-x)^2}, \frac{-4x}{(1-x)^2}\right]$$

$$= (1-x)^{1+z_1+z_2} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(d)_{m_1+m_2} (z_1)_{m_1} (z_2)_{m_2} x^{m_1+m_2}}{(d-1)_{m_1+m_2}\, m_1! m_2!} \; {}_3F_2\left[\begin{matrix} c_1, z_1+m_1, -m_1 \\ \rho_{1,1}, \rho_{2,1} \end{matrix} \;\middle|\; 1\right]$$

$${}_3F_2\left[\begin{matrix} c_2, z_2+m_2, -m_2 \\ \rho_{1,2}, \rho_{2,2} \end{matrix} \;\middle|\; 1\right]$$

Let $\rho_{1,1} = 2c_1 + 1$, $\rho_{1,2} = 2c_2$, $\rho_{2,1} = \frac{1}{2}z_1 + \frac{1}{2}$ and $\rho_{2,2} = \frac{1}{2}z_2 + \frac{1}{2}$, we get

$$F^{1:3}_{1:2}\left[\begin{matrix} d; c_1, \frac{1}{2}z_1, \frac{1}{2}z_1+\frac{1}{2}; c_2, \frac{1}{2}z_2, \frac{1}{2}z_2+\frac{1}{2} \\ d-1; 2c_1+1, \frac{1}{2}z_1+\frac{1}{2}; 2c_2, \frac{1}{2}z_2+\frac{1}{2} \end{matrix} \;\middle|\; \frac{-4x}{(1-x)^2}, \frac{-4x}{(1-x)^2}\right] - \frac{(d-z_1-z_2-1)}{(d-1)}x$$

$$F^{1:3}_{1:2}\left[\begin{matrix} z_1+z_2-d+2; c_1, \frac{1}{2}z_1, \frac{1}{2}z_1+\frac{1}{2}; c_2, \frac{1}{2}z_2, \frac{1}{2}z_2+\frac{1}{2} \\ z_1+z_2-d+1; 2c_1+1, \frac{1}{2}z_1+\frac{1}{2}; 2c_2, \frac{1}{2}z_2+\frac{1}{2} \end{matrix} \;\middle|\; \frac{-4x}{(1-x)^2}, \frac{-4x}{(1-x)^2}\right]$$

$$= (1-x)^{1+z_1+z_2} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{(d)_{m_1+m_2} (z_1)_{m_1} (z_2)_{m_2} x^{m_1+m_2}}{(d-1)_{m_1+m_2}\, m_1! m_2!} \; {}_3F_2\left[\begin{matrix} -m_1, z_1+m_1, c_1 \\ \frac{1}{2}z_1+\frac{1}{2}, 2c_1-1 \end{matrix} \;\middle|\; 1\right]$$

$${}_3F_2\left[\begin{matrix} -m_2, z_2+m_2, c_2 \\ \frac{1}{2}z_2+\frac{1}{2}, 2c_2 \end{matrix} \;\middle|\; 1\right]$$

Using the results (1.8) and (1.9), we have

$$F^{1:2}_{1:1}\left[\begin{matrix} d; c_1, \frac{1}{2}z_1; c_2, \frac{1}{2}z_2 \\ d-1; 2c_1+1; 2c_2 \end{matrix} \;\middle|\; \frac{-4x}{(1-x)^2}, \frac{-4x}{(1-x)^2}\right] - \frac{(d-z_1-z_2-1)}{(d-1)}x$$

$$F^{1:2}_{1:1}\left[\begin{matrix} z_1+z_2-d+2; c_1, \frac{1}{2}z_1; c_2, \frac{1}{2}z_2 \\ z_1+z_2-d+1; 2c_1+1; 2c_2 \end{matrix} \;\middle|\; \frac{-4x}{(1-x)^2}, \frac{-4x}{(1-x)^2}\right]$$

$$= (1-x)^{1+z_1+z_2} \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{(d)_{m_1+m_2}(z_1)_{m_1}(z_2)_{m_2} x^{m_1+m_2}}{(d-1)_{m_1+m_2}\, m_1!m_2!} \frac{2^{z_1-2}\Gamma(\frac{1}{2}z_1+\frac{1}{2})\Gamma(c_1+\frac{1}{2})\Gamma(c_1-\frac{1}{2}z_1+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(-m_1)\Gamma(z_1+m_1)}$$

$$\left[\frac{\Gamma(-\frac{1}{2}m_1)\Gamma(\frac{1}{2}z_1+\frac{1}{2}m_1)}{\Gamma(c_1+\frac{1}{2}m_1+\frac{1}{2})\Gamma(c_1-\frac{1}{2}z_1-\frac{1}{2}m_1+\frac{1}{2})} - \frac{\Gamma(\frac{1}{2}z_1+\frac{1}{2}m_1+\frac{1}{2})\Gamma(\frac{1}{2}-\frac{1}{2}m_1)}{\Gamma(c_1+1+\frac{1}{2}m_1)\Gamma(c_1-\frac{1}{2}z_1+1-\frac{1}{2}m_1)}\right]$$

$$\frac{\Gamma(\frac{1}{2})\Gamma(c_2+\frac{1}{2})\Gamma(\frac{1}{2}z_2+\frac{1}{2})\Gamma(\frac{1}{2}-\frac{1}{2}z_2+c_2)}{\Gamma(\frac{1}{2}-\frac{1}{2}m_2)\Gamma(\frac{1}{2}+\frac{1}{2}z_2+\frac{1}{2}m_2)\Gamma(\frac{1}{2}+\frac{1}{2}m_2+c_2)\Gamma(\frac{1}{2}-\frac{1}{2}z_2-\frac{1}{2}m_2+c_2)}$$

Replacing m_1 by $2\,m_1$ and m_2 by $2\,m_2$ and after making little simplification, we have

$$= (1-x)^{1+z_1+z_2} \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{(d)_{2m_1+2m_2}(z_1)_{2m_1}(z_2)_{2m_2} x^{2m_1+2m_2}}{(d-1)_{2m_1+2m_2}\, 2m_1!2m_2!}$$

$$\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\frac{1}{2}z_2+c_2)}{\Gamma(\frac{1}{2}-m_2)(\frac{1}{2}+c_2)_{m_2}(\frac{1}{2}+\frac{1}{2}z_2)_{m_2}\Gamma(\frac{1}{2}-\frac{1}{2}z_2+c_2-m_2)}$$

$$\left[\frac{(\frac{1}{2})_{m_1}(\frac{1}{2}-c_1+\frac{1}{2}z_1)_{m_1}}{(c_1+\frac{1}{2})_{m_1}(\frac{1}{2}+\frac{1}{2}z_1)_{m_1}} + \frac{(\frac{3}{2})_{m_1}(\frac{1}{2}-c_1+\frac{1}{2}z_1)_{m_1}}{2(c_1+\frac{3}{2})_{m_1}(c_1+\frac{1}{2})(\frac{1}{2}+\frac{1}{2}z_1)_{m_1}}\right]$$

On using the results (2.1), (2.2) and (2.3), we have

$$F_{1:1}^{1:2}\left[\begin{matrix} d;c_1,\frac{1}{2}z_1;c_2,\frac{1}{2}z_2 \\ d-1;2c_1+1;2c_2 \end{matrix} \mid \frac{-4x}{(1-x)^2},\frac{-4x}{(1-x)^2}\right] - \frac{(d-z_1-z_2-1)}{(d-1)}x$$

$$F_{1:1}^{1:2}\left[\begin{matrix} z_1+z_2-d+2;c_1,\frac{1}{2}z_1;c_2,\frac{1}{2}z_2 \\ z_1+z_2-d+1;2c_1+1;2c_2 \end{matrix} \mid \frac{-4x}{(1-x)^2},\frac{-4x}{(1-x)^2}\right]$$

$$= (1-x)^{1+z_1+z_2} \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{2^{2m_1+2m_2}(\frac{1}{2}d)_{m_1+m_2}(\frac{1}{2}d+\frac{1}{2})_{m_1+m_2}}{2^{2m_1+2m_2}(\frac{1}{2}d)_{m_1+m_2}(\frac{1}{2}d-\frac{1}{2})_{m_1+m_2}}$$

$$\frac{2^{2m_1}(\frac{1}{2}z_1)_{m_1}(\frac{1}{2}+\frac{1}{2}z_1)_{m_1}2^{2m_2}(\frac{1}{2}z_2)_{m_2}(\frac{1}{2}+\frac{1}{2}z_2)_{m_2}x^{2m_1+2m_2}}{2^{2m_1}m_1!(\frac{1}{2})_{m_1}2^{2m_2}m_2!(\frac{1}{2})_{m_2}}$$

$$\left[\frac{(\frac{1}{2})_{m_1}(\frac{1}{2}-c_1+\frac{1}{2}z_1)_{m_1}}{(c_1+\frac{1}{2})_{m_1}(\frac{1}{2}+\frac{1}{2}z_1)_{m_1}} + \frac{(\frac{3}{2})_{m_1}(\frac{1}{2}-c_1+\frac{1}{2}z_1)_{m_1}}{2(c_1+\frac{3}{2})_{m_1}(c_1+\frac{1}{2})(\frac{1}{2}+\frac{1}{2}z_1)_{m_1}}\right]$$

$$\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\frac{1}{2}z_2+c_2)(\frac{1}{2})_{m_2}(\frac{1}{2}+\frac{1}{2}z_2-c_2)_{m_2}}{(-1)^{m_2}\Gamma(\frac{1}{2})(\frac{1}{2}+\frac{1}{2}z_2)_{m_2}(\frac{1}{2}+c_2)_{m_2}(-1)^{m_2}\Gamma(\frac{1}{2}-\frac{1}{2}z_2+c_2)}$$

After summing up the series, finally we get the desired result (3.1). Clearly our main result (3.1) is closely related to (1.7).

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GRAVITATIONAL EFFECT ON THERMAL INSTABILITY OF MAXWELL VISCO-ELASTIC FLUID IN POROUS

Dr. RAMESH CHAND

Assistant Professor,

Deptt. of Mathematics, Govt. P.G. College

Dhaliara Distt. Kangra (H.P.) India177103

Email: rameshnahan@yahoo.com

ABSTRACT

The effect of variable gravity on the thermal instabilityof Maxwellvisco-elastic fluid in porous medium is investigated. A linear stability analysis based upon normal mode analysis is used to find solution of the fluid layer confined between twofree boundaries.It is found that in case of stationary convection, Maxwell visco-elastic fluid behaves like an ordinary Newtonian fluid. The effects of the variable and medium permeabilityon stationary convection are investigated. The principle of exchange of stabilities for the problem is satisfied under certain condition.

1.0 INTRODUCTION

The subject of thermal instability in porous medium has been studied extensively in recent years. The problem of convective instability of visco-elastic fluid heated from below was first studied by Green (1968). Vest and Arpaci (1969) have investigated the problem of overstability in a horizontal layer of a Visco-elastic fluid heated from below. The problem of thermal convection in fluids in porous medium is of considerable importance in geophysics, soil sciences, found water hydrology and astrophysics. The physical properties of comets, meteorites and interplanetary dust strongly suggest the importance of porosity in astrophysical context [McDonnel (1978)]. The physics of flow through porous medium has been given in a treatise by Scheidegger (1960). The Rayleigh instability of a thermal boundary layer in flow in porous medium is studied by Wooding (1960). Such problem arises in oceanography, limnology and engineering. The idealization of uniform gravity assumed in theoretical investigations, although valid for laboratory purposes, can scarcely be justified for large-scale convection phenomena occurring in atmosphere, the ocean or mantle of the earth. It then becomes imperative to consider gravity as variable quantity varying with distance from surface or reference point. G.K. Pradhan et. al (1989) studied the thermal instability of a fluid layer in a variable gravitational field and found that variable gravity has destabilizing effect on the fluid layer. In the present paper an attempt has been made to effect of variable gravity on the thermal instability of Maxwell visco-elastic fluid in porous medium.

2.0 FORMULATION OF PROBLEM AND PERTURBATION EQUATIONS

Consider an infinite horizontal layer of Maxwellvisco- elastic fluid of thickness'd' bounded by plane $z = 0$ and $z = d$ in porous medium of porosity ε and medium permeability k_1. The layer is heated from below such that a uniform temperature gradient $\beta\left(= \left|\frac{dT}{dz}\right|\right)$, where T is temperature. The system is acted upon by linear variable gravity force $\vec{g}\,(0,0,g(z))$, where $g(z) = g_0(1 + Mz) > 0$, M is gravity parameter and g_0 is the value of g at $z = 0$.

Let $p,\rho,T,\alpha,\mu,\upsilon$ and κ be the pressure, density, temperature and thermal coefficient of expansion, viscosity, kinematic viscosity and thermal diffusivity of fluid respectively.

As the fluid flow through a porous medium the gross effect is represented by Darcy's law. According to which the usually viscous term is replaced by the resistance term $-\left(\frac{\mu}{k_1}\right)\vec{q}$, in the equation of motion,where $\vec{q}$ is filter velocity of fluid.The fluid velocity $\vec{v}$ and filter velocity $\vec{q}$ are connected by relation $\vec{v} = \frac{\vec{q}}{\varepsilon}$.

The equation of motion, continuity and heat conduction for Maxwellvisco-elastic fluid through porous medium are

$$\frac{\rho}{\varepsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{d\vec{q}}{dt} = \left(1+\lambda\frac{\partial}{\partial t}\right)[-\nabla p + \rho\vec{g}] - \frac{\mu}{k_1}\vec{q}, \qquad \ldots(1)$$

$$\nabla.\vec{q} = 0, \qquad \ldots(2)$$

$$E\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T = \kappa\nabla^2 T, \qquad \ldots(3)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon}(\vec{q}\nabla)$ stands for convection derivative.

The equation of state is

$$\rho = \rho_0[1 - \alpha(T - T_0)] \qquad \ldots(4)$$

where the suffix zero refers to values at reference level $z = 0$, i.e. ρ_0, T_0 stands for density, temperature at lower boundary $z = 0$.

The steady state solution is $\vec{q} = (0,0,0), T = T_0 - \beta z, \rho = \rho_0(1 + \alpha\beta z)$,

Let $\delta\rho, \delta p, \theta$ denote respectively the perturbation in density, pressure and temperature.

Then the linearised perturbations equation of flow through porous medium, following the Boussineq approximations are,

$$\frac{\rho_0}{\varepsilon}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{d\vec{q}}{dt} = \left(1+\lambda\frac{\partial}{\partial t}\right)[-\nabla\delta p + \vec{g}\delta\rho] - \frac{\mu}{k_1}\vec{q}, \qquad \ldots(5)$$

$$\nabla.\vec{q} = 0, \qquad \ldots(6)$$

$$E\frac{\partial\theta}{\partial t} = \beta w + \kappa\nabla^2\theta, \qquad \ldots(7)$$

The change in density $\delta\rho$ caused by the perturbation in temperature θ is given by

$$\delta\rho = -\rho_0\alpha\theta. \qquad \ldots(8)$$

In the cartesian form equation (5) – (7) can be written as

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial}{\partial x}\delta p\right] = -\frac{\nu}{k_1}u, \qquad \ldots(9)$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial v}{\partial t} + \frac{1}{\rho_0}\frac{\partial}{\partial y}\delta p\right] = -\frac{\nu}{k_1}v, \qquad \ldots(10)$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial w}{\partial t} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\delta p - g_0\alpha\theta(1+Mz)\right] = -\frac{\nu}{k_1}w, \qquad \ldots(11)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad \ldots(12)$$

$$E\frac{\partial\gamma}{\partial t} = \beta w + \kappa\nabla^2\theta, \qquad \ldots(13)$$

Operating equation (9) by $\frac{\partial}{\partial x}$ and equation (10) by $\frac{\partial}{\partial y}$; then adding and making use of equation (12), we get

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right)-\frac{1}{\rho_0}\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\delta p\right]=-\frac{\nu}{k_1}\frac{\partial w}{\partial z}. \quad ...(14)$$

Now eliminating δp from (11) and (14), we get

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{1}{\varepsilon}\frac{\partial}{\partial t}\left(\nabla^2 w\right)-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)g_0\alpha\theta\,(1+Mz)\right]=-\frac{\nu}{k_1}\nabla^2 w. \quad ...(15)$$

Equations (13) can be written as

$$\left\{E\frac{\partial}{\partial t}-\kappa\nabla^2\right\}\theta=\beta w. \quad ...(16)$$

3.0 DISPERSION RELATION

Analyzing the disturbances into the normal modes and assuming that the perturbed quantities are of the form

$$[w,\theta]=[W(z),\Theta(z)]\exp\left(ik_x x+ik_y y+nt\right), \quad ...(17)$$

where k_x, k_y are horizontal wave numbers in x and y direction respectively, $k^2=k_x^2+k_y^2$ is the resultant wave number, n is growth rate of disturbances.

Using equation (18), equations (15) – (17) becomes

$$(1+\lambda n)\left[\frac{1}{\varepsilon}n\left(\frac{d^2}{dz^2}-k^2\right)W+g_0k^2\alpha\theta\,(1+Mz)\right]=-\frac{\nu}{k_1}\left\{\frac{d^2}{dz^2}-k^2\right\}w, \quad ...(18)$$

$$\left\{En-\kappa\left(\frac{d^2}{dz^2}-k^2\right)\right\}\Theta=\beta W, \quad ...(19)$$

Expressing the coordinate (x, y, z) = (x*d, y*d, z*d), D* = d/dz* in new unit of length'd' thereafter dropping the superscript for simplicity and also putting

a = kd, $\sigma=\frac{nd^2}{\nu}$, $p_1=\frac{\nu}{\kappa}$ is the Prandtl number, $P_l=\frac{k_1}{d^2}$ is the dimensionless medium permeability and $F=\frac{\lambda\nu}{d^2}$.

Equations (18) – (19) with the help of equation (17) in non-dimensional form can be written as

$$\left[\frac{\sigma}{\varepsilon}+\frac{(1+F\sigma)^{-1}}{P_l}\right](D^2-a^2)W=-\frac{g_0a^2d^2}{\nu}(1+Mz)\alpha\Theta, \quad ...(20)$$

$$\left[D^2-a^2-\sigma E_1p_1\right]\Theta=-\frac{\beta d^2}{\kappa}W \quad ...(21)$$

we consider the case where both the boundaries are free and perfect conductor of heat, while adjoining medium is assumed to be electrically non-conducting. Thus boundary conditions for this case are

$W=D^2W=\Theta=0$ at z = 0 and z = 1. ...(22)

Eliminating Θ between (20) – (21), we get

$$\left\langle(D^2-a^2-\sigma E_1p_1)(D^2-a^2)\left[\frac{\sigma}{\varepsilon}+\frac{(1+F\sigma)^{-1}}{P_l}\right]\right\rangle W=a^2R(1+Mz)W. \quad ...(23)$$

where $R=\frac{g_0\alpha\beta d^4}{\kappa\nu}$ is the thermal Rayleigh number.

Using the boundary conditions (22) it can be shown that all the even order derivative of W vanish at the boundary and hence the proper solution of equation (23) characterizing lowest mode is

$$W = W_0 \sin \pi z, \qquad \ldots(24)$$

where W_0 is constant. Substituting the (24) in equation (23) and letting

$$a^2 = \pi^2 x, R_1 = \frac{R}{\pi^4}, i\sigma = \frac{\sigma}{\pi^2} \text{ and } P = \pi^2 P_l \cdot$$

We obtain the following dispersion relation

$$R_1 = \frac{(1 + x + i\sigma_1 E_1 p_1)(1 + x)\left\{\left(\frac{i\sigma_1}{\varepsilon} + \frac{(1 + i\sigma_1 \pi^2 F)^{-1}}{P}\right)\right\}}{x\left(\frac{M}{4} + \frac{1}{2}\right)}. \qquad \ldots(25)$$

4.0 STATIONARY CONVECTION

When the instability sets in as a stationary convection, the marginal state will be characterized by $\sigma = 0$. On putting $\sigma = 0$ ($\sigma_1 = 0$) in equation (25) it reduces to $R_1 = \frac{(1+x)^2}{x}\left(\frac{4}{2+M}\right)\frac{1}{P}$ ...(26)

Thus in the stationary convection the visco- elastic parameter F vanishes with σ and thus Maxwellian Visco-elastic fluid behaves like an ordinary Newtonian fluid.

To study the effect of variable gravity field, rotation and medium permeability, we examine the nature of $\frac{\partial R_1}{\partial M}$ and $\frac{\partial R_1}{\partial P}$ analytically.

Equation (26) yield,

$$\frac{\partial R_1}{\partial M} = -\left(\frac{2}{2 + M}\right)^2 \frac{(1 + x)^2}{xP} < 0,$$

thus variable gravity has destabilizing effect on the thermal convection in porous medium. This destabilizing effect is an agreement of the earlier work of G.K. Pradhan et. al (1989)for the Newtonian fluids.

$$\frac{\partial R_1}{\partial P} = -\frac{(1 + x)^2}{x}\left(\frac{4}{2 + M}\right)\frac{1}{P^2} < 0,$$

thus medium permeability have destabilizing effect on the thermal convection in porous medium.

5.0 OSCILLATORY MODES AND THE 'PRINCIPLE OF EXCHANGE OF STABILITIES'

Here we examine the possibility of oscillatory modes, if any, on the stability problem due to suspended particles, variable gravity field and medium permeability. Multiplying the equation (23) by W* (the complex conjugate of W), integrating over range of z and making use boundary condition (22); we get

$$\left[\frac{\dot{\sigma}}{\varepsilon} + \frac{(1 + F\sigma)^{-1}}{P_l}\right]\left\{\int_0^1 |(D^2 - a^2)W|^2 dz + \sigma E_1 p_1 \int_0^1 (|DW|^2 + a^2|W^2|)dz\right\} - Ra^2\int_0^1 (1 + Mz)|W|^2 dz = 0 \cdot \qquad \ldots(27)$$

Now for neutral mode, we must have $\sigma = i\sigma_i$ with σ_i is real. The real and imaginary part of equation(27)

$$\frac{1}{P_l}\left(\frac{1}{1+\sigma_i^2 F^2}\right)\left\{\int_0^1 (|(D^2 - a^2)W|^2)dz + (F - 1)\sigma_i^2 E_1 p_1 \int_0^1 (|DW|^2 + a^2|W|^2)dz\right\} - Ra^2\int_0^1 (1 + Mz)|W|^2 dz = 0, \qquad (28) \text{ and}$$

$$\sigma_i\left[\frac{1}{\varepsilon} - \frac{F}{P_l}\left(\frac{1}{1+\sigma_i^2 F^2}\right)\int_0^1 |(D^2 - a^2)W|^2 dz + \frac{E_1 P_1}{P_l}\left(\frac{1}{1+\sigma^2 F^2}\right)\int_0^1 |DW|^2 + a^2|W|^2 dz\right] = 0. \qquad \ldots(29)$$

Equation (29) it follow that σ_i=0 or σ_i # 0 which mean that modes may be non oscillatory or oscillatory.

The term inside the bracket is non zero if $\frac{1}{\varepsilon} > \frac{F}{P_l}$, which implies that σ_i =0, thus the mode are non oscillatory and principle of exchange of stabilities is satisfied. Thus $\frac{1}{\varepsilon} > \frac{F}{P_l}$ is the necessary condition for the validity of principle of exchange of stabilities for Maxwellian visco-elastic fluid in porous.

CONCLUSION

The effect of linear variable gravitational field of a rotating Maxwellvisco-elastic fluid heated from below porous medium has been investigated. From the analysis, the main conclusions are as follows:

(i) In case of stationary convection, Maxwellian visco-elastic fluid behaves like an ordinary Newtonian fluid.

(ii) The variable gravity field and medium permeability have destabilizing effect on the system.

(iii) It is also found that modes may be non oscillatory or oscillatory.

(iv) The principle of exchange of stabilities is valid if $\frac{1}{\varepsilon} > \frac{F}{P_l}$.

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THE SUMMABILITY OF CESARO MEAN OF THE ULTRASPHERICALSERIES

By

ALOK VERMA* & K. AHMAD**

**Department of Mathematics*
Govt. J.P. Verma P.G. College,
Bilaspur (C.G) 495001
&
***Department of Mathematics*
Govt. Agrasen College,
Bilha(C.G)

ABSTRACT

In the present paper we have obtained a theorem for Cesaro means of ultraspherical series which extend and generalize the results of Wang [5 and 6] of Fourier series.

1. Let $f(\theta,\phi)$ be a function defined for the range $0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$ on a sphere S. The ultraspherical series associated with the function is

$$f(\theta,\phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+\lambda) \iint_s \frac{P_n^{(\lambda)}(\cos\omega) f(\theta',\phi') d\sigma'}{\left[\sin^2\theta' \sin^2(\phi-\phi')\right]^{1/2-\lambda}}, \lambda > 0 \qquad \text{...(1.1)}$$

where w is the spherical distance between the points (θ',ϕ'); i.e.

$$\cos w = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')$$

and $$d\sigma' = \sin\theta' d\theta' d\phi'.$$

The Laplace series is a particular case of the series of (1.1) for $\lambda = \frac{1}{2}$, while it reduces to the trigonometric series in the limit as $\lambda \to 0$, because

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} p_n^{(\lambda)}(\cos\theta) = \frac{2}{n} \cos n\theta, \lambda \ge 1 \qquad \text{...(1.2)}$$

The ultraspherical polynomials $p_n^{(\lambda)}(x)$ are defined by the following expansion.

$$\left[1 - 2xz + z^2\right]^{-\lambda} = \sum_{n=0}^{\infty} z^n p_n^{(\lambda)}(x),\ \lambda > 0 \qquad \text{...(1.3)}$$

We suppose throughout that the function

$$f(\theta',\phi')\left[\sin^2\theta' \sin^2(\phi - \phi')\right]^{\lambda - \frac{1}{2}} \qquad \text{...(1.4)}$$

is integrable (L) over the whole surface of the unit sphere and following Kogbetliantz[2], we define the generalized mean value of $f(\theta,\phi)$ as follows:

$$f(\omega)=\frac{1}{2\pi(\sin\omega)^2}\int_{C_\omega}\frac{f(\theta',\phi')dS'}{\left[\sin^2\theta'\sin^2(\phi-\phi')\right]^{\lambda-1/2}} \qquad \text{...(1.4)}$$

where the integral is taken along the small circle C_w, where dS' in element of the arc of C_w, where centre is (θ,ϕ) on the sphere and where curvilinear radius is ω.

We write

$$\phi(\omega)=\left[f(\omega)-\frac{A\Gamma(\lambda)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)}\right](\sin\omega)^{2\lambda}; \qquad \text{...(1.5)}$$

$$\Phi_p(x)=\frac{1}{\Gamma(p)}\int_0^x (x-t)^{p-1}\phi(t)dt;$$

$$\Phi_p(x)=\phi(x);$$

$$\phi_p(x)=\Gamma(p+1)x^{-p}\Phi_p(x),\ p\geq 0;$$

and

$$\Phi_p(x)=\frac{d}{dx}\Phi_{p+1}(x), \qquad -1<p<0$$

We have obtained a theorem for Cesaro summability of the series (1.1) analogous to there of Izumi and Sonouchi [1]. The object of this paper is to extend and generalize the result of Wang [5& 6] for the same series.

We prove the following:

Theorem: If $x\geq 1$ and

$$\Phi_x(t)=o\left(t^{\frac{2\lambda+x}{\alpha}}\right)\ \text{for } x>\alpha \text{ and } 0<\lambda<1$$

then the series (1.1) is summable $(C,\alpha+\lambda)$ at the point (θ,ϕ) to the sum A.

2. For the proof of the theorem we require the following lemmas:

Lemma:1 Let $S_n^k(\omega)$ denote the n^{th} Cesaro mean of order K of the series//3//

$$\sum(n+\lambda)p_n^{(\lambda)}(\cos\omega) \qquad \text{...(2.1)}$$

Then we have, for $\lambda>0$ and $p\geq 0$

$$S_n^p(\omega)=\frac{d^p\left(S_n^k(\omega)\right)}{d\omega^p}=\begin{cases}O\left(n^{2\lambda+p+1}\right), \text{for } 0\le\omega\le\pi, k>0\\ O\left(\dfrac{n^{\lambda+p+k}}{\omega^{\lambda+k+1}}\right)+O\left(n\omega^{1/2\lambda+p+2}\right), \text{for } 0<\omega\le\alpha\le\pi\\ O\left(\dfrac{n^{\lambda+p-k}}{\omega^{k+\lambda+1}}\right), \text{for } 0<\omega\le\alpha\le\pi \text{ and } \lambda+1+[p]\ge k\end{cases} \qquad \dots(2.2)$$

Lemma:2 In order that the series (1.1.) be summable (C, k), it is sufficient that the integral

$$i=\int_0^{\delta}\phi(\omega)S_n^k(\omega)\,d\omega=o(1) \qquad \dots(2.3)$$

for $0<\delta<\pi$ and for each $k>\lambda$

Lemma:3 For a non-integral

$$\delta=m+\sigma, \qquad (o<\sigma<1)$$

We have

$$\int_0^{\Delta}\Phi_s(u)S_n^{(\sigma)}(u)du=\Phi_{m+1}(\Delta)S_n^{(m)}(\Delta)x\int_0^{\Delta}\Phi_m(t)S_n^{(m)}(t)dt \qquad \dots(2.4)$$

Lemma:4 If $0\le u\le\dfrac{1}{n}$

$$F(n,u)=O\left(n^{2\lambda+m+1}u^{m-x}\right)+O\left(n^{2\lambda+m+1}u^{m-x-1}\right) \qquad \dots(2.5)$$

where

$$F(n,u)=\left|\left(\frac{1}{m-x}\right)\right|\int_u^{1/n}(t-u)^{m-x-1}S_n^{(m)}(t)\,dt \qquad \dots(2.6)$$

Lemmas 1, 2, 3 & 4 are due to Obrechkoff [3] and Singhai [4], respectively.

3. Proof of the Theorem:

If we put

$$x=1+\delta$$

and suppose for a non-integral δ

$$\delta=1+\sigma \quad (0<\sigma<1),$$

also let us first take the case when $m\ge1, then$

$$= \left[\sum_{\rho=1}^{m} (-1)^{\rho-1} \Phi_\rho(u) (\frac{d}{du})^{\rho-1} s_n^{\alpha+\lambda} \right]_0^\Delta + (-1)^m \int_0^\Delta \Phi_m(t) s_n^m(t)\, dt$$

$$= I_1 + (-1)^m I_2$$

From Lemma 2 we observe that

$s_n^{(q)}(\Delta) = o(1)$ *as* $n \to \infty$ *for* $\alpha > q$,
since $\alpha > \delta$ *and* $\delta > m$, *hence* $\alpha > m$.
Thus

$$I_1 = o(1) \text{ as } n \to \infty . \qquad \text{...(3.1)}$$

Now

$$\mathrm{I}_2 = \int_0^\Delta \Phi_m(\mathrm{t}) \mathrm{S}_n^{(\mathrm{m})}(\mathrm{t})\, \mathrm{dt}$$

$$= \left[\Phi_m(\Delta) \mathrm{S}_n^{(\mathrm{m-1})}(\Delta) \right] - \int_0^\delta \Phi_\delta(u) \mathrm{S}_n^{(\delta)}(\mathrm{u})\, \mathrm{du}, \text{ [by Lemma 3]}$$

$$= I_{2.1} - I_{2.2},$$

But

$$I_{2.1} = o(1) \text{ since } \alpha \to m . \qquad \text{...(3.2)}$$

Now,

$$\mathrm{I}_{2.2} = \int_0^\Delta \Phi_\delta(u) \mathrm{S}_n^{(\delta)}(\mathrm{u})\, \mathrm{du}$$

$$= \int_0^{1/n^r} + \int_{1/n^r}^{\Delta} = \mathrm{I}_{2.1.1} + \mathrm{I}_{2.2.2}$$

$$r = \frac{\alpha - \delta}{\delta + 2\lambda + 1} = \frac{\alpha - \delta}{x + 2\lambda}$$

But we have [1]

$$\varphi_x(t) = \frac{1}{t} \int_0^t \left(1 - \frac{u}{t}\right)^z \varphi(u)\, du = o\left(t^{\frac{2\lambda + x}{\alpha} - x} \right) \qquad \text{...(3.3)}$$

If we put $x = 1 + \delta$

Then (3.3) is equivalent to

$$\varphi_{1+\delta}(t) = o\left(t^{\frac{2\lambda+x}{\alpha}-x}\right) \qquad \text{...(3.4)}$$

So $\Phi_\delta(u)$ is integrable in the sense of Cauchy Lebesgue.

Thus by (3.4), we get

$$\int_0^t \varphi_\delta(u)du = o\left(t^{\frac{2\lambda+x}{\alpha}-x+1}\right) = o\left(t^{\frac{2\lambda+x}{\alpha}-\delta}\right) \qquad \text{...(3.5)}$$

and so.

$$\Phi^*(t) = \int_0^t \left| \Phi_\delta(u) \right| du$$

$$\text{...(3.6)}$$

$$= \frac{1}{\Gamma(1+\delta)} \int_0^t u^\delta \varphi(u) du$$

$$= o\left(t^{\frac{2\lambda+x}{\alpha}}\right) [by\ (3.5)].$$

$$I_{2.2.2} = \int_{\frac{1}{n^r}}^{\Delta} \Phi_\delta(u) S_n^{(\delta)}(u)\, du$$

$$= \int_{\frac{1}{n^r}}^{\Delta} O(1)\left[O\left(\frac{n^{\delta-\lambda}}{u^{\alpha+2\lambda+1}}\right) + O\left(\frac{1}{nu^{2\lambda+\alpha+\delta}}\right)\right] du$$

$$= O\left(n^{\delta-\alpha}\right) + O\left(n^{\delta-\alpha+\frac{\alpha-\delta}{x+2\lambda}(\alpha+2\lambda)}\right) + O\left(\frac{1}{n}\right) + O\left(\frac{1}{n}.n^{r(2\lambda+x)}\right)$$

$$= o(1)\ as\ n \to \infty.$$

$$I_{2.2.1} = \int_0^{\frac{1}{n^r}} \Phi_\delta(u) S_n^{(\delta)}(u)\, du. \qquad \text{...(3.7)}$$

$$I_{2.2.1.1} = \int_0^{\frac{1}{n}} \Phi_\delta(u) S_n^{(\delta)}(u)\, du$$

$$= \left[\Phi^*(u) S_n^{(\delta)}(u)\ \right]_0^{\frac{1}{n}} - \int_0^{\frac{1}{n}} \Phi^*(u) S_n^{(\delta+\lambda)}(u)\, du$$

$$= k_1 - k_2]$$

Now

$$k_1 = o[u^{\frac{2\lambda+x}{\alpha}}.O(n^{(2\lambda+\delta+2)})]_0^{\frac{1}{n}}$$

$$= o(1) \ as \ n \to \infty .$$

$$k_2 = o[\int_0^{\frac{1}{n}} u^{\frac{2\lambda+x}{\alpha}}.O(n^{(2\lambda+\delta+2)})]du$$

$$= o\left(n^{\delta+2\lambda+2}\right).O\left(\frac{1}{n^{\frac{2\lambda+x}{\alpha}+1}}\right) \qquad ...(3.8)$$

$$= o(1) \ as \ n \to \infty .$$

$$I_{2.2.1.2} = \int_{\frac{1}{n}}^{\frac{1}{n^r}} \Phi_\delta(u) S_n^{(\delta)}(u)\, du \qquad ...(3.9)$$

$$= \left[\Phi^*(u) S_n^{(\delta)}(u) \right]_{\frac{1}{n}}^{\frac{1}{n^r}} - \int_{\frac{1}{n}}^{\frac{1}{n^r}} \Phi^*(u) S_n^{(\delta+\lambda)}(u)\, du$$

$$= k_1' - k_2'$$

$$k_1' = o[u^{\frac{2\lambda+x}{\alpha}} \{ \ +O(n^{\frac{\delta-\alpha}{\alpha+2\lambda+1}}) + O(\frac{1}{nu^{2\lambda+2+\delta}}) \ \} \ ]_{\frac{1}{n}}^{\frac{1}{n^r}}$$

$$= o\left(n^{\delta-\alpha}\right)\left[\frac{1}{n^{r\left\{\frac{2\lambda+x}{\alpha}-\alpha-2\lambda-1\right\}}}\right] + o\left(n^{\delta-\alpha}\right)\frac{1}{n^{\frac{2\lambda+x}{\alpha}-\alpha-2\lambda-1}}$$

$$= o(1) \ as \ n \to \infty .$$

$$k_2' = \int_{\frac{1}{n}}^{\frac{1}{n^r}} o[u^{\frac{2\lambda+x}{\alpha}}.\frac{u^{\delta-\alpha+1}}{u^{\alpha+2\lambda+1}} du \qquad ...(3.10)$$

$$= o\left(n^{\delta-\alpha+1}\right).\left[u^{\frac{2\lambda+x}{\alpha}-\alpha-2\lambda}\right]_{\frac{1}{n}}^{\frac{1}{n^r}}$$

$$= o\left(n^{\delta-\alpha+1}\right).\left[\frac{1}{u^{r\left\{\frac{2\lambda+x}{\alpha}-\alpha-2\lambda\right\}}}\right]+o\left[\frac{n^{\delta-\alpha+1}}{n^{\frac{2\lambda+x}{\alpha}-\alpha-2\lambda}}\right]$$

$= o(1)\ as\ n \to \infty$.

Combining (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) (3.9)and (3.10) the result is proved.

When δ is an integer , say $\delta = m$.

$$i = \left[\sum_{\rho=1}^{m}(-1)^{\rho-1}\Phi\rho(u)\left(\frac{d}{du}\right)^{\rho-1}S_n^{\alpha+\lambda}(u).\right]_0^{\Delta} + (-1)^m\int_0^{\Delta}\Phi_\delta(u)S_n^{\delta}(u)du \qquad \text{...(3.11)}$$

$$= o(1)$$

When m = o

$$\int_0^{\Delta}\Phi_0(u)S_n^0(u)du = \Phi_1(\Delta)S_n^0(\Delta) - \int_0^{\Delta}\Phi_\delta(u)S_n^{\delta}(u)du \qquad \text{...(3.12)}$$

$$= o(1) \quad \text{as before.}$$

When x =1 i.e. $\delta = 0$

$$i = \int_0^{\Delta}\Phi_0(u)S_n^0(u)du$$

and

$$\int_0^t \Phi_0(t)dt = o\left(t^{\frac{1+2\lambda}{\alpha}}\right)$$

By the help of (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) (3.9) and (3.10) we have

$$i = o(1). \qquad \text{...(3.13)}$$

This completes the proof of the theorem.

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M^X/M/1/N QUEUEING SYSTEMS WITH LINEARLY DEPENDENT SERVICE RATE WITH DISCOURAGEMENT AND REFLECTING BARRIERS

D.C. SHARMA

School of Mathematics, Statistics and Computational Sciences

Central University of Rajasthan, Kishangarh, Ajmer-India

dcsharma_1961@yahoo.co.in ; dcsharmacuraj@gmail.com

ABSTRACT

We consider a finite capacity queueing system in which the arrival of customers are in batches of size K. We take general balk function b_n and reneging with reflecting barriers at n= N. Service rate is linearly dependent on the number of customers in the system. Provision of an additional server for longer queue is also considered.

1.0 INTRODUCTION

Many researchers have gone through the queueing systems with linearly dependent service rates. Abou-El-Ata, Al-Seedy and Kotb(1989) studied the state-dependent queue M/M/1/N with reflecting barriers and general balk function. M/M/C/N queueing system has been discussed by Gross and Harris (1985) again without any additional concept in it. Abou-El-Ata (1987) developed a new approach for the moments of the single birth-death process and discrete distribution H. Abou-El-Ata and Kotb (1992) studied a linearly dependent service rate for the queue M/M/1/N with general balk function, reflecting barrier, reneging and an additional server for longer queues. Blackburn (1972) studied the optimal control of a single server queue with balking and reneging. Varshney, Jain and Sharma (1988) studied a multiserver queueing system with balking and reneging via diffusion – approximation approach for G/G/1 double ended queue with balking and finite capacity. Varshney, Jain and Sharma (1989) gave attention on multiserver queueing system with additional server. Varshney, Jain and Sharma (1987) studied diffusion-approximation for G/G/1 queueing system with discouragement.Dequan, Wuyi and Hongjuan (2008) perform analysis of machine repair system with warm spares and N-policy vacation.Zhang and Tian (2003) gave analysis on queuing system with synchronous vacation of partial servers.Ke and Wang (2007) studied machine repair problem with two type spares and multi-server vacations. They solved the steady-state probabilities equations iteratively and derived the steady-state probabilities in matrix form.Wang and Sivazlian [1989] considered the reliability characteristics of a repairable system with m operating units, s warm spares and R repairmen. They obtain the expressions of the reliability and the mean time to system failure.Wang and Ke (2003) extend this model to consider the balking and reneging of failed units.Jain and Maheshwari (2004) extended the model of Hsieh and Wang (1995) to analyze the repairable system in transient by incorporating reneging behaviour of failed units.

Here in this paper our aim is to discuss a finite capacity queueing system in which the arrival of customers are in batch of size k, with the addition of general balk function, reflecting barrier with reneging and an additional server for longer queues. The service rate is linearly dependent on the number of customers present in the system.

2.0 ANALYSIS AND FORMULATION

Arrival of customers in the system are batches of size k. If λ_k is the arrival rate of the Poisson process of batches of size k then clearly $c_k = \lambda_k / \lambda$, where λ is the composite arrival rate of all batches and is equal to $\sum_{i=1}^{\infty} \lambda_i = \lambda$. Units in the system are served on FIFO basis, with different service rates depending on the number of customers in the system. Customers in the system are served with rate μ_1 if $1 \le n < 1$ and the with μ_2 service rate if customers lies in $1 \le n < m$ and beyond m there is a facility of an additional server with service rate μ i.e., $\mu_3 = \mu_2 + \mu$ if $m \le n < N$.

Assuming the balk concept b_n such that $0 \le b_{n+1} \le b_n < 1,\ 1 \le n \le N$ and $b_n = 0$ if n =0 where b_n = Prob. (a unit joins the queue).

Any unit in the system rengees after certain time t, where t is the random variable with exponential distribution.

Let $g(n)=\alpha(n-1)$, $n \ge 1$ be the reneging function when n number of units are their in the system where g(n)=0, if n=0.

The system reflects any unit if n=N. Assume r is the probability of reflecting at n=N.

Now for $M^X/M/1/N$ queueing systems, we have the following steady state-difference equations

$$\lambda p_0 = \mu_1 p_1. \qquad \text{where } n = 0 \qquad \ldots(2.1)$$

$$[b_n\lambda+\mu_1 + (n\text{-}1)\alpha]p_n = \lambda\sum_{k=1}^{n} = b_k c_k p_{n\text{-}k} + (\mu_1 + n\alpha)p_{n+1} \qquad 1 \le n < 1 \qquad \ldots(2.2)$$

$$[b_1\lambda+\mu_1 + (l\text{-}1)\alpha]p_1 = \lambda\sum_{k=1}^{n} = b_k c_k p_{l\text{-}k} + (\mu_1 + 1\alpha)p_{n\text{-}1} \qquad n = 1 \qquad \ldots(2.3)$$

$$[b_n\lambda+\mu_2 + (n\text{-}1)\alpha]p_n = \lambda\sum_{k=1}^{n} = b_k c_k p_{n\text{-}k} + (\mu_2 + n\alpha)p_{n+1} \qquad 1 < n < m \qquad \ldots(2.4)$$

$$[b_m\lambda+\mu_3 + (m\text{-}1)\alpha]p_m = \lambda\sum_{k=1}^{m} = b_k c_k p_{m\text{-}k} + (\mu_3 + m\alpha)p_{m+1} \qquad n = m \qquad \ldots(2.5)$$

$$[b_n\lambda+\mu_3 + (n\text{-}1)\alpha]p_n = \lambda\sum_{k=1}^{m} = b_k c_k p_{n\text{-}k} + (\mu_3 + n\alpha)p_{n+1} \qquad m < n < N\text{-}1 \qquad \ldots(2.6)$$

$$[b_{N\text{-}1}\lambda+\mu_3 + (N\text{-}2)\alpha]p_{N\text{-}1} = \lambda\sum_{k=1}^{N\text{-}1} = b_k c_k p_{N\text{-}k+1} + (\mu_3 + N\text{-}1\alpha)r.p_n \qquad n = N\text{-}1 \qquad \ldots(2.7)$$

$$[\mu_3 + (N\text{-}1)\alpha]r.p_n = \lambda\sum_{k=1}^{N} = b_k c_k p_{N\text{-}k} \qquad n = N\text{-}1 \qquad \ldots(2.8)$$

$$p_1 = \frac{\lambda}{\mu_1} p0 \Rightarrow p_1 = \rho_1 p_0 \qquad ...(2.9)$$

$$\left\{ \frac{\prod_{l-1}^{n-1} b_i}{(1+\delta_1)_{n-1}} r^{n-1} + \sum_{i=1}^{n-2} f_i(b_i, c_i, \delta_1) r^i + \frac{1 - \sum_{i=1}^{n-1} b_i c_i}{(n-1)+\delta_1} \delta_1 \right\} p_0 \rho_1 \text{ for } 1 \le n < 1 \qquad ...(2.10)$$

$$\left\{ \frac{r^l \left(\prod_{l-1}^{n-1} b_i \right) r^{n-l+1}}{(1+\delta_2)(1+\delta_1)_{n-1}((1+1)+\delta_2)_{n-l+1}} + \sum_{i=1}^{n-2} f_i(b_i, c_i, \delta_1, \delta_2) r^i + \frac{1 - \sum_{i=1}^{n-1} b_i c_i}{(m-1)+\delta_1} \delta_2 \right\} p_0 \rho_1$$

$$1 \le n < m \qquad ...(2.11)$$

$$Pn = \left\{ \frac{r^l \left(\prod_{l-1}^{n-1} b_i \right) r^{n-l+1}}{(m+\delta_3)(1+\delta_2)_{m-1}(1+\delta_1)_{l-1}(\delta_3+m+1)_{n-m-1}} r^{n-m-1} + \sum_{i=1}^{n-2} f_i(b_i, c_i, \delta_1, \delta_2, \delta_3) r^i + \frac{1 - \sum_{i=1}^{N-2} b_i c_i}{(N-2+\delta_3)} \delta_3 \right\} p_0 \rho_1$$

$$m < n < N-1 \qquad ...(2.12)$$

$$\left\{ \frac{\left(\prod_{l-1}^{N-1} b_i \right) r^{N-1}}{r(m+\delta_3)_{n-m}(1+\delta_2)_{m-1}(1+\delta_1)_{l+1}} + \sum_{i=1}^{N-2} f_i(b_i, c_i, \delta_1, \delta_2, \delta_3, r) r^i + \frac{1 - \sum_{i=1}^{N-2} b_i c_i}{(N-2+\delta_3)} \delta_3 \right\} p_0 \rho_1$$

$$n + N \qquad ...(2.13)$$

where $\rho_1 = \frac{\lambda}{\mu_1}$, $\delta i = \frac{\mu_1}{\lambda}$ and $\frac{\lambda}{\alpha} = r$ for $i = 1,2,3$

If the arrival rate of customers is λ then the problem reduces to Abou-El-Ata and Kotb type of problem.

Now we are considering certain particular cases.

Case 1 : If the problem is with general balk function with reflecting barriers and if we consider uniform service rate μ_1 for every $1 \le n \le N$ then the problem reduces to the form.

$$\lambda p_0 = \mu_1 p_1 \qquad \text{where } n = 0 \qquad ...(2.14)$$

$$(b_n \lambda + \mu_1) p_n = \lambda \sum_{k=1}^{n} = b_k c_k p_{n-k} + \mu_1 r.p_{n+1} \qquad 1 \le n \le N\text{-}1 \qquad ...(2.15)$$

$$\mu_1 r.p_n = \lambda \sum_{k=1}^{n} = b_k c_k p_{n-k} \qquad n = N \qquad ...(2.16)$$

$$p_1 = \rho_1 p_0 \quad \text{for } n = 0$$

$$p_n = \begin{cases} \left(\prod_{1-1}^{n-1} b_i\right)\rho_1^n + \sum_{i=1}^{n-1} = f_i(b_k, c_k)\rho_1^i + (1 - \sum_{i=1}^{n-1} = f_i(b_i c_i)\rho_1)\}p_0 \quad 1 \le n \le N-1 & \text{...(2.17)} \\ 1 \le k \le i & \\ \frac{p_0}{r}\left[\left\{\prod_{1-1}^{N-1} b_i\right\}\rho_1^{N-1}\right] + \left[\sum_{i=1}^{N-2} = f_i(b_k, c_k)\rho_1^i + \frac{1}{r}(1 - \sum_{i=1}^{N-2}(b_i, c_i)\rho_1\right]p_0 & \\ n=N \qquad 1 \le k \le i & \text{...(2.18)} \end{cases}$$

Case 2 : If in the main problem if we take bn = 1 for all n that is it is without a balk function then pn converted to the following form.

$$p_1 = \rho_1 p_0 \quad \text{for } n = 0$$

$$p_n = \begin{cases} \left[\frac{r^{n-1}}{(1+\delta_1)_{n-1}} + \sum_{i=1}^{n-2} f_i(c_k, \delta_1) r^i + \frac{1 - \sum_{i=1}^{i-1} b_i c_i}{(n-1)+\delta_1}\delta_1\right] p_0 \rho_1 \quad 1 \le n \le 1 & \text{...(2.19)} \\ \left[\frac{r^{n-1}}{r(m+\delta_3)_{n-m}(1+\delta_2)_{m-1}(1+\delta_1)_{l+1}} + \sum_{i=1}^{n-2} f_i(c_k, \delta_1, \delta_2) r^i + \frac{1 - \sum_{i=1}^{i-1} b_i c_i}{(n-1)+\delta_2}\delta_3\right] p_0 \rho_1 & \\ 1 \le n \le m & \text{...(2.20)} \\ \left[\frac{r^{n-1}}{r(m+\delta_3)(1+\delta_2)_{m-1}(\delta_3 + m + 1)_{n+m+1}} + \sum_{i=1}^{n-2} f_i(c_k, \delta_1, \delta_2, \delta_3) r^i + \frac{1 - \sum_{i=1}^{n-2} b_i c_i}{(n-2)+\delta_3}\delta_3\right] p_0 \rho_1 & \\ m \le n \le N-1 & \text{...(2.21)} \\ \left[\frac{r^{n-1}}{r(m+\delta_3)_{n-m}(1+\delta_2)_{m-1}(1+\delta_1)_{l+1}} + \sum_{i=1}^{N-2} f_i(c_k, \delta_1, \delta_2, \delta_3, r) r^i + \frac{1 - \sum_{i=1}^{N-1} c_i}{r(m-\delta_3)}\delta_3\right] p_0 \rho_1 & \\ n = N-1 & \text{...(2.22)} \end{cases}$$

Here in all above equations (2.19) – (2.22) $1 \le k \le 1$.

3.0 CONCLUSION

Here in this paper we have found out the state dependent solution for the system with general balk function, reflecting barrier, reneging and an additional server for longer queues. We have also determined two cases (i) when uniform service rate has been considered, and (ii) when there is no balk function in the main problem we have taken.

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SOME RESULTS ASSOCIATED WITH A GENERALIZED GAMMA-TYPE FUNCTIONS INVOLVING KUMMER'S CONFLUENT HYPEREOGMETRIC FUNCTION AND ASSOCIATED PROBABILITY DISTRIBUTIONS

CHENA RAM* AND NARESH**

Department of Mathematics and Statistics,JaiNarainVyas University, Jodhpur 342005(Raj) Department of Mathematics,Marwar Engineering College and ResearchCentre, Jodhpur(Raj)*

ABSTRACT

This paper deals with the study of a generalized gamma type functions involving Kummar's confluent hypergeometric function. Certain properties of this new function are investigated which include its integralrepresentations.Corresponding incomplete generalized gamma function and its complementary function are also defined and their properties are derived. The results presented in this paper are of general character and results reported earlier by Saxena and Kalla[2],Kobayashi[5,6],AL-Musallam and Kalla[3] follows,as special cases.

1.0 INTRODUCTION

This present paper introduces and study a new generalization of the generalized gamma-type function in the form

$$S^{*}\begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta; & \\ u,v & p,k \end{pmatrix} := v^{-\lambda}\int_0^{\infty} t^{u-1}\,{}_1\Phi_1(\alpha,\beta;-pt)\,{}_3R_2(\lambda,a,b;c,d;k-\frac{t}{v})dt,$$

where ${}_1\Phi_1(\alpha,\beta;z)$ is the well knownKummer'sConfluent Hypergeometric Function and a new probability density function involving generalized gamma function associated with the function ${}_3R_2^k(z)={}_3R_2(\lambda,a,b;c,d;k;z)$ which has been defined and studied by Saxena, Ram and Naresh[1].This generalization provides unification and extension of the various generalization given earlier by Kobayashi[6,7],Al-Musallum, Kalla[4,5]and Virchenko et al.[8,9]. A probability density function associated with the generalized gamma type function investigated in this paper,together with several other related results in the theory of probability and statistics and also considered.

2.0 GENERALIZED GAMA FUNCTION

The present paper deals with a generalization of the gamma–type function associated with Kummer's Confluent Hypergeometric Function in the form

$$S^{*}\left(\begin{matrix}\lambda,a,b;c,d;\alpha,\beta; & p,k\\ u,v & \end{matrix}\right):=v^{-\lambda}\int_0^\infty t^{u-1}{}_1\Phi_1\left(\alpha,\beta;-pt\right)_3R_2(\lambda,a,b;c,d;k-\frac{t}{v})dt, \quad ...(2.1)$$

whereRe(u) > 0, Re(p)>0,k>0 |arg v| <π,and ${}_3R_2^k(z)={}_3R_2(\lambda,a,b;c,d;k;z)$ is the generalized hypergeometric function studied by Saxen,ChenaRam and Naresh [1].

Some special cases:

Case(i) Whenb=d,(2.1) reduces to the results given by Saxena, Kalla Chena Ram and Naresh[3].

Case(ii) For $k=1$ and $\alpha=\beta$,(2.1) reduces to the generalized gamma function involving clausenianhypergeometric series recently introduced and studied by Saxena and Kalla[2].

Case(iii) For $b=d$ and $\alpha=\beta$,(2.1)reduces to the generalized gamma function discussed by Virchenko et al.[8].

Case(iv) For $b=d,k=1$ and $\alpha=\beta$, (2.1) reduces to the generalized gamma function studied by Al-Musallam and Kalla[4,5].

Case(v) For $a=c,b=d$,$p=k=1,\alpha=\beta$ and $\lambda=m\in N_0$, (2.1) reduces to the generalized gamma function studied by Kobayashi [6,7].

Case(vi) It we set $a=c,b=d$,$p=k=1,\alpha=\beta$ and $\lambda=0\in N_0$, (2.1) reduces to the well-known gamma function studied by Kobayashi [6,7].

Theorem 1. S^* is analytic in the domain $\Omega_u\times\Omega_v$.

The proof is similar to the corresponding theorem for the generalized gamma function given by Saxena and Kalla [2, pp.191-192], if we employ the asymptotic estimate [Al Musallamand Kalla (4)]

$${}_3R_2\left(\lambda,a,b;c,d;k;z\right)=A_1z^{-\lambda}+A_2z^{-\frac{a}{k}}+A_3z^{-\frac{b}{k}}+O\left(z^{-\lambda-1}\right)+O\left(z^{-\frac{a}{k}-1}\right)+O\left(z^{-\frac{b}{k}-1}\right),(2.2)$$

which holds for large z, $\left|\arg(-z)\right|<\pi$. Here A_1,A_2,A_3 are numerical constants.

Lemma 1. The partial derivatives of S^* are:

$$\frac{\partial^n}{\partial u^n}S^*=v^{-\lambda}\cdot\int_0^\infty t^{u-1}{}_1\Phi_1\left(\alpha,\beta;-\mathrm{pt}\right)\left(\log t\right)^n{}_3R_2\left(\lambda,a,b;c,d;k;\frac{-t}{v}\right)dt, \quad ...(2.3)$$

and

$$\frac{\partial^n}{\partial v^n} S^* = (-1)^n (\lambda)_n S^* \begin{pmatrix} \lambda+n,a,b;c,d;\alpha,\beta; & \\ u,v & p,k \end{pmatrix}. \qquad ...(2.4)$$

The proof of (2.3) and (2.4) is trivial.

Lemma 2. Let $\lambda,\alpha,\beta,a,b,c,d,p \in C$ with $\beta,c,d \neq 0,-1,-2,\cdots$; $k>0$ and $\mathrm{Re}(p)>0$, then following relation is valid:

$$S^* \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta; & \\ u,v & p,k \end{pmatrix} = \frac{p\alpha}{u\beta} S^* \begin{pmatrix} \lambda,a,b;c,d;\alpha+1,\beta+1; & \\ u+1,v & p,k \end{pmatrix}$$

$$+\frac{\lambda\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}{u\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)} S^* \begin{pmatrix} \lambda+1,a+k,b+k;c+k,d+k;\alpha,\beta; & \\ u+1,v & p,k \end{pmatrix}. \qquad ...(2.5)$$

Proof. If we use [1,equation (3.23)] for $\frac{d}{dz}\left[{}_3R_2^k(z)\right]$ and integrate by parts, then (2.1) reduces to (2.5).

3.0 THE GENERALIZED INCOMPLETE GAMA FUNCTIONS

For $x,k>0$, we introduce the generalized incomplete gamma function in the form

$$S^{*x}_{\ 0} \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta; & \\ u,v & p,k \end{pmatrix} = v^{-\lambda} \cdot \int_0^x t^{u-1} {}_1\Phi_1(\alpha,\beta;-\mathrm{pt})\, {}_3R_2\left(\lambda,a,b;c,d;k;\frac{-t}{v}\right) dt, \qquad ...(3.1)$$

where $\mathrm{x,k}>0, \mathrm{Re}(u)>0$, $\mathrm{Re}(p)>0$ and $|\arg v|<\pi$.

The generalized complementary incomplete gamma function is defined

$$S^{*\infty}_{\ x} \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta; & \\ u,v & p,k \end{pmatrix} = v^{-\lambda} \cdot \int_x^\infty t^{u-1} {}_1\Phi_1(\alpha,\beta;-\mathrm{pt})\, {}_3R_2\left(\lambda,a,b;c,d;k;\frac{-t}{v}\right) dt, \qquad ...(3.2)$$

where $\mathrm{x,k}>0, \mathrm{Re}(u)>0$, $\mathrm{Re}(p)>0$, $|\arg v|<\pi$.

Thus, the definitions (3.1) and (3.2) yield

$$S^* \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta; & \\ u,v & p,k \end{pmatrix} = S^{*x}_{\ 0} \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta; & \\ u,v & p,k \end{pmatrix} + S^{*\infty}_{\ x} \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta; & \\ u,v & p,k \end{pmatrix}, (3.3)$$ special cases:

Case(i) When b=d equations (3.1) and (3.2) reduce to the results given by Saxena, Kalla, Chena Ram and Naresh[3].

Case(ii) *For* $\alpha=\beta$ *and* $b=d$, (3.1) and (3.2) reduce to the generalized incomplete gamma functions developed by Virchenko et al. [9, p.98]. Case(iii) Further for $b=d$, $\alpha=\beta$ and $k=1$, (3.1) and (3.2) reduce to the incomplete gamma functions given by Al-Musallam and Kalla [4].

Remark. If we set $a=c, b=d, \alpha=\beta$ and $p=k=1$ in (3.1) and (3.2) and $\lambda \to 0$, then we find that

$$\operatorname*{Lt}_{\lambda\to 0} S^{*x}_{\ 0}\left(\begin{matrix}\lambda,a,b;a,b;\alpha,\alpha; \\ u,v\end{matrix}\ 1,1\right)=\gamma(u,x)=\int_0^x t^{u-1}e^{-t}\,dt, \qquad \ldots(3.4)$$

where $\gamma(u,x)$ is the incomplete gamma function of the first kind, and

$$\operatorname*{Lt}_{\lambda\to 0} S^{*\infty}_{\ x}\left(\begin{matrix}\lambda,a,b;a,b;\alpha,\alpha \\ u,v\end{matrix}\ 1,1\right)=\Gamma(u,x)=\int_x^\infty t^{u-1}e^{-t}\,dt, \qquad \ldots(3.5)$$

where $\Gamma(u,x)$ is the incomplete gamma function of the second kind.

4.Probability densityfunctions:

From (2.1), we have

$$S^*\left(\begin{matrix}\lambda,a,b;c,d;\alpha,\beta; \\ u,v\end{matrix}\quad p,k\right):=v^{-\lambda}\int_0^\infty t^{u-1}\,{}_1\Phi_1(\alpha,\beta;-pt)\,{}_3R_2(\lambda,a,b;c,d;k-\frac{t}{v})dt, \qquad (4.1)$$

wherek,λ, Re(u,p) > 0, |arg v| <π.

The substitution $t=\sigma x^\delta$ and $dt=\sigma\delta\,x^{\delta-1}dx$, with $p=\frac{\gamma}{\sigma}\,(\gamma>0;\sigma>0)$,

$$u=\frac{m+\delta}{\delta}(\mathrm{m}+\delta>0), \text{and } v=\mathrm{n}\,(\mathrm{n}>0)$$

transform (4.1) into the form

$$\delta\,\sigma^{\frac{m}{\delta}+1}\int_0^\infty x^{m+\delta-1}\,{}_1\Phi_1(\alpha,\beta;-\gamma\,\mathrm{x}^\delta)\,{}_3R_2(\lambda,a,b;c,d;k;-\frac{\sigma\,x^\delta}{n})dx$$
$$=n^\lambda S^*\left(\begin{matrix}\lambda,a,b;c,d;\alpha,\beta; \\ m/\delta+1,n\end{matrix}\quad \frac{\gamma}{\sigma},k\right)\quad(\min\{\gamma,\sigma,\mathrm{m}+\delta,\mathrm{n}\}>0). \qquad \ldots(4.2)$$

By virtue of integral formula (4.2),a class of probability density functions associated with the S^*-function can be defined by

$$f(x):=\begin{cases}\dfrac{\delta\sigma^{\frac{m}{\delta}+1}x^{m+\delta-1}\ {}_1\Phi_1(\alpha,\beta;-\gamma\,\mathrm{x}^\delta)\,{}_3R_2\left(\lambda,a,b;c,d;k;\dfrac{-\sigma\,x^\delta}{n}\right)}{n^\lambda S^*\left[\begin{matrix}\lambda,a,b;c,d;\alpha,\beta; \\ \frac{m}{\delta}+1,n\end{matrix}\ \dfrac{\gamma}{\sigma},k\right]} & (x>0), \\ 0,\ elsewhere\end{cases} \qquad \ldots(4.3)$$

provided that the various parameters and variable x occurring in equation (4.3) are so constrained that the density function is always non-negative. It is evident that

$$\int_{-\infty}^{\infty} f(x)dx=1.$$

We note that the behaviour of f(x) at zero depends on m+δ.

$$f(0) = \delta\,\sigma^{1/\delta} n^{-\lambda} \left\{ S^{*}\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ 1/\delta, n & \end{pmatrix} \right\}^{-1} \quad (m+\delta=1)$$

$$f(0) = 0 \quad (m+\delta>1)$$

$$f(x) \to \infty \text{ as } x \to 0+ \text{ when } m+\delta<1, \qquad \dots(4.4)$$

$$\lim_{x\to\infty} f(x) = 0 \ (\delta > 0), \qquad \dots(4.5)$$

It can be seen that

$$f'(x) = \left(\frac{m+\delta-1}{x} - \gamma\,\delta\, x^{\delta-1} - \frac{\sigma\,\delta}{n} x^{\delta-1}\,\Psi \right) f(x), \qquad \dots(4.6)$$

Where, for convenience,

$$\Psi := \frac{\lambda\,\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)} \frac{{}_3R_2\left(\lambda+1, a+k, b+k; c+k, d+k; k; -\frac{\sigma x^{\delta}}{n}\right)}{{}_3R_2\left(\lambda, a, b; c, d; k; -\frac{\sigma x^{\delta}}{n}\right)} \qquad \dots(4.7)$$

the formula (4.6) can be derived, if we differentiate both the sides of equation (4.3) with respect to x logarithmically and apply the following formula

$$\frac{d}{dx}\left\{ {}_3R_2\left(\lambda, a; b; c, d; k - \frac{\sigma x^{\delta}}{n}\right)\right\} = -\frac{\sigma\,\delta\,\lambda}{n} \frac{\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)} x^{\delta-1}$$
$${}_3R_2\left(\lambda+1, a+k, b+k; c+k, d+k; k; -\frac{\sigma x^{\delta}}{n}\right) \qquad (4.8)$$

Particular cases:

Case(i) Note that for $b = d$ *and* $\alpha = \beta$, the results of this section reduce to Virchenko et al.[9].

Case(ii) If we set b=d equation (4.8),reduce to results given by Saxena,Kalla,ChenaRam and Naresh[3].

5.0 SOME STATISTICAL FUNCTIONS

In this section, several basic statistical functions associated with the probability density function f(x), defined by equation (4.3), will be evaluated.

5.1 The r^{th} moment

The r^{th} moment μ_r^1 about the origin of a continuous real random variable X with the probability density function f(x) is given by

$$\mu_r^1 := \int_{-\infty}^{\infty} x^r f(x)dx =: \mathrm{E}\left[\mathrm{X}^r\right] \; (\mathrm{r} \in N), \qquad \ldots(5.1)$$

which on using equation (4.2) and definition(4.3) gives

$$\mu_r^1 = \sigma^{-r/\delta} S^*\begin{pmatrix} \lambda, a, b; c, d : \alpha, \beta; & \\ \frac{m+r}{\delta}+1, n & \frac{\gamma}{\sigma}, k \end{pmatrix} \left\{ S^*\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \\ \frac{m}{\delta}+1, n & \frac{\gamma}{\sigma}, k \end{pmatrix} \right\}^{-1} \qquad \ldots(5.2)$$

In particular, for r=1,the expected value of the random variable X(also referred to as the mean or the first moment of X) is obtained as

$$E(x) := \int_{-\infty}^{\infty} x\, f(x)\, dx = \sigma^{-1/\delta} S^*\begin{pmatrix} \lambda, a, b; c, d : \alpha, \beta; & \\ \frac{m+1}{\delta}+1, n & \frac{\gamma}{\sigma}, k \end{pmatrix}$$

$$\times \left\{ S^*\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \\ \frac{m}{\delta}+1, n & \frac{\gamma}{\sigma}, k \end{pmatrix} \right\}^{-1} \qquad \ldots(5.3)$$

5.2 The moment generating function:

The moment generating function $M(t;\delta)$ of a continuous random variable X having the probability density function $f(x)$ is defined by

$$M(t;\delta) = E\left[e^{tX}\right] := \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \frac{\delta \sigma^{\frac{m}{\delta}+1} \int_0^{\infty} x^{m+\delta-1} e^{tx}\, {}_1\Phi_1\left(\alpha, \beta; -\gamma x^{\delta}\right) {}_3R_2\left(\lambda, a, b; c, d; k; -\frac{\sigma x^{\delta}}{n}\right) dx}{n^{\lambda} S^*\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \\ \frac{m}{\delta}+1, n & \frac{\gamma}{\sigma}, k \end{pmatrix}} \qquad \ldots(5.4)$$

which itself is a generalization of a result given by Saxena,Kalla,Chena Ram and Naresh[3].

If we set $\alpha = \beta, b = d$ and k=1 (5.4) reduces to the moment generating function studied by Kalla et al.[10].

5.3 The Hazard rate function:

For a continuous random variable X having the probability density function f(x), the commulative distribution function F(t) is given by

$$F(t) := \int_{-\infty}^{t} f(x)\,dx =: \Pr ob\{X \in (-\infty, t)\}, \qquad \text{...(5.5)}$$

that is, by

$$F(t) = S^{*\sigma t^{\delta}}_{0}\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ \frac{m}{\delta}+1, n & \end{pmatrix}\left\{S^{*}\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ \frac{m}{\delta}+1, n & \end{pmatrix}\right\}^{-1}, \qquad \text{...(5.6)}$$

where $S^{*\sigma t^{\delta}}_{0}$ is the generalized incomplete gamma function defined by Virchenko et al.[9,p.98].By virtue of the result given by Virchenko et al.[9],we can express (5.6) in terms of generalized complementary incomplete gamma function $S^{*\infty}_{\sigma t^{\delta}}$ as

$$F(t) = 1 - S^{*\sigma t^{\delta}}_{0}\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ \frac{m}{\delta}+1, n & \end{pmatrix}\left\{S^{*}\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ \frac{m}{\delta}+1, n & \end{pmatrix}\right\}^{-1}, \qquad \text{...(5.7)}$$

thus the survivor function S(t) becomes

$$S(t) = S^{*\infty}_{\sigma t^{\delta}}\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ \frac{m}{\delta}+1, n & \end{pmatrix}\left\{S^{*}\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \frac{\gamma}{\delta}, k \\ \frac{m}{\delta}+1, n & \end{pmatrix}\right\}^{-1}, \qquad \text{...(5.8)}$$

and the Hazard rate function h(x),defined by (5.5), can be expressed as

$$h(t) = \frac{\delta\, \sigma^{\frac{m}{\delta}+1} t^{m+\delta-1}\, {}_1\Phi_1\left(\alpha, \beta; -\gamma\, x^{\delta}\right)\, {}_3R_2(\lambda, a, b; c, d; k; -\frac{\sigma x^{\delta}}{n})}{n^{\lambda} S^{*\infty}_{\sigma t^{\delta}}\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ \frac{m}{\delta}+1, n & \end{pmatrix}}, \ (t>0) \qquad \text{...(5.9)}$$

5.6 The mean residual life (or remaining life expectancy) function:

For a continuous random variable X, the mean residual life (or remaining life expectancy) function K(t) is given by

$$K(t) := E[X - t | X \geq t] = \frac{1}{S(t)}\int_{t}^{\infty}(x-t)f(x)\,\mathrm{d}x, \quad \text{(5.10)}$$

$$= \frac{1}{S(t)}\int_{t}^{\infty} x\, f(x)\,dx - t\ , \qquad \text{...(5.11)}$$

since $S(t)$ denotes the survivor (or reliability) function denoted by equation (5.8).

By virtue of the definition (4.3), if we usethe substitution $z=\sigma x^{\delta}$ *and* $dz=\sigma\,\delta\,x^{\delta-1}dx$, theequation(5.11) reduces as

$$\int_t^{\infty} x\, f(x)\,dx = \sigma^{-1/\delta} S^{*\infty}_{\sigma t^{\delta}}\left(\begin{matrix}\lambda,a,b;c,d;\alpha,\beta; \\ \frac{m+1}{\delta}+1,n\end{matrix}\quad \frac{\gamma}{\sigma},k\right)\left\{S^{*}\left(\begin{matrix}\lambda,a,b;c,d;\alpha,\beta; \\ \frac{m}{\delta}+1,n\end{matrix}\quad \frac{\gamma}{\sigma},k\right)\right\}^{-1} \quad ...(5.13)$$

so that

$$K(t) = \sigma^{-1/\delta} S^{*\infty}_{\sigma t^{\delta}}\left(\begin{matrix}\lambda,a,b;c,d;\alpha,\beta; \\ \frac{m+1}{\delta}+1,\nu\end{matrix}\quad \frac{\gamma}{\sigma},k\right)\left\{S^{*}\left(\begin{matrix}\lambda,a,b;c,d;\alpha,\beta; \\ \frac{m}{\delta}+1,n\end{matrix}\quad \frac{\gamma}{\sigma},k\right)\right\}^{-1} - t. \quad ...(5.14)$$

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KANTOWASKI-SACHS INFLATIONARY COSMOLOGICAL MODEL WITH VARYING Λ-TERM IN GENERAL THEORY OF RELATIVITY

VIMAL CHAND JAIN

Department of Humanities and Sciences,
Government Engineering College,
Badlia Chauraha, N.H.-8, Ajmer-305025, INDIA.
E-mail : vcjeca@gmail.com

ABSTRACT

Inflationary cosmological model with varying Λ- term is investigated in Kantowski-Sachs space-time. To obtain a determinate solution, it is assumed that the scalar of expansion θ is proportional to the shear scalar σ, which leads to a relation between metric potentials $A = kB^2$. A detail study of physical and geometrical parameters is also discussed. The results of the model are consistent within the observational limit.

Keywords Inflationary cosmological model. Kantowski-Sachs space-time. Time dependent Λ-term.

Mathematics Subject Classification (2000): 83D05, 83F05

1.0 INTRODUCTION

The inflation will occur by using the concept of Higg's field ϕ with potential $V(\phi)$ if potential has flat region and the scalar field ϕ· evolves slowly but the universe expands in an exponential way due to matterless scalar field [18]. In general relativity, scalar fields help in explaining the creation of matter in cosmological theories and can also describe the uncharged field. Scalar field is minimally coupled to the gravitational field. In particular, self interacting scalar fields play a very vital role in the study of inflationary cosmological model. Several versions of the inflationary models are studied by Guth [4], Linde [10], Abbott and Wise [1], Mataresse and Luechin [11] and La and Steinhardt [8]. Bali and Jain [2] have presented Bianchi type-I inflationary universe in general relativity. Inflationary cosmological models in four and five dimensions in general relativity have been studied by Reddy et al. [15] and Reddy and Naidu [14].

The cosmological term-Λ provides a repulsive force opposing the gravitational pull between the galaxies. Linde [9] has suggested that Λ is a function of temperature and is related to the spontaneous symmetry breaking process, and therefore it could be a function of time. The existence of the cosmological term-Λ is favourable to recent supernovae (SNe) Ia observations [7, 17] and which is also consistent with the recent anisotropy measurements of the cosmic microwave background (CMB) made by WMAP experiment [3]. Pradhan and Otarod [12, 13] have obtained the solution of Einstein's field equations with time dependent deceleration parameter and Λ-term in presence of perfect and bulk viscous fluid.

Jain et al. [5] have presented Bianchi type–I cosmological model with a varying Λ-term in self creation theory. Recently the inflationary Kantowski-Sachs cosmological model in general relativity is investigated by Katore and Rane [6]. Reddy et al. [16] have studied about plane symmetric Bianchi type-I inflationary universe in general relativity. Motivated by these above arguments, in this paper, Kantowski-Sachs inflationary cosmological model in presence of cosmological term-Λ is investigated.

This paper is organized as follows: The metric and field equations are considered in Sect.2. Solutions of field equations are obtained in Sect.3. Some important physical and geometrical features of the model are discussed in Sect.4. In last Sect.5, conclusions are given.

2.0 THE METRIC AND FIELD EQUATIONS

We consider the Kantowski-Sachs metric in the form

$$ds^2 = dt^2 - A^2 dr^2 - B^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \qquad ...(1)$$

Where the metric potentials A and B are functions of cosmic time t only.

In the case of gravity minimally coupled to a scalar field $V(\phi)$ (18), the Lagrangian L is

$$L = \int \sqrt{-g}\,(R - \frac{1}{2} g^{ij} \phi_{,i}\, \phi_{,j} - V(\phi))\, d^4x \qquad ...(2)$$

Which on variation of L, with respect to dynamical fields, leads to Einstein field equations

$$G_i^j = -T_i^j + \Lambda(t)\, g_i^j \qquad ...(3)$$

Where $G_i^j = R_i^j - \frac{1}{2} R g_i^j$ is an Einstein's tensor and the contracted tensor T is trace of the energy momentum tensor that describes all non-gravitational and non- scalar field matter and energy.

The energy momentum tensor has the from

$$T_i^j = \phi_{,i}\, \phi^{,j} - [\frac{1}{2} \phi_{,k}\, \phi^{,k} + V(\phi)] g_i^j \qquad ...(4)$$

and

$$\frac{1}{\sqrt{-g}} \partial_{,i} (\sqrt{-g}\, \partial^{,i} \phi) = -\frac{dV(\phi)}{d\phi} \qquad ...(5)$$

Where comma (,) and semicolon (;) indicate ordinary and covariant differentiation respectively. The function ϕ depends on t only due to homogeneity. Other symbols have their usual meaning and units are taken such that

$$8\pi G = c = 1 \qquad ...(6)$$

By adoption of co-moving coordinates the field equations (3) for the line element (1) can be written as

$$2\frac{B_{44}}{B}+\frac{B_4^2}{B^2}+\frac{1}{B^2}=-[\frac{1}{2}\phi_4^2+V(\phi)]+\Lambda(t) \qquad ...(7)$$

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}+\frac{1}{B^2}=-[\frac{1}{2}\phi_4^2+V(\phi)]+\Lambda(t) \qquad ...(8)$$

$$2\frac{A_4B_4}{AB}+\frac{B_4^2}{B^2}+\frac{1}{B^2}=\frac{1}{2}\phi_4^2-V(\phi)+\Lambda(t) \qquad ...(9)$$

and the scalar field is

$$\phi_{44}+\phi_4(\frac{A_4}{A}+2\frac{B_4}{B})+\frac{dV(\phi)}{d\phi}=0 \qquad ...(10)$$

Here the suffix 4 indicates ordinary differentiation with respect to cosmic time t only.

3.0 SOLUTIONS OF THE FIELD EQUATIONS

Stein-Schabes [18] has shown that the scalar field ϕ will take sufficient time to cross the flat region so that the universe expands sufficiently to become homogeneous and isotropic on the scale of the order of the horizon size. Thus, we are interested here, in inflationary solutions of the field equations (7)-(10).

The flat region is considered where potential is constant i.e.

$$V(\phi)=cons\tan t=V_0 \text{ (say)} \qquad ...(11)$$

since the field equations are highly non-linear, here, we also assume the relation between potentials, i.e.

$$A=kB^2 \qquad ...(12)$$

Where k is constant.

From equations (7), (8) and (12), we have

$$2BB_{44}-B_4^2=1 \qquad ...(13)$$

Let us use the transformation

$$B_4=f(B)\ \therefore B_{44}=ff',\ f'=\frac{df}{dB} \qquad ...(14)$$

Equation (13) leads to

$$f=\frac{dB}{dt}=[k_1B-1]^{1/2} \qquad ...(15)$$

Equation (15) yields

$$B=(at+b)^2 \qquad ...(16)$$

where

$$a = \frac{k_1}{2} \text{ and } b = \frac{k_1 k_2}{2}$$

Equation (12) leads to

$$A = k(at+b)^4 \quad ...(17)$$

Therefore, the metric (1) reduces to the form

$$ds^2 = dt^2 - k^2(at+b)^8 dr^2 - (at+b)^4(d\theta^2 + \sin^2\theta\, d\phi^2) \quad ...(18)$$

After suitable transformation of coordinates, metric (18) reduces into the form

$$dS^2 = \frac{1}{a^2} dT^2 - k^2 T^8 dr^2 - T^4(d\theta^2 + \sin^2\theta\, d\phi^2) \quad ...(19)$$

Which represents Kantowski-Sachs inflationary cosmological model with varying Λ- term in general relativity.

Using equations (11) and (12) in equation (10), we get

$$\phi_{44} + 4\frac{B_4}{B}\phi = 0 \quad ...(20)$$

Which on integration gives

$$\phi = k_4 - \frac{k_5}{T^7} \quad ...(21)$$

Where

$$k_5 = \frac{k_3}{7a}$$

4.0 SOME PHYSICAL AND GEOMETRICAL FEATURES

After using equations (16), (17) and (21) in (9), the time dependent cosmological term Λ is given by

$$\Lambda = \frac{20a^2}{T^2} + \frac{1}{T^4} + \frac{k_3^2}{2T^{16}} + V_0 \quad ...(22)$$

The scalar of expansion θ calculated for the flow vector v^i is given by

$$\theta = \frac{8k}{T} \quad ...(23)$$

The Hubble parameter H is given by

$$H = \frac{8a}{3T} \quad ...(24)$$

$$V = R^3 = kT^8 \quad \text{...(25)}$$

Where R is a average scale factor.

The expansion velocity is given by

$$R_4 = \frac{8a}{3} k^{1/3} T^{5/3} \quad \text{...(26)}$$

For the model (19) deceleration parameter is calculated as

$$q = -\frac{5}{8} \quad \text{...(27)}$$

For the model (19), the particle horizon exist because

$$\int_{T_0}^{T} \frac{dt}{R(t)} = \int_{T_0}^{T} \frac{dt}{k^{1/3}(at+b)^{8/3}}$$

$$= -\frac{3}{5a} [(at+b)^{-5/3}]_{T_0}^{T} \quad \text{...(28)}$$

is a convergent integral.

In the model we observe that the spatial volume V is zero at $T = 0$ or $t = -b/a = t_0$ and scalar expansion θ is infinite at initial singularity $t = t_0$ which shows that the universe starts evolving with zero volume and infinite rate of expansion at $t = t_0$. As T increases, the spatial volume V increases but the scalar expansion decreases. Thus, the expansion rate decreases as time increases. As $T \to \infty$ the spatial volume V becomes infinitely large.

.Clearly $T \to 0$ gives $\Lambda \to \infty$ and $T \to \infty$ gives $\Lambda \to V_0$. The cosmological term-Λ has constant value with in the range $0 < T < \infty$. The value of cosmological constant Λ is in an excellent agreement with observations [7, 17] of type Ia Supernovae (SNe). The main conclusion of these observations is that the expansion of the universe is accelerating and the cosmological term was very large at initial times which relaxes to a genuine cosmological constant with due course of time.

Scalar field ϕ is constant when $T \to \infty$. The expansion velocity R_4 diverges as $T \to 0$, hence the expansion of the universe is infinite as we approach towards $t \to t_0$.

5.0 CONCLUDING REMARKS

In this paper, Kantowski-Sachs inflationary cosmological model with varying Λ- term in general relativity is investigated. Equation (19) shows that the model will represent an expanding universe. The anisotropic expansion of the universe with time is evident from the model. The value of cosmological constant Λ is in an excellent agreement with observations [7, 17] of type Ia Supernovae (SNe). The main conclusion of these observations is that the expansion of the universe is accelerating and the cosmological term was very large at initial times which relaxes to a genuine cosmological constant with due course of time. The model obtained in this paper is of considerable interest and may be useful in general theory of relativity.

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