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FRACTIONAL INTEGRATION OF MULTIVARIABLE H-FUNCTION VIA PATHWAY OPERATOR

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ABSTRACT

On account of the importance of pathway model introduced by Mathai [4] *in statistical distributions, the authors are motivated to derive the fractional integration of the multi variable H-function via pathway operator. The results are of generalized nature and some known and unknown results are deduced, as special cases.*

Mathematics Subject Classification :- 33C60 , 26A33.

Key words and Phrases :- Multivariable H-function , Pathway model, probability density function, fractional integrals.

1. INTRODUCTION

Fractional integration operators play an important role in the solution of several problems of diversified fields of science and engineering. Many fractional integral operators like Riemann-Liouville , Weyl, Kober, Erdély-Kober and Saigo operators are studied by various workers due to their applications in the solution of integral equations arising in several problem of many areas of physical, engineering & Technological sciences, such as reaction, diffusion reactun-diffusion, Viscoelasticity, Rheology etc. A detailed description of these operators can be found in the survey paper by Srivastava and Saxena [22].

Let $f(x) \in L(a,b)$, $\alpha \in C$, $\Re(\alpha) > 0$, then left sided Riemann-Liouville fractional integral operator is defined as

$$\left(I_{0+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \qquad \ldots(1.1)$$

where $\Re(\alpha) > 0$.

DEFINITION 1 Let $f(x) \in L(a,b)$, $\eta \in C$, $\Re(\eta) > 0$, $a > 0$ and let us take a "pathway parameter" $\alpha < 1$. Then the pathway fractional integration operator is defined by Nair [9] as

$$\left(P_{0+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_{0}^{\left[\frac{x}{a(1-\alpha)}\right]} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)\, dt \qquad \text{...(1.2)}$$

For further details of pathway model and its applications one can refer to paper by Mathai and Haubold [5, 6]. For $\Re(\alpha) > 0$, the pathway model for scalar random variables is represented by the following probability density function (p.d.f.)

$$f(x) = c\,|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} \qquad \text{...(1.3)}$$

$$-\infty < x < \infty, \quad \delta > 0, \quad \beta \geq 0, \; 1 - a(1-\alpha)|x|^{\delta} > 0, \; \gamma > 0,$$

where c is the normalizing constant and α is the pathway parameter. For real α, the normalizing constant is as follows:

$$c = \frac{1}{2} \frac{\delta\left[a(1-\alpha)\right]^{\frac{\gamma}{\delta}} \Gamma\left[\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1\right]}{\Gamma\left[\frac{\gamma}{\delta}\right] \Gamma\left[\frac{\beta}{1-\alpha} + 1\right]}, \text{ for } \alpha < 1. \qquad \text{...(1.4)}$$

$$= \frac{1}{2} \frac{\delta\left[a(\alpha-1)\right]^{\frac{\gamma}{\delta}} \Gamma\left[\frac{\beta}{\alpha-1}\right]}{\Gamma\left[\frac{\gamma}{\delta}\right] \Gamma\left[\frac{\beta}{\alpha-1} - \frac{\gamma}{\delta}\right]}, \textit{ for } \frac{1}{\alpha-1} - \frac{\gamma}{\delta} > 0, \; \alpha > 1. \qquad \text{...(1.5)}$$

$$= \frac{1}{2} \frac{\delta(a\,\beta)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)} \text{ for } \alpha \to 1. \qquad \text{...(1.6)}$$

It may be observed that for $\alpha < 1$, it is a finite range density with $1 - a(1-\alpha)|x|^{\delta} > 0$ and (1.3) remains in the extended generalized type-1 beta family. The pathway density in (1.3), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

When $\alpha > 1$, we write $1-\alpha = -(\alpha-1)$, then

$$\left(P_{0+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_{0}^{\left[\frac{x}{-a(\alpha-1)}\right]} \left[1 + \frac{a(\alpha-1)t}{x}\right]^{\frac{\eta}{-(\alpha-1)}} f(t)\, dt \qquad \text{...(1.7)}$$

$$f(x)=c\,|x|^{\gamma-1}\left[1+a(\alpha-1)\,|x|^{\delta}\right]^{-\frac{\beta}{\alpha-1}} \quad \ldots (1.8)$$

$$-\infty < x < \infty, \quad \delta > 0, \ \beta \geq 0, \ \alpha > 1,$$

which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F density, the student-t density, the Cauchy density.

Here we consider the case of pathway parameter for $\alpha < 1$. For $\alpha \to 1$ both (1.3) and (1.8) take the exponential form, since

$$\lim_{\alpha\to 1} c\,|x|^{\gamma-1}\left[1-a(1-\alpha)\,|x|^{\delta}\right]^{\frac{\eta}{1-\alpha}} = \lim_{\alpha\to 1} c\,|x|^{\gamma-1}\left[1+a(\alpha-1)\,|x|^{\delta}\right]^{-\frac{\eta}{\alpha-1}}$$

$$= c\,|x|^{\gamma-1}\,e^{-a\eta\,|x|^{\delta}} \quad \ldots (1.9)$$

when $\alpha \to 1$, $\left(1-\frac{a\,(1-\alpha)\,t}{x}\right)^{\frac{\eta}{1-\alpha}} \to e^{-\frac{a\eta}{x}t}$, then operator (1.2) reduces to

$$\left(P_{0+}^{(\eta,1)} f\right)(x) = x^{\eta} \int_0^{\infty} e^{-\frac{a\eta}{x}t} f(t)\,dt = x^{\eta}\, L_f\left(\frac{a\eta}{x}\right). \quad \ldots (1.10)$$

that is, it reduces to the Laplace integral transform of f with parameter, $\frac{a\eta}{x}$:

$$L_f(x) = \int_0^{\infty} e^{-xt} f(t)\,dt.$$

When $\alpha = 0$, $a = 1$ and η is replaced by $\eta - 1$ in (1.2), it yields

$$(I_{0+}^{\eta} f)(x) = \frac{1}{\Gamma(\eta)} \int_0^{x} (x-t)^{\eta-1} f(t)\,dt \quad \ldots (1.11)$$

which is the left-sided Riemann-Liouville fractional integral discussed in the monograph by Samko et al [14]

In a recent paper integration of Aleph function by means of pathway model is demonstrated by Saxena et al [15]. In this paper we will integrate the multivariable H-function by means of pathway model.

The multivariable H-function is defined and studied by Srivastava & Panda [20 , p 271, Eq. (4.1)] in term of a multiple Mellin-Bernes type contour integral as

$$H[Z_1,\ldots,Z_r] = H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix}\left(a_j;\zeta_j^{(1)},\ldots,\zeta_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\left\{\prod_{i=1}^{r}\phi_i(\xi_i)\,Z_i^{\xi_i}\right\}d\xi_1\ldots d\xi_r \qquad \ldots(1.12)$$

where $\omega = \sqrt{-1}$

$$\psi(\xi_1,\ldots,\xi_r) = \frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}\zeta_j^{(i)}\xi_i\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\zeta_j^{(i)}\xi_i\right)\prod_{j=1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i\right)}; \qquad \ldots(1.13)$$

$$\phi_i(\xi_i) = \frac{\prod_{l=1}^{m_i}\Gamma\left(d_l^{(i)}-\delta_l^{(i)}\xi_i\right)\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)}-\gamma_j^{(i)}\xi_i\right)\prod_{l=m_i+1}^{q_i}\Gamma\left(1-d_l^{(i)}+\delta_l^{(i)}\xi_i\right)}\ (i=1,\ldots,r) \qquad \ldots(1.14)$$

For a detailed definition and convergence conditions of the multivariable H-function, the reader is referred to the original paper by Srivastava and Panda [20] (also see Mathai et al [8] and Srivastava et al [17, 21]).

From Srivastava and Panda [21, p. 131], we have

$$H[Z_1, \ldots, Z_r] = O\left(|z_1|^{e_1} \ldots |z_r|^{e_r}\right) \quad \left(\max_{1 \le j \le r} \|Z_j\| \to 0\right) \qquad \ldots(1.15)$$

where

$$e_i = \min_{1 \le j \le m_i} \left[\frac{\text{Re}\,(d_j^{(i)})}{\delta_j^{(i)}}\right] \quad (i = 1, \ldots, r),$$

For $n = p = q = 0$ the multivariable H-function breaks up into product of 'r' H-functions and consequently there holds the following result (See Mathai et al [8])

$$H^{0,0:m_1,n_1;\ldots,m_r,n_r}_{0,0:p_1,q_1;\ldots,p_r,q_r}\left[\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix}\left|\begin{matrix} \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \\ \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right.\right]$$

$$= \prod_{i=1}^{r} H^{m_i,n_i}_{p_i,q_i}\left[z\left|\begin{matrix} \left(c_j^{(i)},\gamma_j^{(i)}\right)_{1,p_i} \\ \\ \left(d_j^{(i)},\delta_j^{(i)}\right)_{1,q_i} \end{matrix}\right.\right] \qquad \ldots(1.16)$$

where $H^{m,n}_{p,q}(\cdot)$ is the familiar H-function [3,7,8,17].

A general class of multivariable polynomials of real or complex variables $x_1, \ldots, x_s$ is defined and studied by Srivastava and Garg [18] in the following form

$$S_L^{h_1,\ldots,h_s}(x_1,\ldots,x_s) = \sum_{k_1,\ldots,k_s=0}^{h_1k_1+\ldots+h_sk_s \le L} (-L)_{h_1k_1+\ldots+h_sk_s}\, A(L;k_1,\ldots,k_s)\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_s^{k_s}}{k_s!}, \qquad \ldots(1.17)$$

where $L, h_1, \ldots, h_s \in \mathbb{N}_0 = \{0,1,\cdots\}$ and the coefficients $A(L;k_1,\ldots,k_s)$ $(k_j \in \mathbb{N}_0;\ j = 1,\cdots,s)$ are arbitrary constants real or complex. Evidently the case $s = 1$ of the above polynomials would correspond to the polynomials (see Srivastava [16])

$$S_L^h(x) = \sum_{k=0}^{[l/h]} \frac{(-l)_{hk}}{k!} A_{l\,k}\, x^k \qquad \ldots(1.18)$$

The coefficients A_{lk} $(l, k \in \mathbb{N}_0)$ are arbitrary constants real or complex.

2. MAIN RESULT

Theorem 2.1 . Let $\eta, \rho \in C$, $\Re(\beta) > 0$, $\Re\left(1+\frac{\eta}{1-\alpha}\right) > 0$, $\Re\left(\rho + \mu_i e_i\right) > 0$ $(i = 1, \ldots, r)$ and $\alpha < 1$, $b \in R$. Then for the pathway fractional operator $P_{0+}^{(\eta,\alpha)}$ the following formula holds:

$$\left(P_{0+}^{(\eta,\alpha)}\, t^{\rho-1}\, H[Z_1 t^{\mu_1}, \ldots, Z_r t^{\mu_r}]\right) = \frac{x^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}}$$

$$H^{0,n+1:m_1,n_1;\ldots,m_r,n_r}_{p+1,q+1:p_1,q_1;\ldots,p_r,q_r}\left[\begin{array}{c} z_1\left[\frac{x}{a(1-\alpha)}\right]^{\mu_1} \\ \vdots \\ z_r\left[\frac{x}{a(1-\alpha)}\right]^{\mu_r} \end{array}\middle|\begin{array}{l} (1-\rho;\mu_1,\ldots,\mu_r),\left(a_j;\zeta_j^{(1)},\ldots,\zeta_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \\ \left(-\rho-\frac{\eta}{1-\alpha};\mu_1,\ldots,\mu_r\right),\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{array}\right]$$

... (2.1)

where e_i is defined in (1.15)

Proof :- Using equation (1.2) and (1.12), it follows that

$$I = x^{\eta}\int_0^{\frac{x}{a(1-\alpha)}} t^{\rho-1}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} dt\ \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)$$

$$\left\{\prod_{i=1}^{r}\varphi_i(\xi_i)\,[z_i t^{\mu_i}]^{\xi_i}\right\} d\xi_1,\ldots,d\xi_r$$

$$= x^{\eta} \int_{0}^{\frac{x}{a(1-\alpha)}} t^{\rho-1} \left[1-\frac{a\,(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} dt \; \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\,(\xi_1, \ldots, \xi_r)$$

$$\left\{ \prod_{i=1}^{r} \varphi_i\,(\xi_i)\; Z_i^{\xi_i} \right\} t^k \; d\xi_1, \ldots, d\xi_r$$

where $k = \sum_{i=1}^{r} \mu_i \xi_i$

$$= x^{\eta} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi\,(\xi_1, \ldots, \xi_r) \left\{ \prod_{i=1}^{r} \varphi_i\,(\xi_i)\; Z_i^{\xi_i} \right\} d\xi_1, \ldots, d\xi_r$$

$$\int_{0}^{\frac{x}{a(1-\alpha)}} t^{\rho+k-1} \left[1-\frac{a\,(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} dt$$

Put $\frac{a\,(1-\alpha)\,t}{x} = V$, interchange the order of integration and evaluate the inner integral by means of beta function formula, it gives

$$= \frac{x^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1, \ldots, \xi_r) \left\{ \prod_{i=1}^{r} \varphi_i\,(\xi_i)\; Z_i^{\xi_i} \right\} d\xi_1, \ldots, d\xi_r$$

$$\chi \quad \left[\frac{x}{a(1-\alpha)}\right]^{k} \frac{\Gamma\,(\rho+k)}{\Gamma\left(1+\frac{\eta}{1-\alpha}+\rho+k\right)} \qquad \ldots(2.2)$$

On interpreting the above result by mean of (1.12), we obtain the desired result (2.1).

It is interesting to observe that for $\alpha=0$, $a=1$, Theorem 2 reduces to the result given by Gupta et al [2].

Note 2.1. We note that the fractional integration of multivariable H-function has been discussed earlier, among others, by Saigo and Saxena [10-12], Saigo et al [13], Srivastava et al [19], Srivastava and Panda [20,21], Gupta et al [2] and others.

If we set $n=p=q=0$ then by virtue of the identity (1.16), we obtain

Corollary 2.1 If $\eta,\rho\in C$, $\Re(\beta)>0$, $\Re\left(1+\frac{\eta}{1-\alpha}\right)>0$, $\Re(\rho+\mu_i e_i)>0$,

$(i=1,\ldots r)$, $\alpha<1, b\in R$, then there holds the result

$$\left(P_{0+}^{(\eta,\alpha)}\, t^{\rho-1}\prod_{i=1}^{r} H_{p_i,q_i}^{m_i,n_i}\left[z_i\, t^{\mu_i}\left|\begin{matrix}\left(c_j^{(i)},\gamma_j^{(i)}\right)_{1,p_i}\\ \\ \left(d_j^{(i)},\delta_j^{(i)}\right)_{1,q_i}\end{matrix}\right.\right]\right)$$

$$=\frac{x^{\rho+\eta}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\left[a(1-\alpha)\right]^{\rho}}\, H_{1,1:p_1,q_1;\ldots;p_r,q_r}^{0,1:m_1,n_1;\ldots;m_r,n_r}\left[z_i\left[\frac{x}{a(1-\alpha)}\right]^{\mu_i}\left|\begin{matrix}(1-\rho;\mu_1,\ldots,\mu_r),:\left(c_j^{(i)},\gamma_j^{(i)}\right)_{1,p_1}\\ \left(-\rho-\frac{\eta}{1-\alpha};\mu_1,\ldots,\mu_r\right),:\left(d_j^{(i)},\delta_j^{(i)}\right)_{1,q_1}\end{matrix}\right.\right]$$

...(2.3)

If we further take $r=1$, we arrive at the result given by Nair [9, p. 242]

In order to obtain the Laplace transform of H-function of several variables, we evaluate the expression

$$\lim_{\alpha\to 1}\frac{x^{\rho+\eta}\,\Gamma(\rho+k)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\left[a(1-\alpha)\right]^{\rho+k}\,\Gamma\left(\rho+k+1+\frac{\eta}{1-\alpha}\right)}$$

which can be seen from (2.2). Expanding the gamma function by Stirling formula for gamma function, namely

$$\Gamma(1-z)\sim(2\pi)^{1/2}\,(z)^{1+z-1/2}\,e^{-z},$$

the above expression becomes

$$= \lim_{\alpha \to 1} \frac{x^{\rho+\eta}\,\Gamma(\rho+k)\left(\frac{\eta}{1-\alpha}\right)^{1+\frac{\eta}{1-\alpha}-1/2}}{[a(1-\alpha)]^{\rho+k}\left(\frac{\eta}{1-\alpha}\right)^{\frac{\eta}{1-\alpha}+\rho+k+1-1/2}} = \lim_{\alpha \to 1} \frac{x^{\rho+\eta}\,\Gamma(\rho+k)}{[a(1-\alpha)]^{\rho+k}\left(\frac{\eta}{1-\alpha}\right)^{\rho+k}}$$

$$= \frac{x^{\rho+\eta}\,\Gamma(\rho+k)}{[a\eta]^{\rho+k}}$$

Now using the limit formula (1.9) we arrive at

Corollary 2.2 If $\Re(\rho+\mu_i e_i) > 0,\ R\left(\frac{a\eta}{x}\right) > 0,\ (i=1,\ldots,r)$, then there holds the formula

$$\int_0^\infty e^{-\frac{a\eta}{x}t}\, t^{\rho-1}\, H[Z_1 t^{\mu_1},\ldots,Z_r t^{\mu_r}]\,dt$$

$$= \frac{x^{\rho+\eta}}{(a\eta)^{\rho}} \cdot H^{0,n+1:m_1,n_1;\ldots,m_r,n_r}_{p+1,q+1:p_1,q_1;\ldots,p_r,q_r}\left[\begin{matrix} z_1\left[\frac{x}{a\eta}\right]^{\mu_1} \\ \vdots \\ z_r\left[\frac{x}{a\eta}\right]^{\mu_r}\end{matrix}\left|\begin{matrix}(1-\rho;\mu_1,\ldots,\mu_r),\left(a_j;\zeta_j^{(1)},\ldots,\zeta_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \\ \left(-\rho-\frac{\eta}{1-\alpha};\mu_1,\ldots,\mu_r\right),\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right]$$

... (2.4)

By following a similar procedure, it is not difficult to establish

Theorem 3.1 . Let $\eta,\rho \in C$, $\Re(\beta) > 0$, $\Re\left(1+\frac{\eta}{1-\alpha}\right) > 0$, $\Re\left(\rho+\mu_i\, e_i\right) > 0\ \ (i=1,\ldots,r)$ and $\alpha < 1,\ b \in R$. Then for the pathway fractional integral $P_{0+}^{(\eta,\alpha)}$, fractional integral of the product of multivariable H-function and $S_L^{h_1,\ldots,h_s}(\cdot)$ exists and there holds formula :

$$\left(P_{0+}^{(\eta,\alpha)}\left[t^{\rho-1}\, S_L^{h_1,\ldots,h_s}\left(y_1 t^{\lambda_1},\ldots,y_s t^{\lambda_s}\right) H\left(Z_1 t^{\mu_1},\ldots,Z_r t^{\mu_r}\right)\right]\right)$$

$$= \frac{x^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\left[a(1-\alpha)\right]^{\rho}} \sum_{k_1,\ldots,k_s=0}^{h_1k_1+\ldots+h_sk_s\le L} (-L)_{h_1k_1+\ldots+h_sk_s}\, A(L;k_1,\ldots,k_s)\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_1!}\, x^{\lambda_1k_1+\ldots+\lambda_sk_s}$$

$$H^{0,n+1:m_1,n_1;\ldots,m_r,n_r}_{p+1,q+1:p_1,q_1;\ldots,p_r,q_r}\left[\begin{matrix} z_1\left[\frac{x}{a(1-\alpha)}\right]^{\mu_1} \\ \vdots \\ z_r\left[\frac{x}{a(1-\alpha)}\right]^{\mu_r}\end{matrix}\,\middle|\, \begin{matrix}\left(1-\rho-\sum_{i=1}^{s}\lambda_i k_i;\mu_1,\ldots,\mu_r\right),\left(a_j;\zeta_j^{(1)},\ldots,\zeta_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \\ \left(-\rho-\frac{\eta^*}{1-\alpha^*}-\sum_{i=1}^{s}\lambda_i k_i;\mu_1,\ldots,\mu_r\right),\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right]$$

... (3.1)

where $S_L^{h_1,\ldots,h_s}(\cdot)$ is defined in (1.17) and e_i is defined in (1.15).

When $n=p=q=0$ then by virtue of the identity (1.16), we obtain

Corollary 3.1 If $\eta,\rho\in C$, $\Re(\beta)>0$, $\Re\left(1+\frac{\eta}{1-\alpha}\right)>0$, $\Re(\rho+\mu_i e_i)>0$,

$(i=1,\ldots r)$, $\alpha<1, b\in R$, then there holds the following result :

$$\left(P_{0+}^{(\eta,\alpha)}\left[t^{\rho-1}\, S_L^{h_1,\ldots,h_s}\left(y_1 t^{\lambda_1},\ldots,y_s t^{\lambda_s}\right)\prod_{i=1}^{r} H^{m_i,n_i}_{p_i,q_i}\left[z_i\, t^{\mu_i}\,\middle|\,\begin{matrix}\left(c_j^{(i)},\gamma_j^{(i)}\right)_{1,p_i} \\ \left(d_j^{(i)},\delta_j^{(i)}\right)_{1,q_i}\end{matrix}\right]\right]\right)$$

$$= \frac{x^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\left[a(1-\alpha)\right]^{\rho}} \sum_{k_1,\ldots,k_s=0}^{h_1k_1+\ldots+h_sk_s\le L} (-L)_{h_1k_1+\ldots+h_sk_s}\, A(L;k_1,\ldots,k_s)\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_1!}\, x^{\lambda_1k_1+\ldots+\lambda_sk_s}$$

$$H^{0,1:m_1,n_1;\ldots;m_r,n_r}_{1,1:p_i,q_i;\ldots;p_r,q_r}\left[z_i\left[\frac{x}{a(1-\alpha)}\right]^{\mu_i} \middle| \begin{matrix} \left(1-\rho-\sum_{i=1}^{s}\lambda_i k_i;\mu_1,\ldots,\mu_r\right):(c_j^{(1)},\gamma_j^{(1)})_{1,p_i} \\ \left(-\rho-\frac{\eta}{1-\alpha}-\sum_{i=1}^{s}\lambda_i k_i;\mu_1,\ldots,\mu_r\right):(d_j^{(1)},\delta_j^{(1)})_{1,q_i}\end{matrix}\right]$$

...(3.2)

which holds under the same conditions as given in (3.1)

Remark 9 The condition of convergence, as given by Nair [9, p.212], are erroneous. The

condition $\mathrm{Re}(\rho)$ should be corrected to $\mathrm{Re}\,(\rho)+\beta \min_{1\le j\le m} \mathrm{Re}\left(\frac{b_j}{\beta_j}\right) > 0$.

CONCLUSION

On account of the most general character of the H-function of several variables appearing in Theorem 2.1 and Theorem 3.1 numerous other special cases associated with potentially useful higher transcendental functions like Kampé de Fériet function, Lauricella function of several variables, H-function of two variables etc. can be deduced but for the sake of brevity, they are not presented. Further due to presence of the pathway parameter α, the results of this paper may find some applications in statistical distributions.

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R-WEAK COMMUTATIVITY AND COMMON FIXED POINT IN FUZZY METRIC SPACE

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ABSTRACT

In this note we shall prove common fixed point of four self maps in a fuzzy metric space in the setting of R-weak commutativity. Our results generalize the results of Singh, Jain and Jain [10].

Mathematics Subject Classification: 54A35, 54A40, 54H25.

Keywords: - Fuzzy metric space; R-weakly commuting maps; fixed point.

INTRODUCTION

Fuzzy set was defined by Zadeh [13] in 1965. Kramosil and Michalek [5] introduced fuzzy metric space, George and Veeramani [3] modified the concept of fuzzy metric spaces with the help of continuous t-norms. They also showed that every metric space induces a fuzzy metric. Vasuki [12] proved fixed point theorems for R-weakly commuting mappings. Cho, Sharma and Sahu [1] introduced the concept of semi compatibility of maps in d-complete metric spaces Singh and Chouhan [8] prove the existence of a unique common fixed point of four self maps A, B, S and T in a fuzzy metric space taking two of the maps S and T to be continuous and assuming pairs (A, S) and (B, T) to be compatible.

Singh, Jain and Jain [10] prove the existence of a unique common fixed point of four self maps A,B, S and T in a fuzzy metric space taking one of the four maps to be continuous and assuming pairs (A,S) and (B,T) to be semi compatible.

Here we prove a similar theorem on existence of unique common fixed point of the four maps in a fuzzy metric space in which we drop the assumption of continuity and take R-weak commutativity in place of semi compatibility.

PRELIMINARIES

Definition 1: [13] A fuzzy set A in X is a function with domain X and values in [0, 1]

Definition 2:[11] A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t- norm if {[0, 1], *} is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d, \in [0, 1]$.

Examples of t- norm are $a * b = ab$ and $a* b = \min \{a, b\}$.

Definition 3: [5] the triplet (X, M, *) is a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions;

(i) $M(x, y, 0) = 0$.

(ii) $M(x, y, t) = 1 \ \forall\ t > 0$ if and only if $x = y$.

(iii) $M(x, y, t) = M(y, x, t)$.

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \ \forall\ x, y, z \in X$ and $t, s, > 0$.

(v) $M(x, y, .) : [0, \infty) \rightarrow [0, 1]$ is left continuous.

(vi) $\lim_{t\rightarrow\infty} M(x, y, t) = 1 \ \forall\ x, y, \in X$.

Definition 4: [5] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is called Cauchy if $\lim_{n\rightarrow\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and each $p > 0$.

Definition 5: [5] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to be convergent to $x \in X$ if $\lim_{t\rightarrow\infty} M(x_n, x, t) = 1$, for each $t > 0$.

Definition 6: [5] A fuzzy metric space (X, M, *) is said to be complete if every Cauchy sequence in X converges in X.

Definition 7: [6] Two mappings f and g of a fuzzy metric space (X, M, *) into itself are said to be weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for every $x \in X$.

Definition 8: [12] The mappings f and g of a fuzzy metric space (X, M, *) into itself are R- weakly commuting provided there exists some positive real number R such that $M(fgx, gfx, t) \geq M(fx, gx, t/R)$ for all $x \in X$.

Weak commutativity implies R-weak commutativity and the converse is true for $R \leq 1$.

Lemma 1: [4] Let (X, M, *) be a fuzzy metric space. Then for all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.

Lemma 2: [2] Let (X, M, *) a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$, $M(x, y, kt) \geq M(x, y, t) \quad \forall\ t > 0$, then $x = y$.

Lemma 3: [9] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number $k \in (0, 1)$ such that $M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)\ \forall\ t > 0$ and $n \in N$. Then $\{x_n\}$ is a Cauchy sequence in X.

Proposition 1: [10] In a fuzzy metric space (X, M, *) limit of a sequence is unique.

Proof: Let $\{x_n\}$ be a sequence in X such that $\{x_n\} \to x$ and $\{x_n\} \to y$ then $\lim_{n\to\infty} M(x_n, x, t) = 1 = \lim_{n\to\infty} M(x_n, y, t)$.

Now, $M(x, y, t) \geq M(x, x_n, t/2) * M(y, x_n, t/2)$.

Taking limit as $n\to\infty$, we get $\quad M(x, y, t) \geq 1 * 1$

I.e. $M(x, y, t) = 1$ for all $t > 0$. Thus $x = y$ and hence the limit is unique.

Now, we give an example of a pair of maps (A, S) which is semi compatible but not compatible.

Example- Let X = [0, 2], define

$$Sx = 1,\ x \in [0, 1) \qquad Ax = 2,\ x \in [0, 1]$$

$$= 2,\ x = 1; \qquad = x/2,\ x \in (1, 2]$$

$$= (x+3)/5,\ x \in (1, 2]$$

and $\quad x_n = 2 - 1/(2n)$ and $M(x, y, t) = t/[t+|x-y|]$

We have $\quad S(1) = A(1) = 2$ and $S(2) = A(2) = 1$.

Also $\quad SA(1) = AS(1) = 1$ and $SA(2) = AS(2) = 2$.

Hence $\quad Ax_n \to 1$ and $Sx_n \to 1$, $ASx_n \to 2$ and $SAx_n \to 1$.

Now, $\quad \lim_{n\to\infty} M(ASx_n, Sy, t) = (2, 2, t) = 1$

and $\quad \lim_{n\to\infty} M(ASx_n, SAx_n, t) = (2, 1, t) = t/\{t+1\} < 1$.

Hence (A, S) is semi compatible but not compatible.

MAIN RESULTS

Theorem 1: Let A, B, S and T be self mappings of a complete fuzzy metric space (X, M, *) with continuous t-norm defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$ satisfying

(1.1) $A(X) \subset T(X), B(X) \subset S(X)$.

(1.2) Pairs (A, S) and (B, T) are R-weakly commuting.

(1.3) $\exists$ some $k \in (0, 1)$ such that for all $x, y \in X, t > 0$

$$M(Ax, By, Bt) \geq \text{Min}\{M(Sx, Ty, t), M(Sx, Ax, t),$$

$$M(Ty, By, t), M(Sx, By, 2t), M(Ty, Ax, t)\}$$

Then A, B, S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be any point. As $A(X) \subset T(X)$ and $S(X) \subset B(X)$, $\exists$ $x_1 \in X$ and $x_2 \in X$ such that

$Ax_0 = Tx_1$, and $Bx_1 = Sx_2$. Inductively we construct a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}, (y_{2n} = Sx_{2n}), n = 0, 1...$$

Using (1.3), we have,

$$M(y_{2n+1}, y_{2n+2}, kt) = M(Ax_{2n}, Bx_{2n+1}, kt)$$

$$\geq \text{Min}\{M(Sx_{2n}, Tx_{2n+1}, t), M(Sx_{2n}\ Ax_{2n}, t), M(Tx_{2n+1}, Bx_{2n+1}, t),$$

$$M(Sx_{2n}, Bx_{2n+1}, 2t), M(Tx_{2n+1}, Ax_{2n}, t)\}$$

$$= \text{Min}\{M(y_{2n}, y_{2n+1}, t), M(y_{2n}\ y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t),$$

$$M(y_{2n}, y_{2n+2}, 2t), M(y_{2n+1}, y_{2n+1}, t)\}$$

$$= \text{Min}\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}\ y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t)$$

$$* M(y_{2n+1}, y_{2n+2}, t), 1\}$$

$$= \text{Min}\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t)\}$$

$= M(y_{2n+1}, y_{2n}, t)$

As M(x, y, t) is non- decreasing (George, and Veeramani, 1994).

Similarly, $M(y_{2n+1}, y_{2n}, kt) \geq M(y_{2n}\, y_{2n-1}, t)$.

Hence $M(y_{n+1}, y_n, kt) \geq M(y_n, y_{n-1}, t)\ \forall\ n$

We show that

$$\lim_{n\to\infty} M(y_{n+p}, y_n, t) = 1 \text{ for all p and } t > 0$$

Now, $M(y_{n+1}, y_n, t) \geq M(y_n, y_{n-1}, t/k)$

$$\geq M(y_{n-1}, y_{n-2}, t/k^2)$$

$$\geq \ldots$$

$$\geq M(y, y_0, t/k^n) \to 1 \text{ as } t/k^2 \to \infty \text{ as } n \to \infty.$$

Thus the result holds for $p = 1$.

By induction hypothesis suppose that the result holds for $p = r$.

Now, $M(y_n, y_{n+r+1}, t) \geq M(y_n, y_{n+r}, t/2) * M(y_{n+r}, y_{n+r+1}, t/2)$

$$\to 1 * 1 = 1$$

Thus the result holds for $p = r + 1$

Hence $\{y_n\}$ is a Cauchy sequence in X and as X is complete we get $\{y_n\} \to z \in X$.

Hence,

$$Ax_{2n} \to z, \qquad Sx_{2n} \to z \qquad \ldots (1)$$

$$Tx_{2n+1} \to z, \qquad Bx_{2n+1} \to z \qquad \ldots (2)$$

Since $A(X) \subset T(X)$ $\exists$ for $p \in X$ such that $p = T^{-1} z$ i.e. $Tp = z$.

By (1.3) we have; (at $x = x_{2n}$ and $y = p$)

$$M(Ax_{2n}, Bp, kt) \geq \text{Min}\{M(Sx_{2n}, Tp, t), M(Sx_{2n}, Ax_{2n}, t),$$

$$M(Tp, Bp, t), M(Sx_{2n}, Bp, 2t), M(Tp, Ax_{2n}, t)\}$$

$$M(Ax_{2n}, Bp, kt) \geq \text{Min}\{M(Sx_{2n}, z, t), M(Sx_{2n}, Ax_{2n}, t),$$

$$M(z, Bp, t), M(Sx_{2n}, Bp, 2t), M(z, Ax_{2n}, t)\}$$

Taking the $\text{limit}_{n\to\infty}$, we have;

$$M(z, Bp, kt) \geq \text{Min}\{M(z, z, t), M(z, z, t), M(z, Bp, t),$$

$$M(z, Bp, 2t), M(z, z, t)\}$$

$$M(z, Bp, kt) \geq \text{Min}\{M(z, Bp, t), M(z, Bp, 2t)\}$$

$$M(z, Bp, kt) \geq M(z, Bp, t)$$

This implies that $z = Bp$. Since $Tp = z$ therefore $Bp = Tp = z$. i.e. p is a coincidence point of B and T.

Similarly since $B(X) \subset S(X)$ $\exists$ $q \in X$ such that $q = S^{-1}z$ i.e. $Sq = z$. By (1.3) we have (at $x = q$ and $y = x_{2n+1}$)

$$M(Aq, Bx_{2n+1}, kt) \geq \text{Min}\{M(Sq, Tx_{2n+1}, t), M(Sq, Aq, t),$$

$$M(Tx_{2n+1}, Bx_{2n+1}, t), M(Sq, Bx_{2n+1}, 2t), M(Tx_{2n+1}, Aq, t)\}$$

$$M(Aq, Bx_{2n+1}, kt) \geq \text{Min}\{M(z, Tx_{2n+1}, t), M(z, Aq, t),$$

$$M(Tx_{2n+1}, Bx_{2n+1}, t), M(z, Bx_{2n+1}, 2t), M(Tx_{2n+1}, Aq, t)\}$$

Taking the $\lim_{n\to\infty}$ we have;

$$M(Aq, z, kt) \geq \text{Min}\{M(z, z, t), M(z, Aq, t),$$

$$M(z, z, t), M(z, z, 2t), M(z, Aq, t)\}$$

$$M(Aq, z, kt) \geq M(z, Aq, t)$$

This implies that $z = Aq$, since $Sq = z$.

Therefore $Aq = Sq = z$.

i.e. q is a coincidence point of A and S.

Since $\{A, S\}$ is R-weakly commuting by definition we have;

$$M(ASq, SAq, t) \geq M(Aq, Sq, t/R) \text{ for all } q \in X.$$

$$M(Az, Sz, t) \geq M(z, z, t/R)$$

$$M(Az, Sz, t) \geq 1$$

implies that $Az = Sz$.

By (1.3) we have;

$$M(Az, Bx_{2n+1}, kt) \geq \text{Min}\{M(Sz, Tx_{2n+1}, t), M(Sz, Az, t),$$

$$M(Tx_{2n+1}, Bx_{2n+1}, t), M(Sz, Bx_{2n+1}, 2t), M(Tx_{2n+1}, Az, t)\}$$

Taking the $\lim_{n\to\infty}$, we have,

$$M(Az, z, kt) \geq \text{Min}\{M(Az, z, t)\ M(Az, Az, t), M(z, z, t),$$

$$M(Az, z, 2t), M(z, Az, t)\}$$

$$M(Az, z, kt) \geq \text{Min}\{M(Az, z, t), M(Az, z, 2t)\}$$

$$M(Az, z, kt) \geq M(Az, z, t).$$

This implies that $Az = z$.

Since $Az = Sz$ therefore $Az = Sz = z$.

Since $\{B, T\}$ is R-weakly commuting therefore by defination we have,

$$M(BTp, TBp, t) \geq (Bp, Tp, t/R) \text{ for all } p \in X.$$

$$M(Bz, Tz, t) \geq (z, z, t/R)$$

$$M(Bz, Tz, t) \geq 1$$

implies that $Bz = Tz$.

By (1.3) we have, (at $p = z$)

$$M(Ax_{2n}, Bz, kt) \geq \text{Min}\{M(Sx_{2n}, Tz, t), M(Sx_{2n}, Ax_{2n}, t),$$

$$M(Tz, Bz, t), M(Sx_{2n}\ Bz, 2t), M(Tz, Ax_{2n}, t)\}$$

Taking the $\lim_{n\to\infty}$, we have;

$$M(z, Bz, kt) \geq \text{Min}\{M(z, Bz, t), M(z, z, t) M(Bz, Bz, t), M(z, Bz, 2t), M(Bz, z, t)\}$$

$$M(z, Bz, kt) \geq \text{Min}\{M(z, Bz, t), M(z, Bz, 2t)\}$$

$$M(z, Bz, kt) \geq M(z, Bz, t)$$

This implies that $z = Bz$.

Since $Bz = Tz$ therefore $Bz = Tz = z$.

Thus we have;

$$Az = Bz = Sz = Tz = z.$$

This means that z is a common fixed point of A, B, S and T.

Uniqueness- Let z and z' be two common fixed points of the maps A, B, S and T. Then

$$Az = Bz = Sz = Tz = z \text{ and } Az' = Bz' = Sz' = Tz' = z'$$

Using (1.3) we have;

$$M(Az, Bz', kt) \geq \text{Min}\{M(Sz, Tz', t), M(Sz, Az, t), M(Tz', Bz', t), M(Sz, Bz', 2t), M(Tz', Az, t)\}$$

$$M(z, z', kt) \geq \text{Min}\{M(z, z', t), M(z, z, t), M(z', z', t), M(z, z', 2t), M(z', z, t)\}$$

$$M(z, z', kt) \geq M(z, z', t)$$

This implies that $z = z'$.

Hence z is the unique common fixed point of the four maps A, B, S and T.

This completes the proof.

Theorem 2: Let A, B, S and T be four self maps of a complete fuzzy metric space (X, M, *) with continuous t-norm defined by $a * b = \min\{a, b\}$ satisfying 1.1, 1.2, and (2.1) $\forall$ x, y $\in$ X, $\forall$ $t > 0$ $\exists$ some $k \in (0, 1)$, such that $M(Ax, By, kt) \geq M(Sx, Ty, t)$. Then A, B, S and T have a unique common fixed point.

Proof: The proof can be given on the line of that of theorem 1. Here we have only one factor in condition (2.1) as against 4 factors as in theorem 1 condition (1.3).

Corollary 1: Let A, B, S and T be self mappings of a complete fuzzy metric space (X, M, *) with continuous t-norm defined by a * b = min {a, b} satisfying;

(1.1) $AS = SA$, $TB = BT$

(1.2) $A^m(X) \subset T^p(X)$, $B^n(X) \subset S^q(X)$, where $n, m, p, q \in N$

(1.3) for all x, y in X, $\exists$ some $k \in (0, 1)$ and for all $t > 0$

$$M(A^m x, B^n y, kt) \geq \text{Min}\{M(S^q x, T^p y, t), M(B^n y, T^p y, t),$$

$$M(A^m x, S^q x, t), M(B^n y, S^q x,, 2t), M(A^m x, T^p y, t)\}$$

Then A, B, S and T have a unique common fixed point.

Proof: As AS = SA and BT = TB we get $A^m S^q = S^q A^m$ and $B^n T^p = T^p B^n$

We want to show that (A^m, S^q) and (B^n, T^p) are R-weakly commuting.

Let $x_0 \in X$ be any point, As $A^m(X) \subset T^p(X)$, and $B^n(X) \subset S^q(X)$, $\exists\ x_1 \in X$ and $x_2 \in X$ such that

$A^m x_0 = T^p x_1$ and $B^n x_1 = S^q x_2$. Inductively we can construct a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = A^m x_{2n} = T^p x_{2n+1} \text{ and } y_{2n+2} = B^n x_{2n+1} = S^q x_{2n+2}, (y_{2n} = S^q x_{2n}), n = 0, 1\ldots$$

Using 1.3 of theorem 1we have,

$$M(y_{2n+1}, y_{2n+2}, kt) = M(A^m x_{2n}, B^n x_{2n+1}, kt)$$

$$\geq \text{Min}\{M(S^q x_{2n}, T^p x_{2n+1}, t), M(B^n x_{2n+1}, T^p x_{2n+1}, t),$$

$$M(A^m x_{2n}, S^q x_{2n}, t), M(B^n x_{2n+1}, S^q x_{2n}, 2t), M(A^m x_{2n}, T^p x_{2n+1}, t)\}$$

$$= \text{Min}\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+2}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t),$$

$$M(y_{2n+2}, y_{2n}, 2t), M(y_{2n+1}, y_{2n+1}, t)\}$$

$$= \text{Min}\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+2}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t) *$$

$M(y_{2n+1}, y_{2n+2}, t), 1\}$

$= \text{Min}\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t)\}$

$= M(y_{2n+1}, y_{2n}, t)$

As $M(x, y, t)$ is non- decreasing, then similarly $M(y_{2n+1}, y_{2n}, kt) \geq M(y_{2n}, y_{2n-1}, t)$

Hence $\quad M(y_{n+1}, y_n, kt) \geq M(y_n, y_{n-1}, t), \ \forall\ n.$

Now by using similar process of theorem 1, we have $\{y_n\}$ is a Cauchy sequence in X and as X is complete.

We get $\{y_n\} \to z \in X$.

Hence

$$A^m x_{2n} \to z,\ S^q x_{2n} \to z \quad \dots (1)$$

$$T^p x_{2n+1} \to z,\ B^n x_{2n+1} \to z \quad \dots (2)$$

Since $A^m(X) \subset T^p(X)\ \exists$ for $p \in X$ such that $p = (T^p)^{-1} z$

implies that $\quad T^p_p = z.$

By (1.3) we have;

$$M(A^m x_{2n}, B^n p, kt) \geq \text{Min}\{M(S^q x_{2n}, T^p_p, t), M(B^n p, T^p p, t),$$

$$M(A^m x_{2n}, S^q x_{2n}, t), M(B^n p, S^q x_{2n}, 2t), M(A^m x_{2n}, T^p_p, t)\}$$

$$M(A^m x_{2n}, B^n p, kt) \geq \text{Min}\{M(S^q x_{2n}, z, t), M(B^n p, z, t),$$

$$M(A^m x_{2n}, S^q x_{2n}, t)\ M(B^n p, S^q x_{2n}, 2t), M(A^m x_{2n}, z, t)\}.$$

Taking the limit $n \to \infty$ and using (1) and (2) we have $z = B^n p$ (By using similar process of theorem 1)

Since $T^p_p = z$, therefore $\quad B^n p = T^p_p = z.$

i.e. p is a coincidence point of B^n and T^p.

Now, $B^nT^pp = T^pB^np$ (By using $BT = TB$ or $B^nT^p = T^pB^n$)

i.e. B^n and T^p are commute at their coincidence point p.

Thus (B^n, T^p) is R- weakly commuting.

Similarly we can show that (A^m, S^q) is R-weakly commuting.

Hence by theorem 1, A^m, B^n, S^q and T^p have a unique common fixed point z.

i.e. $$A^mz = B^nz = S^qz = T^pz = z.$$

Now; $Az = A(A^mz) = A^m(Az)$ and $Az = A(S^qz) = S^qAz$.

Hence Az is a common fixed point of A^m and S^q.

Similarly Bz is a common fixed point of B^n and T^p.

Put $x = Az$ and $y = Bz$ in (1.3), we get

$$Az = Bz. \text{ Hence } z = Az = Sz.$$

Similarly we can prove $z = Sz = Tz$.

Thus we get $z = Az = Sz = Bz = Tz$.

Hence z is the unique common fixed point of A, B, S and T.

Corollary 2: The corollary 1 remains true if the condition (1.3) is replaced by $M(A^mx, B^ny, kt) \geq M(S^qx, T^py, t)$ $\forall$ $x, y \in X$, $t > 0$ and for some $k \in (0, 1)$.

Proof: It follows from Theorem 2 by similar reasoning as in corollary 1.

If we take $S = T = I$ in corollary 2 then the condition (1.1), (1.2) of corollary 1 are satisfied trivially and we get.

Corollary 3: A and B be self maps of a complete fuzzy metric space (X, M, *) with continuous t-norm defined by $a * b = \min\{a, b\}$ satisfying $M(A^mx, B^ny, kt) \geq M(x, y, t)$ for al x, y in X, $t > 0$, for some $k \in (0, 1)$.

Then A and B have a unique common fixed point.

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A NOTE ON DUALS OF SOME PBIB DESIGNS

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ABSTRACT

If D is a block design with v treatments arranged in b blocks each of size k such that each treatments appears once in each of r different blocks ($b \geq r$), then the design D obtained from D by interchanging the blocks and treatments is called the dual of the original design D. Dualization is a very useful technique. By dualization sometimes we get the new designs and sometimes already known designs. If the dual of any design D turns out to be a higher associate class partially balanced incomplete block (PBIB) design then the analysis of these designs by the dual method becomes extremely simple whereas the direct analysis of these as a PBIB design would be very tedious. In this paper we investigate the duals of some two associate class PBIB designs and association scheme of these dual designs. We also obtained the variances of the estimated elementary treatment contrast and efficiency factor of these dual designs.*

Keywords: Group divisible design, Association scheme, Partially balanced incomplete block (PBIB) design, Elementary treatment contrast

Mathematics Subject Classification (2000):62K10

1. INTRODUCTION

In plane projective geometry if the roles of lines and points are interchanged the dual geometry is obtained. The statements, "there is one point common to any two lines" and "there is one straight line common to any two points" are duals. In block design the dual design is obtained by interchanging blocks and treatments. Thus by interchanging blocks and treatments in a given class of block design, we get a new class of designs. The design so obtained is called the dual of the original design.

Many authors like Youden (1951) ,Roy and Laha(1956) , Ramakrishna(1956) , Cerenka and Meza(1978) , Mohan (1983) , Chaudhary and Singh (1986) , Pratap and Singh (1988), Bayrak and Gonen (2002), Singh(2009) have considered the duals of incomplete block design.

Clatworthy (1973) has listed more than 800 actual plans of PBIB(2) designs for which $2 \leq k \leq 10$ and $2 \leq r \leq 10$, where k is the number of experimental units (block size) and r is the number of blocks (replication of a treatment). In the tables of Clatworthy these 800 designs are classified into the following type:

(a) Group divisible design (GD)
The GD designs are further classified into three categories viz. singular (S), semi-regular (SR) and regular (R)

(b) Triangular

(c) Latin-Square type

(d) Cyclic

(e) Partial geometry and

(f) Miscellaneous

In these tables Clatworthy(1973) also mentioned something about the dual of each 800 designs. It is worthwhile to note that these tables are a revised and enlarged version of the earlier tables given by Bose, Clatworthy and Shrikhande (1954).

In this paper, we study the duals of some regular group divisible designs R18, R34 , R38 and R41[cf ; Clatworthy(1973)]. The designs R18, R34, R38 and R41 are all two associate partially balanced incomplete block (PBIB(2)) design. According to the Clatworthy(1973) duals of these regular group divisible PBIB(2) design are neither balanced incomplete block (BIB) design nor PBIB designs with two associate classes. According to our investigation these dual designs turns out to be four associate class PBIB design having a block contents based association scheme which is also discussed in this paper. Thus this investigation will augment the tables of Clatworthy (1973).

2. DUALS OF R 18, R 34, R 38 AND R 41

The parameters of the designs R18, R34, R38 and R41 are

$$v = 3a,\ b = 3a^2,\ r = 2a,\ k = 2 \text{ and efficiency factor}$$

$$E = 3(3a-1) / (18a-10),$$

for a = 2,3,4 and 5 respectively. The 3a treatments are grouped as below :

1	2	3 ...	a	I group
a+1	a+2	a+3 ...	2a	II group
2a+1	2a+2	2a+3 ...	3a	III group

The duals of the designs R18, R34, R38 and R41 will be denoted by R*18, R*34, R*38 and R*41 respectively. Our investigation reveals that the designs R*18, R*34, R*38 and R*41 are PBIB designs with four associate classes (PBIB(4)) having a block- contents – based association scheme mentioned here.

The parameters of these designs are

$$v^* = 3a^2,\ b^* = 3a,\ r^* = 2,\ k^* = 2a,\ \lambda_1^* = 1 = \lambda_2^*,$$

$$\lambda_3^* = 0 = \lambda_4^*,\ n_1^* = 2a,\ n_2^* = 2(a-1),\ n_3^* = 2a(a-1),$$

$$n_4^* = (a-1)^2,\quad \text{for a = 2,3,4 and 5 respectivel..}$$

Below we define the association scheme of these PBIB(4) designs and obtain the variances of the estimates of the elementary contrasts of the treatment effects.

3. ASSOCIATION SCHEME

We define the association scheme as follows: Let p and q be the blocks in which a particular treatment (for which have to found various associates) occurs in the dual design. For p,q as the treatments of the original design, let p_1 , p_2 ..., p_{a-1} be the other treatments of the group to which p belongs and $q_1, q_2, ..., q_{a-1}$ be the other treatments of the group to which q belongs. We now define the association scheme in terms of treatments occurring in the blocks p,q, p_1, p_2 ..., p_{a-1}, q_1, q_2 ..., q_{a-1} , of the dual design. We denote the set of treatments present in the jth block of the dual design by B(j) and S stands for the set of all treatments. Now

$$S_1 = {}^{B}(p) \cup {}^{B}(q) - {}^{B}(p) \cap {}^{B}(q)$$

= set of those treatments which are either I or II associates.

$$S_2 = S - {}^{B}(p) \cup {}^{B}(q)$$

= set of those treatments which are either III or IV associates.

Further we find that

$$S_3 = [\,[\,{}^{B}(p) \cap {}^{B}(q_1)\,], [\,{}^{B}(p) \cap {}^{B}(q_2)\,], \ldots, [\,{}^{B}(p) \cap {}^{B}(q_{a-1})\,],$$

$$[\,{}^{B}(q) \cap {}^{B}(p_1)\,], [\,{}^{B}(q) \cap {}^{B}(p_2)\,], \ldots, [\,{}^{B}(q) \cap {}^{B}(p_{a-1})\,]\,]$$

is the set of 2(a-1) treatments which are II associates.

Thus the I associates can be obtained by $S_1 - S_3$.

Finally the set S_4 of $(a-1)^2$ treatments which are fourth associates is obtained as below :

$${}^{B}(q_1) \cap {}^{B}(p_1), {}^{B}(q_1) \cap {}^{B}(p_2) \ldots, {}^{B}(q_1) \cap {}^{B}(p_{a-1})$$

....

....

$${}^{B}(q_{a-1}) \cap {}^{B}(p_1), {}^{B}(q_{a-1}) \cap {}^{B}(p_2), \ldots {}^{B}(q_{a-1}) \cap {}^{B}(p_{a-1}).$$

Now the third associates can easily be obtained by $S_2 - S_4$.

The variances of the estimates of the elementary contrasts of treatment effects are as below :

$$V(\hat{t}u - \hat{t}u') = \begin{cases} \{(6a^2+3a-1)/6a^2\}\ \sigma^2 & \text{for I associates} \\ \{(2a+1)/2a\}\ \sigma^2 & \text{for II associates} \\ \{(6a^2+6a-1)/6a\}^2\ \sigma^2 & \text{for III associates} \\ \{(a+1)/a\}\ \sigma^2 & \text{for IV associates} \end{cases}$$

The efficiency factor E* is

$$E^* = 3(3a^2-1)/(9a^2+9a-10)$$

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ON CERTAIN CUBIC SUBGRAPHS OF THE MIDDLE TWO LAYER'S GRAPH USING MODULAR MATCHINGS

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ABSTRACT

The n-dimensional discrete cube, denoted by $\boldsymbol{Q_n}$, is defined as the graph (V, E), where the vertex set V consists of all the subsets of $\{1,2, \dots n\}$ and there is an edge between two subsets if the size of their symmetric difference is 1. Thus, $(A, B) \in E$, if $|A\Delta B| = 1$. If n is odd, say $2k + 1$, then the bipartite subgraph induced by the sets of sizes k and $k + 1$ is called the middle two layer's graph. The long standing middle two layers' conjecture is that this graph is hamiltonian. Modular matchings were introduced by Duffus, Kierstead and Snevily [1] as a tool to study the middle two layers' graph and its conjecture. In this paper, we show some properties regarding the connectivity of the cubic subgraph of $\boldsymbol{Q_n}$ formed by three consecutive modular matchings.

Keywords: middle two layer's conjecture, modular matchings, hamiltonian, factorization.

PRELIMINARY NOTATION AND DEFINITIONS

We denote the collection of all j-element subsets of $[n]$ as $\boldsymbol{R_j}$ and call it the j'th layer of the discrete cube $\boldsymbol{Q_n}$ of dimension n. When n is odd, say $2k + 1$, clearly the middle two layers of $\boldsymbol{Q_{2k+1}}$ have the same size, since $\binom{2k+1}{k+1} = \binom{2k+1}{k}$ and there is an edge between any two vertices of the middle two layers if one vertex is contained in the other. The graph induced by these middle two layer's R_k and R_{k+1} of the discrete cube Q_{2k+1} is called the middle two layer's graph, denoted by B_k. The number of vertices adjacent to a given vertex v is called the *degree* of v. If all the vertices have the same degree k, the graph is called k-regular. A 3-regular graph is called a *cubic* graph. A 1-*factor* of a graph G is a perfect matching of G, or a spanning 1-regular subgraph of G. Decomposition of the edge-set $E(G)$ of a graph G into a collection of 1-factors is called a 1-*fatcorization* of G. Thus when we form a 1-factorization of the middle two layer's graph B_k, it will be a collection of $k+1$ disjoint perfect matchings. One such decomposition is given by the *modular matchings*.

Definition 1.1 Given $n = 2k+1$, the i-th modular matching $\boldsymbol{m_i}$, for $i = 1,2, \ldots k+1$ is defined as the function $\boldsymbol{m_i}: R_k \to R_{k+1}$ with $\boldsymbol{m_i}(A) = A \cup \{x\}$ where x is the j'th largest element of A^c and j is given by $j = i + \sum_{a \in A} a \pmod{k+1}$.

Note that in any modular matching, the index i is computed modulo $(k+1)$ and we usually replace $i = 0$ with $i = k+1$. It will be convenient for us to consider a perfect matching in $\boldsymbol{B_k}$ as an injection $\boldsymbol{m}: \boldsymbol{R_k} \to \boldsymbol{R_{k+1}}$ such that A is adjacent to $\boldsymbol{m}(A)$. Thus, if A is any k-element subset of $[n]$, then the i'th modular matching $\boldsymbol{m_i}$ of $\boldsymbol{B_k}$ consists of the edges $(A, \boldsymbol{m_i}(A))$. We denote the subset of contiguous elements $\{x-r, x-r+1, \ldots x-1\}$ of the set $[1, 2, \ldots 2k+1]$ by any of the following ways: $[x-r, x-1]$, $(x-r-1, x-1]$, $(x-r-1, x-1]$ or $(x-r-1, x)$, where the addition and subtraction of elements is done modulo $2k+1$. In this notation, we write the *segment* of a set A as the maximal interval $[x, y]$ contained in the set respecting the underlying circular ordering as mentioned above. A *path* P is a sequence of vertices $v_0, v_1, \ldots v_k$, where all the vertices are distinct and (v_{i-1}, v_i) is an edge for $i = 1, 2 \ldots k$. We denote the various paths that are the subgraph of P as follows: $Pv_i = v_0, v_1, \ldots v_i$, $v_iP = v_i, v_{i+1}, \ldots v_k$ and $v_iPv_j = v_i, v_{i+1}, \ldots v_j$. If we want to reverse the path, we will refer to it as $v_j\bar{P}v_i = v_j, v_{j-1}, \ldots v_i$, or simply as $\bar{P}$ if it is from v_k to v_0. Duffus, Kierstead and Snevily in [1] showed that, that $\boldsymbol{m_i}$ is a well-defined matching and $\{\boldsymbol{m_1}, \boldsymbol{m_2}, \ldots, \boldsymbol{m_{k+1}}\}$ forms a 1-factorization of B_k. They did this by introducing the following inverse function $\boldsymbol{b_i}$ from R_{k+1} to R_k and proving that $\boldsymbol{m_i} \circ \boldsymbol{b_i} = \boldsymbol{id}$.

Definition 1.2 Given $n = 2k + 1$, for $i = 1,2, \dots k + 1$ let b_i be the function $b_i: R_{k+1} \to R_k$ given by $b_i(B) = B \backslash \{y\}$ where y is the l'th smallest element of B and $l = i + \sum_{b \in B} b \pmod{k+1}$.

It is also shown in [1] that the modular matchings are essentially invariant under the rotation and reflexive permutations and that no other permutation preserves m_i.

Let $H_{a,b,c}(k)$ denote the spanning subgraph of B_k whose edge set is $m_a \cup m_b \cup m_c$. If the three matchings are consecutive, i.e. $a = b - 1$ and $c = b + 1$, we will use the notation $H_b(k)$ to denote the resulting subgraph. Both these subgraphs are cubic. Whenever the value of k is clear from the context or it is true for all values of k, we will write $H_{a,b,c}$ and H_b , instead of $H_{a,b,c}(k)$ and $H_b(k)$ respectively. For a set A of size k, let $A(i)$ denote the element $x \in A^c$ such that $m_i(A) = A \cup \{x\}$. The elements $A(1), A(2), \dots A(k + 1)$ satisfy the property $A(i + 1) < A(i)$ unless $A(i)$ is the smallest element of A^c, whence $A(i + 1)$ becomes the largest element of A^c.Hence the decreasing sequence of $A(i)$'s have an underlying circular ordering. This observation, which follows from the definition of modular matchings was made in [2]. Similarly, if B is a set of size $k + 1$, we denote the element $B(i)$ to be the element $y \in B$ such that $b_i(B) = B - \{y\}$ and we observe that the elements $B(1), B(2), \dots B(k + 1)$ satisfy the property that $B(i + 1) > B(i)$ unless $B(i)$ is the largest element of B, in which case $B(i + 1)$ becomes the smallest element of B. Once we know the position of $A(i)$ for any k-set A and any i, it becomes fairly easy to locate the edges of any remaining modular matchings which are incident with the vertex $B = m_i(A)$. This becomes even easier if the matchings are consecutive or almost consecutive. This is the underlying idea used in the following, which is one of the two main results of this paper.

***Proposition* 1.** *The distance between any set of size k, A_0 and $m_{i+2}(A_0)$ in H_i is at most* 11.

Proof: Let $x = A_0(i + 2) \bmod (2k + 1)$ so that $m_{i+2}(A_0) = A \cup \{x\}$, which we denote by B. Also let c be the largest element of the segment of A_0, which is to the left of x in the underlying circular ordering as described above. We will show that there exists a path from A_0 to B using only the edges of the matchings m_{i-1}, m_i, m_{i+1}. The paths can be shown in tables where the rows of the table denote the adjacent vertex sets of the middle two layer's graph B_k, represented as binary sequences $[\alpha_1, \alpha_2, \dots \alpha_{2k+1}]$ of length $2k+1$and an element j belongs to the set if and only if $\alpha_j = 1$. We have also used the convention that A_j's represents sets of size k and B_j's represents sets of size $k+1$. Only some of the crucial elements of the underlying set $[2k+1]$ are shown on the top.

Case 1: $A_0(i+1) = x+1 \bmod (2k+1)$. We add this element $x+1$ to A_0 to obtain the vertex- set $B_0 = \boldsymbol{m}_{i+1}(A_0)$. The element c can now be easily identified as $B_0(i)$ and we remove it to reach the next vertex-set $A_1 = \boldsymbol{b}_i(B_0)$. The following two subcases arise.

Subcase 1a: $c = x - 1$. This is shown in the following Table 1. We add the element x using the edge from the modular matching $\boldsymbol{m}_{i-1}$ to reach the vertex set B_1. Then we undo the changes we made by removing the element $x+1$ with the matching $\boldsymbol{m}_i$ and adding the element c back in with $\boldsymbol{m}_{i+1}$ and we are done. As Table 1 shows, we reach the desired vertex by a path of length 5.

Table 1. Subcase 1 a $A_0(i+1) = x+1,\ c = x-1$

Vertex-sets ↓		Location of c	Location of x	$x+1$	
A_0	...	1	0	$\underline{0}$ $A_0(i+1)$	...
B_0	...	$\underline{1}$ $B_0(i)$	0	1	...
A_1	...	0	$\underline{0}$ $A_1(i-1)$	1	...
B_1	...	0	1	$\underline{1}$ $B_1(i)$	...
A_2	...	$\underline{0}$ $A_2(i+1)$	1	0	...
B_2=B	...	1	1	0	...

Subcase 1b: $c \neq x - 1$. Here, the required element x is greater than $c+1$, but still we add $c+1$ using the matching $\boldsymbol{m}_{i-1}$ to reach $B_1 = m_{i-1}(A_0)$ and then remove $x+1$ with $\boldsymbol{m}_i$ to get A_2. Now we are in a position to

add the required element x to get $B_2 = \boldsymbol{m}_{i+1}(A_0)$. Then we correct the changes we made by removing $c+1$ with $\boldsymbol{m}_i$ and adding c back with $\boldsymbol{m}_{i+1}$. In this case we have to take two additional steps, so we reach the desired vertex by a path of length 7. This completes the first case.

Case 2: $A_0(i+1) \neq x+1 \mod (2k+1)$. In this case, there are elements lying between x and $A_0(i+1)$. In fact, there is exactly one segment , say $[a_1, b_1]$, between them which could very well be of size one, as indicated in the table by dots. Hence a_1 could be equal to b_1, but a segment will still exist. As we will see, the length of this segment does not affect the path that we take. We first add $A_0(i+1) = b_1 + 1$ to reach the vertex set $B_0 = \boldsymbol{m}_{i+1}(A_0)$. Now $B_0(i) = b_1$ which we remove using the modular matching $\boldsymbol{m}_i$ to reach A_1. Then we add the required element $a_1 - 1$ to get $B_1 = \boldsymbol{m}_{i+1}(A_1)$ and remove c with $\boldsymbol{m}_i$ to get A_2. Again we have two possibilities as in Case 1.

Subcase 2a: $c = x - 1$. We add the element b_1 back using the edge from the modular matching $\boldsymbol{m}_{i-1}$ to reach the vertex set B_2. Then we remove the element $b_1 + 1$ with the matching $\boldsymbol{m}_i$ and finally add the element c back in with $\boldsymbol{m}_{i+1}$ to reach A_3 followed by B_3 which is our required set B. This takes 7 steps, as shown in Table 2.

Table 2. Subcase 2 a. $A_0(i+1) \neq x+1,\ c = x-1$

Vertex-sets↓		Location of c	Location of x	a_1	b_1	b_1+1	
A_0	...	1	0	...	1	$\underline{0}$ $A_0(i+1)$	...
B_0	...	1	0	...	$\underline{1}$ $B_0(i)$	1	...
A_1	...	1	$\underline{0}$ $A_1(i+1)$	...	0	1	...
B_1	...	1 $B_1(i)$	1	...	0	1	...
A_2	...	0	1	...	$\underline{0}$ $A_2(i-1)$	1	...
B_2	...	0	1	...	1	$\underline{1}$ $B_2(i)$	...
A_3	...	$\underline{0}$ $A_3(i+1)$	1	...	1	0	...
$B_3 = B$	...	1	1	...	1	0	...

Subcase 2b: $c \neq x-1$. The second possibility of Case 2 has $x > c$. In this case, we add $c+1$ to get $B_2 = \boldsymbol{m}_{i-1}(A_2)$ and again remove $x = a_1 - 1$ using the modular matching $\boldsymbol{m}_i$, although that is the target to be added. The set reached is $A_3 = \boldsymbol{b}_i(B_2)$. We now correct the modifications we made by adding b_1 back in using $\boldsymbol{m}_{i-1}$ to reach B_3 and remove $b_1 + 1$ using $\boldsymbol{m}_i$ to reach A_4. Now we finally add our required candidate $x = a_1 - 1$ to reach $B_4 = \boldsymbol{m}_{i+1}(A_4)$ and revert the changes we had made on its left by removing $c+1$ using $\boldsymbol{m}_i$ to reach A_5 and then adding c with the modular matching $\boldsymbol{m}_{i+1}$ to reach the final required set $B_5 = B$ in 11 steps.

***Corollary* 1.** *The distance between any set of size* $k+1$, B *and* $b_{i+2}(B)$ *in* H_i *is at most* 11.

Proof: Let $A = \boldsymbol{b}_{i+2}(B)$. Then by the above proposition, we have a path P from A to $\boldsymbol{m}_{i+2}(A)$. But since, Duffus, Kierstead and Snevily in [1] showed that $\boldsymbol{m}_i \circ \boldsymbol{b}_i = id,\ \forall\, i$, we have that $\boldsymbol{m}_{i+2}(A) = \boldsymbol{m}_{i+2}(\boldsymbol{b}_{i+2}(B)) = B$. Hence by reversing the path, we obtain the required path $\bar{P}$, from $\boldsymbol{m}_{i+2}(A) = B$ to the required vertex $\boldsymbol{b}_{i+2}(B) = A$ of distance at most 11.

We will use the above lemma and its corollary to give the alternate proof of the theorem from [2] that the subgraph of $\boldsymbol{B}_k$ formed by three consecutive modular matchings is connected.

***Proposition* 2.** *H_i is connected* $\forall\, i$.

***Proof*:** We will first show that for any given i, there is a path in the subgraph H_i between any k-set A and $\boldsymbol{m}_{i+2+j}(A)$, for all $j = 0, 1, \ldots k-1$. To do this, we will apply induction on j. For $j = 0$, the result follows directly from the above Proposition 1. Now suppose that the result is true for some $j \geq 0$. Applying the induction hypothesis to the subgraph H_{i+1}, we have a path P from A to $\boldsymbol{m}_{i+3+j}(A)$ in this subgraph. So, P uses only the edges from $\boldsymbol{m}_i, \boldsymbol{m}_{i+1}$ and $\boldsymbol{m}_{i+2}$. But then, again by the above Proposition 1, the edges from the matching $\boldsymbol{m}_{i+2}$ can be replaced by paths using only the edges of H_i of length at most 11. Hence there is a path from A to $\boldsymbol{m}_{i+3+j}(A)$ in H_i itself and the induction hypothesis is true for $j+1$ also. This completes the proof by induction for the first part. Similarly, using Corollary 1 we can show that there is a path from every $(k+1)$-set B to $\boldsymbol{b}_{i+2+j}(B)$, for all $j = 0, 1, \dots k-1$. Now from [1] we know that $\{\boldsymbol{m}_1, \boldsymbol{m}_2, \dots, \boldsymbol{m}_{k+1}\}$ forms a 1-factorization of the entire middle two layer's graph $\boldsymbol{B}_k$. Since $\boldsymbol{B}_k$ itself is connected, there is a path between any two vertices using these $k+1$ modular matchings and by the above induction result, these matchings can be replaced by paths consisting of edges of H_i only. This completes the proof of the proposition.

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OPTIMUM VALUES OF SECOND ORDER MOMENT WHEN HARMONIC MEAN AND ARITHMETIC MEAN ARE PRESCRIBED

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ABSTRACT

For any given probability distribution, we can obtain all the moments. But if some moments are given, it is not necessary to obtain a probability distribution satisfying all these moments. Such a probability distribution exist only if moments are consistent. To check the consistency of moments we need information about maximum and minimum values of one moment when other moment/moments is/are prescribed. In this paper our aim is to obtain the maximum and minimum values of Second order moment when Harmonic mean and Arithmetic mean are prescribed.

Keywords: Consistency, minimum & maximum values of moment, switching point

INTRODUCTION

If a probability distribution is given, values of different moments can be determined uniquely since for the given distribution network is rigid. But if only partial information is available about the probabilities then the network is flexible. Partial information about probabilities may also be given in the form of values of some moments. There are infinite sets of probability distribution for the given moments and corresponding to these distributions, there may be infinite sets of required moments.

Taglini [6] had given the application of maximum entropy to the moments problem. Traditionally, the method of Relative Entropy Maximization is considered with linear moment constraints. But Grendar [4] studied the method under frequency moment constraints which are non-linear in probabilities. Lutwak, Yang & Zhang [5] considered Moment-Entropy inequalities for a random vector. Anju Rani [1,2] had calculated maximum and minimum values of r^{th} order moment when s^{th} order moment is prescribed for the continuous and discrete variables.

In the present paper, we shall obtain analytical expressions for maximum and minimum values of Second order moment when the Harmonic mean and Arithmetic mean are prescribed.

Further we shall consider special case when n=6 and obtain maximum & minimum values of Second order moment for the given values of Harmonic mean & Arithmetic mean and these values of Second order moment are given in the table form. In the last, we shall give concluding remarks.

ANALYTICAL EXPRESSIONS OF MAXIMUM AND MINIMUM VALUES SECOND OF ORDER MOMENT WHEN HARMONIC MEAN AND ARITHMETIC MEAN ARE GIVEN

Let a discrete random variate x takes values 1, 2,_ _ _, n with probabilities $p_1, p_2, \_\_\_, p_n$. Its Harmonic mean and Arithmetic mean are H and M respectively. There will be many probability distributions with these values of Harmonic mean & Arithmetic mean and for every one of these, there will be a Second order moment say $\left(\mu_2'\right)^{1/2}$. Although Second order moment is not uniquely determined , but the set of all values of Second order moment for given values of Harmonic mean and Arithmetic mean will have a maximum value say $\left(\mu_2'\right)_{max}$ and a minimum value say $\left(\mu_2'\right)_{min}$. Our aim is to obtain these values. In other words, we have to choose $p_1, p_2, \_\_\_, p_n$ to optimize μ_2', where

$$\mu_2' = \sum_{i=1}^{n} i^2 p_i \qquad \dots (1)$$

subject to

$$\sum_{i=1}^{n} p_i = 1 \ , \quad \sum_{i=1}^{n} \frac{p_i}{i} = \frac{1}{\mathrm{H}} \ , \quad \sum_{i=1}^{n} i\, p_i = \mathrm{M} \qquad \dots (2)$$

This is a linear programming problem having three equality constraints, hence maximum and minimum values will occur when at most three of the probabilities are non zero.

Let these be p_h, p_k and p_l. Then

$$p_h + p_k + p_l = 1, \ \frac{p_h}{h} + \frac{p_k}{k} + \frac{p_l}{l} = \frac{1}{\mathrm{H}}\ , \ hp_h + kp_k + lp_l = \mathrm{M} \qquad \dots (3)$$

on solving equation (3), we get

$$p_h = \frac{h\,[kl + \mathrm{H}\ (\mathrm{M} - k - l)]}{\mathrm{H}(k-h)\,(l-h)} \qquad \dots (4)$$

$$p_k = \frac{k\,[\mathrm{H}(h + l - \mathrm{M}) - hl]}{\mathrm{H}(k-h)\,(l-k)} \qquad \dots (5)$$

$$p_l = \frac{l\,[hk + \mathrm{H}\ (\mathrm{M} - h - k)]}{\mathrm{H}(l-h)\,(l-k)} \qquad \dots (6)$$

from equations (1), (4), (5) and (6)

$$\mu_2' = \frac{h^3[kl+\mathrm{H}(\mathrm{M}-k-l)]}{\mathrm{H}(k-h)(l-h)} + \frac{k^3[\mathrm{H}(h+l-\mathrm{M})-hl]}{\mathrm{H}(k-h)(l-k)} + \frac{l^3[hk+\mathrm{H}(\mathrm{M}-h-k)]}{\mathrm{H}(l-h)(l-k)} \quad \dots (7)$$

Now,

$$\frac{d\mu_2'}{dh} = \frac{\{kl+\mathrm{H}(\mathrm{M}-k-l)\}}{\mathrm{H}}\left[\frac{h^2\{h^2-2h(k+l)+3kl\}}{(k-h)^2(l-h)^2} + \frac{1}{(l-k)}\left\{\frac{l^3}{(l-h)^2} - \frac{k^3}{(k-h)^2}\right\}\right] > 0 \quad \dots (8)$$

$$\frac{d\mu_2'}{dk} = \frac{\{\mathrm{H}(h+l-\mathrm{M})-hl\}}{\mathrm{H}}\left[\frac{-k^2\{k^2-2k(h+l)+3hl\}}{(k-h)^2(l-k)^2} - \frac{1}{(l-h)}\left\{\frac{l^3}{(l-k)^2} - \frac{h^3}{(k-h)^2}\right\}\right] < 0 \quad \dots (9)$$

$$\frac{d\mu_2'}{dl} = \frac{\{hk+\mathrm{H}(\mathrm{M}-h-k)\}}{\mathrm{H}}\left[\frac{l^2\{l^2-2l(h+k)+3hk\}}{(l-h)^2(l-k)^2} + \frac{1}{(k-h)}\left\{\frac{k^3}{(l-k)^2} - \frac{h^3}{(l-h)^2}\right\}\right] > 0 \quad \dots (10)$$

Hence μ_2' increases with h, l and decreases with k.

First we calculate feasible range of M for given value of H. So, for this we use following expressions by Anju Rani [3].

(i) When H takes discrete values

$$\mathrm{M}_{\min} = \mathrm{H} \quad \dots (11)$$

and, when H does not take discrete values

$$\mathrm{M}_{\min} = [\mathrm{H}] + \frac{\mathrm{L}[\mathrm{H}+1]}{[\mathrm{H}]+\mathrm{L}} \quad \dots (12)$$

where [H] is integral part of H and $0 < \mathrm{L} < 1$

(ii) The expression for maximum value of M is given as

$$\mathrm{M}_{\max} = 1 + \mathrm{n} - \frac{\mathrm{n}}{\mathrm{H}} \quad \dots (13)$$

For the given values of H and $\mathrm{M}_{\min}$, probability p_h is zero at point (1, a, a +1) or p_l is zero at point (a, a +1, n) & for the given values of H and $\mathrm{M}_{\max}$, probability $p_k = 0$ at point (1, n-1, n) . $\mathrm{H} \in (\mathrm{a}, \mathrm{a}+1]$. Here $p_h = 0$ for $\{1 \le h < k < \mathrm{H} \le l \le n\}$ or $\{1 \le h < k \le \mathrm{H} < l \le n\}$ & $p_l = 0$ for $\{1 \le h \le \mathrm{H} < \mathrm{k} < \mathrm{l} \le \mathrm{n}\}$ or $\{1 \le h < \mathrm{H} \le \mathrm{k} < \mathrm{l} \le \mathrm{n}\}$. For the given values of H and $\mathrm{M}_{\min}$, the values of $\left(\mu_2'\right)^{1/2}$ are the same at all existing points and similarly for the given values of H and $\mathrm{M}_{\max}$, the values of $\left(\mu_2'\right)^{1/2}$ are the same at all existing points.

MINIMUM VALUE OF SECOND ORDER MOMENT

μ_2' is minimum when $h = 1$, $k = a$, $l = a + 1$, then from equation (7)

$$\left[\left(\mu_2'\right)_{min}\right]_{(1,\mathrm{a},\mathrm{a}+1)} = \frac{\{\mathrm{a}(\mathrm{a}+1)+\mathrm{H}(\mathrm{M}-2\mathrm{a}-1)\}}{\mathrm{H}(\mathrm{a}-1)\mathrm{a}} + \frac{\mathrm{a}^3\{\mathrm{H}(\mathrm{a}+2-\mathrm{M})-(\mathrm{a}+1)\}}{\mathrm{H}(\mathrm{a}-1)} + \frac{(\mathrm{a}+1)^3\{\mathrm{a}+\mathrm{H}(\mathrm{M}-\mathrm{a}-1)\}}{\mathrm{Ha}} \quad \dots (14)$$

from this expression, for given value of Harmonic mean we cannot obtain $\left(\mu_2'\right)_{min}$ for all values of Arithmetic mean. Since probability distribution does not exist for such values. For obtaining another points to lie minimum value of Second order moment, we consider those conditions when only two probabilities are non zero. These are:

$$p_k(1, a+b, a+b+1) = 0, \text{ for a=1 and } p_k(1, a+b-1, a+b) = 0, \text{ for a} > 1.$$

This situation occurs at particular value of M say M_{b*}. M_{b*} is switching point for $\left(\mu_2'\right)_{min}$. The expression of M_{b*} for a=1 can be obtained from equation (5) by taking $p_k = 0$

for $h = 1$, $k = a + b$, $l = a + b + 1$.

$$M_{b*} = \frac{(b+3)H-(b+2)}{H} \qquad \dots (15)$$

at switching point M_{b*}

$$\left[\left(\mu_2'\right)_{min}\right]_{(1, b+1, b+2)} = \left[\left(\mu_2'\right)_{min}\right]_{(1, b+2, b+3)}$$

$\left(\mu_2'\right)_{min}$ occurs at point $(1, b+1, b+2)$ for $M_{min} \le M \le M_{b*}$ (for b=1)

or $M_{(b-1)*} \le M \le M_{b*}$ (for b>1)

$\left(\mu_2'\right)_{min}$ occurs at point $(1, b+2, b+3)$ for $M_{b*} \le M \le M_{(b+1)*}$

and for a > 1, the expression of M_{b*} is obtained from equation (5) by taking $p_k = 0$

for $h = 1$, $k = a + b - 1$, $l = a + b$

$$M_{b*} = \frac{(a+b+1)H-(a+b)}{H} \qquad \dots (16)$$

at switching point M_{b*}

$$\left[\left(\mu_2'\right)_{min}\right]_{(1, a+b-1, a+b)} = \left[\left(\mu_2'\right)_{min}\right]_{(1, a+b, a+b+1)}$$

$\left(\mu_2'\right)_{min}$ occurs at point $(1, a+b-1, a+b)$ for $M_{min} \le M \le M_{b*}$ (for b=1)

or $M_{(b-1)*} \le M \le M_{b*}$ (for b>1)

$\left(\mu_2'\right)_{min}$ occurs at point $(1, a+b, a+b+1)$ for $M_{b*} \leq M \leq M_{(b+1)*}$

Similarly, we obtain M_{i*} from equation (5) by equating to zero for $h = 1$, $k = n-2$, $l = n-1$.

$$M_{i*} = \frac{Hn-(n-1)}{H} \qquad \dots (17)$$

where i represents the number of switching points for $\left(\mu_2'\right)_{min}$, I = n – a - 1 for a > 1 and I = n - 3 for a = 1.

For discrete value of Harmonic mean, M_{min} and M_{1*} concide to each other.

for value M_{i*}

$$\left[\left(\mu_2'\right)_{min}\right]_{(1, n-2, n-1)} = \left[\left(\mu_2'\right)_{min}\right]_{(1, n-1, n)}$$

$\left(\mu_2'\right)_{min}$ occurs at point $(1, n-2, n-1)$ for $M_{(i-1)*} \leq M \leq M_{i*}$

$\left(\mu_2'\right)_{min}$ occurs at point $(1, n-1, n)$ for $M_{i*} \leq M \leq M_{max}$

MAXIMUM VALUE OF SECOND ORDER MOMENT

$\left(\mu_2'\right)^{1/2}$ is maximum when $h = a$, $k = a+1$, $l = n$. So, from equation (7)

$$\left[\left(\mu_2'\right)_{max}\right]_{(a, a+1, n)} = \frac{a^3\{n(a+1)+H(M-n-a-1)\}}{H(n-a)} + \frac{(a+1)^3\{H(n+a-M)-na\}}{H(n-a-1)} + \frac{n^3\{a(a+1)+H(M-2a-1)\}}{H(n-a)(n-a-1)} \qquad \dots (18)$$

Now, we are considering those conditions when only two probabilities are non-zero.

$p_k(a-b+1, a-b+2, n) = 0$, for a < n-1 & $p_k(a-b, a-b+1, n) = 0$, for a = n-1.

This situation occurs at particular value of M say M_b^*. M_b^* is switching point for $\left(\mu_2'\right)_{max}$ The expression of M_b^* for a < n-1 is obtained for $h = a-b+1$, $k = a-b+2$, $l = n$ from equation (5) by taking $p_k = 0$.

$$M_b^* = \frac{H(n+a-b+1)-n(a-b+1)}{H} \qquad \dots (19)$$

at switching point M_b^*

$$\left[\left(\mu_2'\right)_{max}\right]_{(a-b+1, a-b+2, n)} = \left[\left(\mu_2'\right)_{max}\right]_{(a-b, a-b+1, n)}$$

$\left(\mu_2'\right)_{max}$ occurs at point $(a-b+1\,, a-b+2, n)$ for $M_{min} \le M \le M_b^*$ (for b=1)

or $M_{(b-1)}^* \le M \le M_b^*$ (for b>1)

$\left(\mu_2'\right)_{max}$ occurs at point $(a-b\,, a-b+1, n)$ for $M_b^* \le M \le M_{(b+1)}^*$

and the expression of M_b^* for a = n-1 is obtained from equation (5) by taking $p_k = 0$

for $h = a-b\,,\ k = a-b+1\,, l = n$

$$M_b^* = \frac{H(2n-b-1)-n(n-b-1)}{H} \qquad \ldots (20)$$

for switching point M_b^*

$$\left[\left(\mu_2'\right)_{max}\right]_{(a-b\,, a-b+1\,, n)} = \left[\left(\mu_2'\right)_{max}\right]_{(a-b-1, a-b\,, n)}$$

$\left(\mu_2'\right)_{max}$ occurs at point $(n-b-1\,, n-b, n)$ for $M_{min} \le M \le M_b^*$ (for b=1)

or $M_{(b-1)}^* \le M \le M_b^*$ (for b>1)

$\left(\mu_2'\right)_{max}$ occurs at point $(n-b-2\,, n-b-1, n)$ for $M_b^* \le M \le M_{(b+1)}^*$

Similarly, we obtain M_j^* from equation (5) by equating to zero for $h = 2\,, k = 3\,, l = n$.

$$M_j^* = \frac{H(n+2)-2n}{H} \qquad \ldots (21)$$

where j represents the number of switching points for $\left(\mu_2'\right)_{max}$, j=a-1 for a < n-1 and j=n-3 for a=n-1.

for value M_j^*

$$\left[\left(\mu_2'\right)_{max}\right]_{(2\,,3\,,n)} = \left[\left(\mu_2'\right)_{max}\right]_{(1,2\,,n)}$$

$\left(\mu_2'\right)_{max}$ occurs at point $(2\,,3, n)$ for $M_{(j-1)}^* \le M \le M_j^*$

$\left(\mu_2'\right)_{max}$ occurs at point $(1\,,2, n)$ for $M_j^* \le M \le M_{max}$

Now, we find variation of μ_2' with respect to H and M.

$$\frac{d\left(\mu_2'\right)_{min}^{1/2}}{dH} = -\frac{1}{2\left(\mu_2'\right)^{1/2}}\left[\frac{a(a+1)}{H^2}\right] < 0$$

$$\frac{d^2\left(\mu_2'\right)_{min}^{1/2}}{dH^2} = \frac{a(a+1)}{H^3}\left[\frac{a(a+1)}{4\mu_2'H}+1\right] > 0 \qquad \ldots (22)$$

$\left(\mu_2'\right)_{min}^{1/2}$ is convex decreasing function of H.

$$\frac{d\left(\mu_2'\right)_{min}^{1/2}}{dM} = \frac{1}{\left(\mu_2'\right)^{1/2}}\,[a+1] > 0$$

$$\frac{d^2\left(\mu_2'\right)_{min}^{1/2}}{dM^2} = -\frac{1}{2\left(\mu_2'\right)^{3/2}}\,[a+1]^2 < 0 \qquad \ldots (23)$$

$\left(\mu_2'\right)_{min}^{1/2}$ is concave increasing function of M.

$$\frac{d\left(\mu_2'\right)_{max}^{1/2}}{dH} = -\frac{1}{2\left(\mu_2'\right)^{1/2}}\left[\frac{na(a+1)}{H^2}\right] < 0$$

$$\frac{d^2\left(\mu_2'\right)_{max}^{1/2}}{dH^2} = \frac{na(a+1)}{H^3}\left[\frac{na(a+1)}{4\mu_2'H}+1\right] > 0 \qquad \ldots (24)$$

$\left(\mu_2'\right)_{max}^{1/2}$ is convex decreasing function of H.

$$\frac{d\left(\mu_2'\right)_{max}^{1/2}}{dM} = \frac{1}{2\left(\mu_2'\right)^{1/2}}\,[n+2a+1] > 0$$

$$\frac{d^2\left(\mu_2'\right)_{max}^{1/2}}{dM^2} = -\frac{1}{4\left(\mu_2'\right)^{3/2}}\,[n+2a+1]^2 < 0 \qquad \ldots (25)$$

$\left(\mu_2'\right)_{max}^{1/2}$ is concave increasing function of M.

MAXIMUM AND MINIMUM VALUES OF SECOND ORDER MOMENT WHEN HARMONIC MEAN AND ARITHMETIC MEAN ARE GIVEN: SIX FACED DICE

Now we calculate maximum and minimum values of second order moment for a six faced dice i.e. n=6. We calculate the values of second order moment for prescribed moments belonging to all possible intervals and observe how the minimum and maximum values of Second order moment shift from one set of (h,k,l) to another set of (h,k,l).

Here we take different intervals in which Harmonic mean lies. These intervals are (1,2], (2,3], (3,4], (4,5], (5,6]. In these intervals, values of Harmonic mean are taken as 1.25, 1.5,....5.75, 6.0. But in the present paper we are considering only for H=1.25. Similarly we can obtain maximum and minimum values of Second order moment for other values of Harmonic mean.

For the given value of H, we obtain maximum and minimum values of M. Values of $\left(\mu_2'\right)^{1/2}$ are given in the following table for the given values of H and M. Out of these values we obtain minimum and maximum say $\left(\mu_2'\right)^{1/2}_{min}$ & $\left(\mu_2'\right)^{1/2}_{max}$ respectively. For $H \in (1,2]$, $h = 1;\ k = 2,3,\_\_\_,5;\ l = 3,4,\_\_\_6$.

H=1.25

First, we calculate values M_{min} from equation (12) and M_{max} from equation (13) for H=1.25

$$M_{min} = 1.4 \text{ and } M_{max} = 2.2$$

M / h,k,l	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.2
1,2,3	**1.4832**	**1.6733**	**1.8439**					
1,2,4	1.483 2	1.703 1	1.897 2	2.073 6	2.236 2			
1,2,5	1.483 2	1.732 1	1.949 2	2.144 8	2.323 9	2.489 8	2.645 8	
1,2,6	***1.4832***	***1.7607***	***2.0000***	***2.2136***	***2.4083***	***2.5884***	***2.7568***	***3.0659***
1,3,4			**1.8439**	**2.0493**	**2.2362**			
1,3,5			1.843 9	2.073 6	2.280 4	2.469 8	2.645 8	3.065 9
1,3,6			1.843 9	2.097 6	2.323 8	2.529 8	2.720 3	
1,4,5					**2.2362**	**2.4494**	**2.6458**	
1,4,6					2.2362	2.4698	2.6832	3.0659
1,5,6							**2.6458**	**3.0659**

Table 1 for $\left(\mu_2'\right)^{1/2}$

$\left(\mu_2'\right)^{1/2}$ is obtained from equation (7) for M_{min} at point (1,2,3)

$$\left(\mu_2'\right)^{1/2} = 1.4832$$

$\left(\mu_2'\right)^{1/2}$ is obtained from equation (7) for M_{max} at point (1,2,6)

$$\left(\mu_2'\right)^{1/2} = 3.0659$$

For $M \gtrsim M_{min}$, $\left(\mu_2'\right)^{1/2}_{min}$ occurs at point $(1,2,3)$. From table 1 we observe that for $M = 1.6$,the values of $\left(\mu_2'\right)^{1/2}_{min}$ are equal at points $(1,2,3)$ and $(1,3,4)$ i.e. $\left(\mu_2'\right)^{1/2}_{min}$ shifts from point $(1,2,3)$ to $(1,3,4)$ at $M = M_{1*} = 1.6$.

M_{1*} is first switching point. The value of M_{1*} can be obtained for a = 1, b = 1, H = 1.25 from equation (15).

for value M_{1*}, $\left[\left(\mu_2'\right)^{1/2}_{min}\right]_{(1,2,3)} = \left[\left(\mu_2'\right)^{1/2}_{min}\right]_{(1,3,4)} = 1.8439$

Now, $\left(\mu_2'\right)_{\min}$ is calculated from equation (7) at point (1,2,3).

$$\left(\mu_2'\right)_{\min} = 6M - 6.2, \qquad \text{for } M \in [1.4,1.6] \text{ or } M_{min} \le M \le M_{1*}$$

Again, $\left(\mu_2'\right)^{1/2}_{min}$ shifts from point $(1,3,4)$ to $(1,4,5)$ for $M = M_{2*} = 1.8$.

M_{2*} , second switching point, can be obtained for a=1, b=2, H=1.25 from equation (15).

for value M_{2*}, $\left[\left(\mu_2'\right)^{1/2}_{min}\right]_{(1,3,4)} = \left[\left(\mu_2'\right)^{1/2}_{min}\right]_{(1,4,5)} = 2.2362$

$\left(\mu_2'\right)_{\min}$ is calculated from equation (7) at point (1,3,4).

$$\left(\mu_2'\right)_{\min} = 8M - 9.4, \qquad \text{for } M \in [1.6,1.8] \text{ or } M_{1*} \le M \le M_{2*}$$

Now, $\left(\mu_2'\right)^{1/2}_{min}$ shifts from point $(1,4,5)$ to point $(1,5,6)$. hence from equation (17), $M_{3*} = 2.0$. M_{3*} is the last switching point. Since I = 3

$$\left[\left(\mu_2'\right)_{min}^{1/2}\right]_{(1,4,5)} = \left[\left(\mu_2'\right)_{min}^{1/2}\right]_{(1,5,6)} = 2.6458, \text{ for value } M_{3*}$$

$\left(\mu_2'\right)_{\min}$ is calculated from equation (7) at point (1,4,5).

$$\left(\mu_2'\right)_{\min} = 10M - 13, \qquad \text{for } M \in [1.8, 2.0],\ M_{2*} \le M \le M_{3*}$$

$\left(\mu_2'\right)_{\min}$ is calculated from equation (7) at point (1,5,6).

$$\left(\mu_2'\right)_{\min} = 12M - 17, \qquad \text{for } M \in [2.0, 2.2] \text{ or } M_{3*} \le M \le M_{\max}$$

Here j = 0 , so there is no switching point for $\left(\mu_2'\right)_{max}^{1/2}$.

$\left(\mu_2'\right)_{max}^{1/2}$ occurs at point $(1, 2, 6)$ for all values of M for H = 1.25 .

$$\left(\mu_2'\right)_{\max} = 9M - 10.4, \qquad \text{for } M \in [1.4, 2.2] \text{ or } M_{min} \le M \le M_{\max}$$

Similarly we can obtain optimum values of Second order moment and switching points for other values of Harmonic mean.

MAXIMUM AND MINIMUM VALUES OF SECOND ORDER MOMENT FOR GIVEN VALUES OF H AND M FOR ALL POSSIBLE INTERVALS

H	M	Min $\left(\mu_2'\right)^{1/2}$	Max $\left(\mu_2'\right)^{1/2}$	H	M	Min $\left(\mu_2'\right)^{1/2}$	Max $\left(\mu_2'\right)^{1/2}$
1.25	1.4	1.4832	1.4832	1.5	2.6	3.2145	3.3764
1.25	1.5	1.6733	1.7607	1.5	2.6667	3.3167	3.4641
1.25	1.6	1.8439	2.000	1.5	2.8	3.5496	3.6332
1.25	1.7	2.0493	2.2136	1.5	3.0	3.8729	3.8729
1.25	1.8	2.2362	2.4083				
1.25	1.9	2.4494	2.5884	1.75	1.8571	1.8899	1.8899

H	M	Min $\left(\mu_2'\right)^{1/2}$	Max $\left(\mu_2'\right)^{1/2}$	H	M	Min $\left(\mu_2'\right)^{1/2}$	Max $\left(\mu_2'\right)^{1/2}$
1.25	2.0	2.6458	2.7568	1.75	2	2.1044	2.2042
1.25	2.1	2.8636	2.9155	1.75	2.2	2.3725	2.5804
1.25	2.2	3.0659	3.0659	1.75	2.2857	2.4785	2.7258
				1.75	2.4	2.6565	2.9084
1.5	1.6667	1.7321	1.7321	1.75	2.6	2.9424	3.2029
1.5	1.7	1.7889	1.8166	1.75	2.7143	3.0937	3.3592
1.5	1.8	1.9494	2.0494	1.75	2.8	3.2291	3.4725
1.5	1.9	2.0976	2.2583	1.75	3.0	3.5253	3.7227
1.5	2.0	2.2361	2.4495	1.75	3.1429	3.7235	3.8913
1.5	2.1	2.4082	2.6268	1.75	3.2	3.8137	3.9571
1.5	2.2	2.5691	2.7928	1.75	3.4	4.1163	4.1783
1.5	2.3333	2.7688	2.9999	1.75	3.5714	4.3589	4.3589
2.0	2.0	2.0000	2.0000	2.5	3.0	3.1306	3.3764
2.0	2.2	2.2804	2.4083	2.5	3.2	3.3763	3.6878
2.0	2.4	2.5298	2.7568	2.5	3.4	3.6056	3.9243
2.0	2.5	2.6458	2.9155	2.5	3.6	3.8729	4.1473
2.0	2.6	2.7928	3.0659	2.5	3.8	4.1232	4.3589
2.0	2.8	3.0659	3.3466	2.5	4.0	4.3589	4.5607
2.0	3.0	3.3167	3.6056	2.5	4.2	4.6260	4.7539
2.0	3.2	3.6056	3.8471	2.5	4.4	4.8785	4.9396
2.0	3.4	3.8729	4.0743	2.5	4.6	5.1186	5.1186
2.0	3.5	4.0000	4.1833				
2.0	3.6	4.1473	4.2895	2.75	2.8182	2.8445	2.8445
2.0	3.8	4.4272	4.4944	2.75	2.9091	2.9387	3.0154
2.0	4.0	4.6904	4.6904	2.75	3.0	3.06	3.1766
				2.75	3.2	3.3112	3.5058

H	M	Min $(\mu_2')^{1/2}$	Max $(\mu_2')^{1/2}$	H	M	Min $(\mu_2')^{1/2}$	Max $(\mu_2')^{1/2}$
2.25	**2.3333**	**2.3804**	**2.3804**	2.75	3.4	3.5444	3.8067
2.25	2.4	2.463	2.5296	**2.75**	**3.5455**	**3.7051**	4.0112
2.25	2.6	2.6956	2.9324	2.75	3.6	3.7781	4.0854
2.25	**2.6667**	**2.7688**	**3.0552**	**2.75**	**3.6364**	3.8257	**4.1342**
2.25	2.8	2.9552	3.2457	2.75	3.8	4.0340	4.3086
2.25	3.0	3.2144	3.5121	2.75	4.0	4.2747	4.5127
2.25	3.2	3.4544	3.7596	**2.75**	**4.1818**	**4.4824**	4.6903
2.25	**3.2222**	**3.48**	3.7858	2.75	4.2	4.5064	4.7079
2.25	3.4	3.6897	3.9918	2.75	4.4	4.7653	4.8953
2.25	3.6	3.9859	4.2113	2.75	4.6	5.0108	5.0759
2.25	**3.7778**	**4.2030**	4.397	2.75	4.8	5.2448	5.2502
2.25	3.8	4.2350	4.4198	**2.75**	**4.8182**	**5.2655**	**5.2655**
2.25	4.0	4.5095	4.6189				
2.25	4.2	4.7682	4.8099	**3.0**	**3.0**	**3.0000**	**3.0000**
2.25	**4.3333**	**4.933**	**4.933**	3.0	3.2	3.2558	3.3466
				3.0	3.4	3.4928	3.6606
2.5	**2.6**	**2.6458**	**2.6458**	3.0	3.6	3.7148	3.9497
2.5	**2.8**	**2.8636**	3.0332	**3.0**	**3.6667**	**3.7859**	4.0412
3.0	3.8	3.9582	4.219	**3.5**	**4.5714**	**4.7359**	**4.9571**
3.0	**4.0**	4.2032	**4.4721**	3.5	4.6	4.7719	4.9827
3.0	4.2	4.4349	4.669	3.5	4.8	5.0171	5.1602
3.0	**4.3333**	**4.5825**	4.7958	3.5	5.0	5.2508	5.3318
3.0	4.4	4.669	4.8579	3.5	5.2	5.4746	5.4979
3.0	4.6	4.9193	5.0398	**3.5**	**5.2857**	**5.5676**	**5.5676**
3.0	4.8	5.1575	5.2154				

H	M	Min $(\mu_2')^{1/2}$	Max $(\mu_2')^{1/2}$	H	M	Min $(\mu_2')^{1/2}$	Max $(\mu_2')^{1/2}$
3.0	5.0	5.3852	5.3852	3.75	3.8	3.8209	3.8209
				3.75	3.9333	3.9582	4.0412
3.25	3.3077	3.3397	3.3397	3.75	4.0	4.0414	4.1473
3.25	3.4	3.4485	3.5148	3.75	4.2	4.2816	4.4497
3.25	3.4615	3.5191	3.6267	3.75	4.4	4.5092	4.6904
3.25	3.6	3.6731	3.8311	3.75	4.6	4.7257	4.9193
3.25	3.7692	3.8531	4.0668	3.75	4.6667	4.7959	4.9932
3.25	3.8	3.8930	4.1082	3.75	4.8	4.9598	5.1381
3.25	4.0	4.1419	4.3678	3.75	5.0	5.1962	5.3104
3.25	4.2	4.3766	4.6128	3.75	5.2	5.4222	5.4772
3.25	4.3077	4.4979	4.7393	3.75	5.4	5.6391	5.6391
3.25	4.4	4.5995	4.8261				
3.25	4.4615	4.6658	4.8834	4.0	4.0	4.0000	4.0000
3.25	4.6	4.8407	5.0092	4.0	4.2	4.2426	4.3128
3.25	4.8	5.0825	5.1857	4.0	4.4	4.4722	4.6043
3.25	5.0	5.3134	5.3565	4.0	4.5	4.5825	4.7434
3.25	5.1538	5.4843	5.4843	4.0	4.6	4.6904	4.8579
				4.0	4.75	4.8477	5.0249
3.5	3.5714	3.6055	3.6055	4.0	4.8	4.9092	5.0794
3.5	3.6	3.6370	3.6568	4.0	5.0	5.1478	5.2915
3.5	3.8	3.8508	3.9965	4.0	5.2	5.3759	5.4589
3.5	3.8571	3.9077	4.0883	4.0	5.4	5.5946	5.6214
3.5	4.0	4.0884	4.276	4.0	5.5	5.7009	5.7009
3.5	4.2	4.326	4.5259				
3.5	4.4	4.5514	4.7628	4.25	4.2941	4.3182	4.3182

H	M	Min $\left(\mu_2'\right)^{1/2}$	Max $\left(\mu_2'\right)^{1/2}$	H	M	Min $\left(\mu_2'\right)^{1/2}$	Max $\left(\mu_2'\right)^{1/2}$
4.25	4.3529	4.3858	4.4193	4.75	5.6	5.7025	5.7381
4.25	4.4	4.4392	4.4878	4.75	5.7368	5.8446	5.8446
4.25	4.6	4.6589	4.7687				
4.25	4.7647	4.8326	4.9881	5.0	5.0	5.0000	5.0000
4.25	4.8	4.869	5.0271	5.0	5.2	5.2345	5.2915
4.25	4.8235	4.8930	5.0525	5.0	5.3	5.3479	5.4129
4.25	5.0	5.1049	5.2413	5.0	5.4	5.4589	5.5317
4.25	5.1765	5.308	5.4232	5.0	5.5	5.5678	5.6303
4.25	5.2	5.3348	5.4428	5.0	5.6	5.6745	5.7271
4.25	5.4	5.5552	5.6057	5.0	5.7	5.7793	5.8052
4.25	5.5882	5.7547	5.7547	5.0	5.8	5.8822	5.8822
4.5	4.5556	4.5826	4.5826	5.25	5.2857	5.3049	5.3049
4.5	4.6	4.6308	4.6545	5.25	5.4	5.4328	5.4644
4.5	4.6667	4.7021	4.7609	5.25	5.4286	5.4634	5.5033
4.5	4.8	4.8419	4.9397	5.25	5.5	5.5421	5.5869
4.5	4.8889	4.9328	5.0554	5.25	5.5714	5.6188	5.6694
4.5	5.0	5.0660	5.1962	5.25	5.6	5.6493	5.6970
4.5	5.2	5.2976	5.4038	5.25	5.7143	5.7694	5.8065
4.5	5.3333	5.4407	5.5377	5.25	5.8	5.8579	5.8725
4.5	5.4	5.5195	5.5915	5.25	5.8571	5.916	5.916
4.5	5.6	5.7328	5.7502				
4.5	5.6667	5.8022	5.8022	5.5	5.5455	5.5678	5.5678

				5.5	5.6	5.6263	5.6408
4.75	4.7895	4.8068	4.8068	5.5	5.6364	5.6647	5.6889
4.75	4.8	4.8177	4.8232	5.5	5.7	5.7319	5.761
4.75	4.9474	4.9682	5.0472	5.5	5.7273	5.7603	5.7919
4.75	5.0	5.0317	5.1145	5.5	5.8	5.8357	5.8607
4.75	5.2	5.2648	5.3626	5.5	5.8182	5.8541	5.8774
4.75	5.2105	5.2766	5.3753	5.5	5.9	5.9376	5.9399
4.75	5.4	5.4880	5.5659	5.5	5.9091	5.9467	5.9467
4.75	5.4736	5.5677	5.6382				
5.75	5.7826	5.7973	5.7973	5.75	5.9	5.9177	5.9295
5.75	5.8	5.8154	5.8198	5.75	5.913	5.9306	5.9419
5.75	5.8261	5.8421	5.8532	5.75	5.9565	5.9746	5.9746
5.75	5.8696	5.8867	5.9013	6.0	6.0	6.0	6.0

Table 2

CONCLUDING REMARKS

We have calculated minimum and maximum values of Second order moment for the given values of Harmonic mean and Arithmetic mean. Hence, we observe that:

1. for given values of H and M_{min}, the values of Second order moment are same at all existing points. Similarly for given values of H and M_{max}, the values of Second order moment are same at all existing points.
2. The value of Second order moment increases with Arithmetic mean for the fixed value of Harmonic mean and decreases with Harmonic mean for the fixed value of Arithmetic mean.
3. Number of switching points decreases with Harmonic mean for minimum value of Second order moment and number of switching points increases with Harmonic mean for maximum value of Second order moment.
4. It is not necessary that a probability distribution exist for all values of prescribed moments. We need to search such values. Any probability distribution lies in a feasible range of given moments. This feasible range can be obtained by the information of maximum and minimum values of moments. If only one

moment is prescribed then the feasible range of another moment can be obtained by Anju Rani [2] and if two moments are given then the feasible range of third moment can be obtained.

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EFFECT OF HEAT TRANSFER ON MHD FLOW OF A DUSTY VISCO-ELASTIC LIQUID THROUGH POROUS MEDIUM PAST AN INCLINED PLANE WITH MASS TRANSFER

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ABSTRACT

The present problem is concerned with the effect of heat transfer on MHD laminar flow of an unsteady visco-elastic (Kuvshiniski type) liquid through porous medium with uniform distribution of dust particles past an inclined plane under the influence of exponential pressure gradient and mass transfer. The liquid is bounded by a parallel upper surface at a distance h from the plane. Analytical expressions for velocities of liquid and dust particles and concentration are obtained which are in elegant forms. It is observed that velocities of liquid and dust particles increase with the increase in Permeability of medium (K_0), *Grashof number* (G_1) *and Prandtl number* (P_r) *for first half cycle, after it the effect of* K_0, G_1 *and* P_r *on both velocities is reversed for second half cycle.*

Keywords : Kuvshinski Fluid, Dusty fluid, Heat transfer, Mass transfer, Porous medium, MHD Flow.

INTRODUCTION

The problem of laminar flow of dusty visco-elastic liquid past an inclined plane has become very important in recent years particularly in the field of industrial and chemical engineering such as latex particles emulsion paints and reinforcing particles in polymer. The study of these problems and rheological aspects of such flows have not received much attention although this has some bearing on the problems of petroleum industry and chemical engineering.

Saffman [8] has expressed a model equation describing the influence of dust particles on the motion of fluids. Bagchi and Maiti [1], Kishore and Pandey [4], Marble [6], Mukherjee et al [7], Sharma and Dubey [9], Singh [10], Srivastava [11] and Vimala [13] using equations of Saffman [8] have investigated a number of dusty gas flow problems in different situations. Mandal et al [5] have considered unsteady flow of dusty visco-elastic (Kuvshiniski type) liquid between two oscillating plates. Chaudhary and Singh [2] have considered the flow of a dusty visco-elastic (Kuvshiniski type) liquid down an inclined plane. Johari and Gupta [3] have studied MHD flow of a dusty visco-elastic (Kuvshiniski type) liquid past an inclined plane. Recently, Varshney et al [12] have discussed effect of porous medium on MHD flow of a dusty visco-elastic liquid past an inclined plane with mass transfer.

In the present section we have considered the problem of Varshney et al [12] by introducing heat transfer through porous medium with mass transfer under the same conditions taken by Varshney et al.

MATHEMATICAL ANALYSIS

Consider the laminar flow of an unsteady visco-elastic (Kuvshiniski type) liquid with uniform distribution of dust particles past an inclined plane of inclination θ to the horizontal. We choose the origin of coordinate system at the bottom of the inclined plane. The x-axis is taken opposite to the direction of the flow and along the greatest slope of the plane and y-axis is taken perpendicular to the plane. The magnetic field of uniform strength is applied along to y-axis. Since both the dust and liquid particles move along the greatest slope of the plane and the flow is laminar, the velocity of the both liquid and dust particles can be defined by the following relations:

$$u_1 = u_1(y, t), \qquad u_2 = 0, \qquad u_3 = 0$$

$$v_1 = v_1(y, t), \qquad v_2 = 0, \qquad v_3 = 0 \qquad \text{... (1)}$$

where (u_1, u_2, u_3), (v_1, v_2, v_3) are the velocity components of liquid and dust particles respectively.

Following Saffman [8] the equations of motion for the flow of dusty visco-elastic liquid (Kuvshiniski type) through porous medium with heat and mass transfer are given by:

$$\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial u_1}{\partial t}=-\frac{1}{\rho}\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u_1}{\partial y^2}+\frac{kN_o}{\rho}\left(1+\alpha\frac{\partial}{\partial t}\right)(v_1-u_1)$$

$$-\frac{\sigma B_o^2}{\rho}u_1-\frac{\nu}{K_o}u_1-g\,sin\theta+g\beta(C_o-C)+g\beta_1(T_o-T) \quad \text{... (2)}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y}+g\,cos\theta=0 \quad \text{... (3)}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}=0 \quad \text{... (4)}$$

$$\frac{\partial v_1}{\partial t}=\frac{k}{m}(u_1-v_1) \quad \text{... (5)}$$

$$\frac{\partial T}{\partial t}=\frac{k_1}{\rho C_p}\frac{\partial^2 T}{\partial y^2} \quad \text{... (6)}$$

$$\frac{\partial C}{\partial t}=D\frac{\partial^2 C}{\partial y^2} \quad \text{... (7)}$$

where p is the pressure, ν is the kinematic coefficient of viscosity of the gas, α is the coefficient of visco-elasticity of the gas, k is the Stoke's resistance coefficient, N_o is the number density of the dust particles which is taken to be constant, ρ is the density of the liquid, m is the mass of a dust particle, β is the coefficient of mass expansion, β_l is the coefficient of temperature expansion, D is the mass diffusivity, K_o is the permeability of medium, k_l is the thermal conductivity, C_p is the specific heat at constant pressure.

The initial boundary conditions are :

$$t \le 0;\quad u_l = 0 = v_l \qquad C = C_o, \qquad T = T_o \qquad \text{at } y = 0$$

$$t > 0;\quad u_l = 0 = v_l \qquad C = C_o, \qquad T = T_o \qquad \text{at } y = 0$$

$$u_l = U \qquad C = C_o - e^{-\lambda^2 t}, \qquad T = T_o - e^{-\lambda^2 t} \qquad \text{at } y = h$$

We express the pressure p as

$$p = -\rho g(x\sin\theta + y\cos\theta) - x\rho\varphi(t) \qquad \text{... (8)}$$

With the help of equation (8), equations (2), (5), (6) and (7) become :

$$\left(1+\alpha\frac{\partial}{\partial t}\right)\frac{\partial u_1}{\partial t} = F(t) + \nu\frac{\partial^2 u_1}{\partial y^2} + \frac{kN_o}{\rho}\left(1+\alpha\frac{\partial}{\partial t}\right)(v_1 - u_1)$$

$$-\frac{\sigma B_o^2}{\rho}u_1 - \frac{\nu}{K_o}u_1 + g\beta C_1 + g\beta_1 T_1 \qquad \text{... (9)}$$

$$\frac{\partial v_1}{\partial t} = \frac{k}{m}(u_1 - v_1) \qquad \text{... (10)}$$

$$\frac{\partial T_1}{\partial t} = \frac{k_1}{\rho C_p}\frac{\partial^2 T_1}{\partial y^2} \qquad \text{... (11)}$$

$$\frac{\partial C_1}{\partial t} = D\frac{\partial^2 C_1}{\partial y^2} \qquad \text{... (12)}$$

where $C_1 = C_o - C,\ T_1 = T_o - T$ and $F(t) = \phi(t) + \alpha\, \phi'(t)$

Corresponding boundary conditions are:

$$t \le 0; \quad u_1 = 0 = v_1 \quad C_1 = 0, \quad T_1 = 0 \quad \text{at } y = 0$$

$$t > 0; \quad u_1 = 0 = v_1 \quad C_1 = 0, \quad T_1 = 0 \quad \text{at } y = 0$$

$$u_1 = U \quad C_1 = e^{-\lambda^2 t}, \quad T_1 = e^{-\lambda^2 t} \quad \text{at } y = h$$

Let us choose u_1, v_1, T_1, C_1 and $F(t)$ as

$$\begin{aligned} u_1(y,t) &= u(y)\, e^{-\lambda^2 t} \\ v_1(y,t) &= v(y)\, e^{-\lambda^2 t} \\ T_1(y,t) &= T_{11}(y)\, e^{-\lambda^2 t} \\ C_1(y,t) &= C_{11}(y)\, e^{-\lambda^2 t} \\ F(t) &= c\, e^{-\lambda^2 t} \end{aligned} \qquad \text{... (13)}$$

Substituting the values of u_1, v_1, T_1, C_1 and $F(t)$ in equation (9) to (12), we get :

$$u'' + \left(A^2 - M_o^2 - \frac{1}{K_o}\right)u = -d - GC_{11} - G_1 T_{11} \qquad \text{... (14)}$$

$$v = \frac{k}{k - m\lambda^2}u \qquad \text{... (15)}$$

$$T''_{11} + \lambda^2 P_r T_{11} = 0 \qquad \text{... (16)}$$

$$C''_{11} + \lambda^2 S_c C_{11} = 0 \qquad \text{... (17)}$$

where
$$A^2 = \frac{\lambda^2}{\nu}(1-\alpha\lambda^2)\left\{1+\frac{Mm}{k-m\lambda^2}\right\},$$

$$M_0^2 = \frac{\sigma B_0^2}{\mu}, \text{ (Hartman number)}$$

$$M = \frac{kN_o}{\rho}, \text{ (Dusty Fluid Parameter)}$$

$$d = \frac{c}{\nu}, \quad \text{(Pressure parameter)}$$

$$G = \frac{g\beta}{\nu}, \quad \text{(Modified Grashof number)}$$

$$G_1 = \frac{g\beta_1}{\nu}, \quad \text{(Grashof number)}$$

$$S_c = \frac{1}{D}, \text{ (Schmidt number)}$$

$$P_r = \frac{\rho C_p}{k_1}, \text{ (Prandtl number)}$$

The boundary conditions are :

$$y = 0;\ u = 0 \qquad C_{11} = 0 \qquad T_{11} = 0$$
$$y = h;\ u = U \qquad C_{11} = 1 \qquad T_{11} = 1 \qquad \text{... (18)}$$

Solution of equation (14), (15), (16) and (17) under boundary conditions (18) is given by

$$u(y) = \frac{d}{p^2}cos(py) + \frac{sin(py)}{sin(ph)}\left\{U + \frac{d}{p^2} + \frac{G}{p^2-q^2} + \frac{G_1}{p^2-r^2} - \frac{d}{p^2}cos(ph)\right\}$$

$$-\frac{d}{p^2} - \frac{G}{p^2-q^2}\frac{sin(qy)}{sin(qh)} - \frac{G_1}{p^2-r^2}\frac{sin(ry)}{sin(rh)} \qquad \text{... (19)}$$

$$v(y) = \frac{k}{k-m\lambda^2}\left[\frac{d}{p^2}cos(py) + \frac{sin(py)}{sin(ph)}\left\{U + \frac{d}{p^2} + \frac{G}{p^2-q^2} + \frac{G_1}{p^2-r^2} - \frac{d}{p^2}cos(ph)\right\}\right.$$

$$\left.-\frac{d}{p^2} - \frac{G}{p^2-q^2}\frac{sin(qy)}{sin(qh)} - \frac{G_1}{p^2-r^2}\frac{sin(ry)}{sin(rh)}\right] \qquad \text{... (20)}$$

$$T_{11} = \frac{sin\,(ry)}{sin\,(rh)} \qquad \text{... (21)}$$

$$C_{11} = \frac{sin(qy)}{sin(qh)} \qquad \text{... (22)}$$

where

$$p^2 = A^2 - M_o^2 - \frac{1}{K_o}, \qquad q^2 = S_c\,\lambda^2, \qquad r^2 = S_c\,\lambda^2$$

The equations of velocities of the liquid and dust particle, temperature and concentration are expressed as

$$u_1 = \left[\frac{d}{p^2} cos(py) + \frac{sin(py)}{sin(ph)} \left\{ U + \frac{d}{p^2} + \frac{G}{p^2 - q^2} + \frac{G_1}{p^2 - r^2} - \frac{d}{p^2} cos(ph) \right\} \right.$$

$$\left. - \frac{d}{p^2} - \frac{G}{p^2 - q^2} \frac{sin(qy)}{sin(qh)} - \frac{G_1}{p^2 - r^2} \frac{sin(ry)}{sin(rh)} \right] e^{-\lambda^2 t} \quad \text{... (23)}$$

$$v_1 = \frac{k}{k - m\lambda^2} \left[\frac{d}{p^2} cos(py) + \frac{sin(py)}{sin(ph)} \left\{ U + \frac{d}{p^2} + \frac{G}{p^2 - q^2} + \frac{G_1}{p^2 - r^2} - \frac{d}{p^2} cos(ph) \right\} \right.$$

$$\left. - \frac{d}{p^2} - \frac{G}{p^2 - q^2} \frac{sin(qy)}{sin(qh)} - \frac{G_1}{p^2 - r^2} \frac{sin(ry)}{sin(rh)} \right] e^{-\lambda^2 t} \quad \text{... (24)}$$

$$T_1 = \frac{sin\,(ry)}{sin\,(rh)} e^{-\lambda^2 t} \quad \text{... (25)}$$

$$C_1 = \frac{sin(qy)}{sin(qh)} e^{-\lambda^2 t} \quad \text{... (26)}$$

The skin friction of liquid is expressed as

$$\tau = \left(\frac{\partial u_1}{\partial y} \right)_{y=0} = \left[\frac{p}{sin(ph)} \left\{ U + \frac{d}{p^2} + \frac{G}{p^2 - q^2} + \frac{G_1}{p^2 - r^2} - \frac{d}{p^2} cos(ph) \right\} \right.$$

$$\left. - \frac{G}{p^2 - q^2} \frac{q}{sin\,(qh)} - \frac{G_1}{p^2 - r^2} \frac{r}{sin\,(rh)} \right] e^{-\lambda^2 t} \quad \text{... (27)}$$

RESULTS AND DISCUSSION

The Velocity Profiles for visco-elastic liquid and dust particles are tabulated in Table-1 and 2 and plotted in Fig.-1 and 2 having Graph-1 to 3 at $\nu = 1$, $U = 1$, $t = 0.4$, $\lambda = 2$, $h = 30$, $M = 1$, $m/k = 0.2$, $c = 2$, $M_o = 1$, $\alpha = 0.2$, $G = 2$, $S_c = 0.2$ and different values of K_o, G_1 and P_r.

	K_o	G_1	P_r
For Graph-1	10	2	0.2
For Graph-2	100	2	0.2
For Graph-3	10	4	0.2
For Graph-4	10	2	0.6

From the Graphs of Fig.-1 and 2 it is noticed that velocity of visco-elastic liquid and dust particles are sinusoidal in nature. It is also observed that velocity of liquid and dust particles increases with the increase in K_o, G_1 and P_r for first half cycle, after it the effect of K_o, G_1 and P_r on both velocities is reversed for second half cycle.

The temperature Profile is tabulated in Table-3 and plotted in Fig.-3 having Graph-1 for $P_r = 0.2$ and Graph-2 for $P_r = 0.6$ at $t = 0.4$, $\lambda = 2$, $h = 30$. From Fig.-3 it is noticed that temperature is also sinusoidal in nature. It is also observed that temperature increases upto $y = 1.7$ with the increase in P_r, after it temperature decreases upto $y = 4$ for first half cycle. This effect is reversed for second half cycle.

The skin friction Profile is tabulated in Table-4 and plotted in Fig.-4 having Graph-1 to 3. It is concluded that skin friction decreases with the increase in t. It is also observed that skin friction increases with the increase in K_o, G_1 and P_r.

PARTICULAR CASE

When G_1 and P_r are equal to zero, this problem reduces to the problem of Varshney et al [12].

ACKNOWLEDGEMENT

The author are very thankful to the Principal and the Head, Department of Mathematics for providing necessary facilities.

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Table-1 : Values of velocity of liquid at $\nu = 1$, $U = 1$, $t = 0.4$, $\lambda = 2$, $h = 30$, $M = 1$, $m/k = 0.2$, $c = 2$, $M_o = 1$, $\alpha = 0.2$, $G = 2$, $S = 0.2$ and different values of K_o, G_1 and P_r

y	Graph-1	Graph-2	Graph-3	Graph-4
0	0.00000	0.00000	0.00000	0.00000
2	0.40884	6.06504	5.49643	0.69170
4	-3.20352	-2.79073	-2.59727	-2.16592
6	-1.92397	-6.76466	-6.39870	-2.23726
8	2.84232	2.53965	1.63867	0.80121

Table-2 : Values of velocity of dust particles at $\nu = 1$, $U = 1$, $t = 0.4$, $\lambda = 2$, $h = 30$, $M = 1$, $m/k = 0.2$, $c = 2$, $M_o = 1$, $\alpha = 0.2$, $G = 2$, $S = 0.2$ and different values of K_o, G_1 and P_r

y	Graph-1	Graph-2	Graph-3	Graph-4
0	0.00000	0.00000	0.00000	0.00000
2	2.04420	30.32518	27.48217	3.45848
4	-16.01760	-13.95365	-12.98633	-10.82960
6	-9.61985	-33.82329	-31.99352	-11.18632
8	14.21158	12.69826	8.19336	4.00605

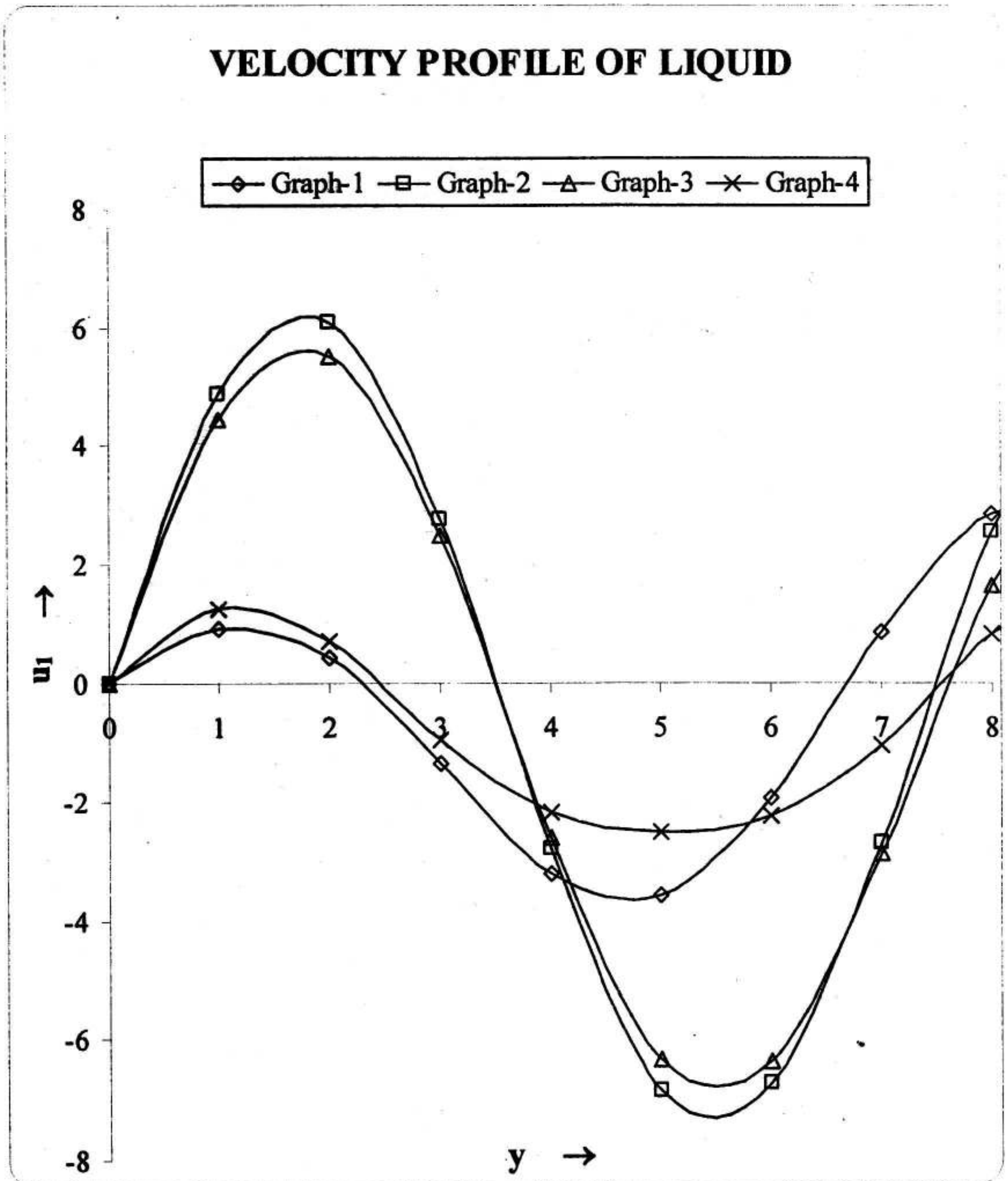
VELOCITY PROFILE OF LIQUID
Graph-1
Graph-2
Graph-3
Graph-4
u1 →
y →
8
6
4
2
0
-2
-4
-6
-8
0
1
2
3
4
5
6
7
8

Fig-1

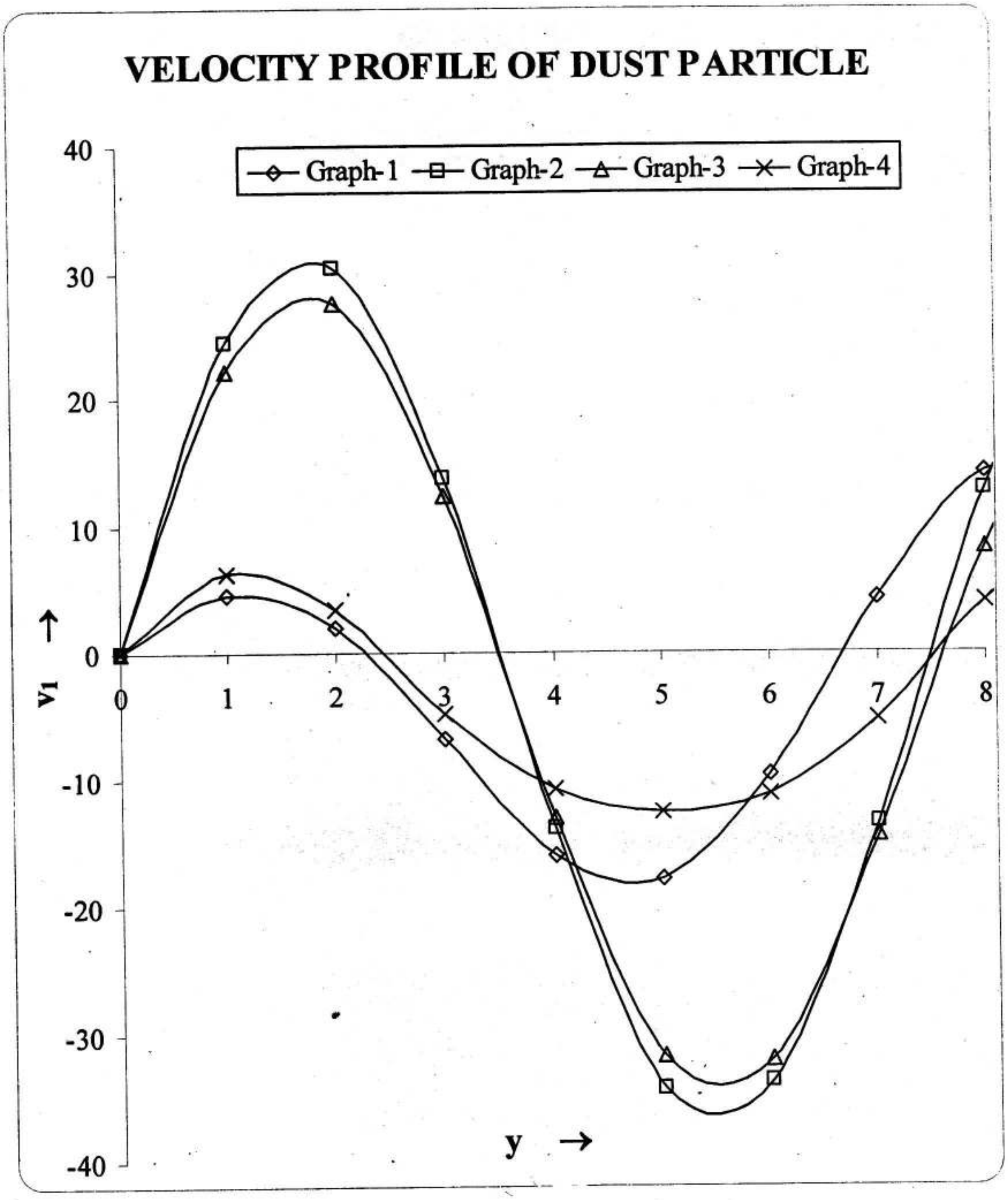
VELOCITY PROFILE OF DUST PARTICLE
Graph-1
Graph-2
Graph-3
Graph-4
40
30
20
10
0
-10
-20
-30
-40
0
1
2
3
4
5
6
7
8
v1 →
y →

Fig.-2

Table-3 : Values of temperature at $t = 0.4$, $\lambda = 2$, $h = 30$ and different values of P_r

y	Graph-1	Graph-2
0	0	0
2	0.198497	0.01437
4	-0.08555	-0.02872
6	-0.16162	0.043011
8	0.155212	-0.05723

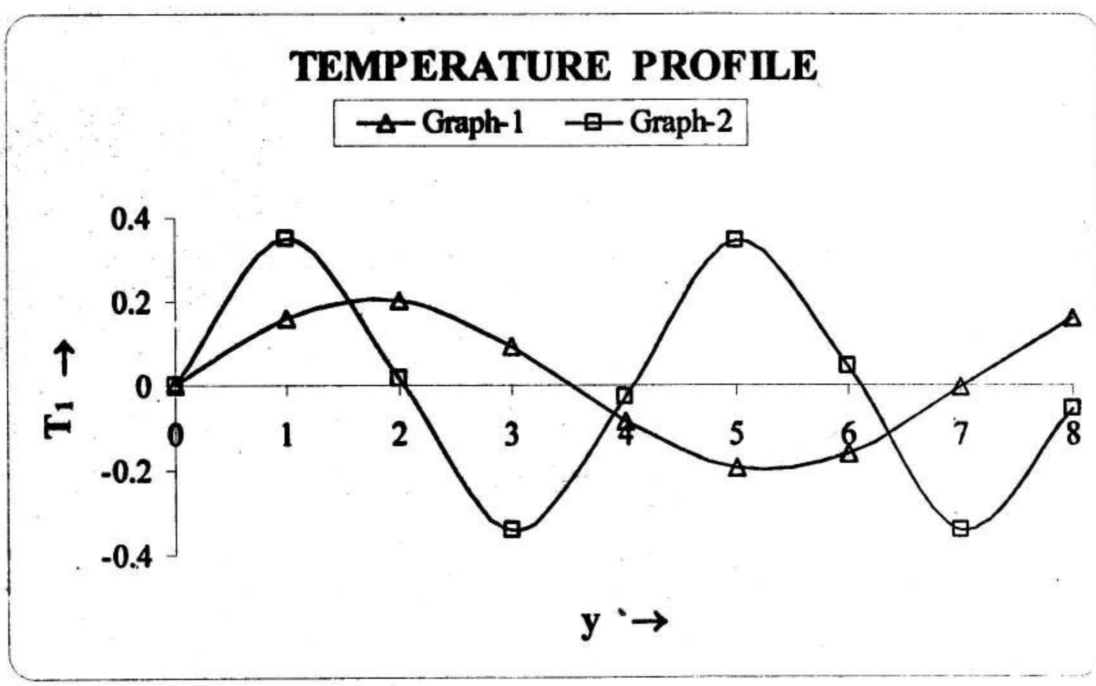

Fig.-3

Table-4 : Values of skin friction at $\nu = 1$, $U = 1$, $\lambda = 2$, $h = 30$, $M = 1$, $m/k = 0.2$, $c = 2$, $M_o = 1$, $\alpha = 0.2$, $G = 2$, $S = 0.2$ and different values of K_o, G_1 and P_r

t	Graph-1	Graph-2	Graph-3	Graph-4
0	6.48503	28.35128	25.82981	8.91044
0.2	2.91391	12.73905	11.60608	4.00372
0.4	1.30931	5.72402	5.21495	1.79899
0.6	0.58831	2.57197	2.34323	0.80834
0.8	0.26434	1.15566	1.05288	0.36321
1	0.11878	0.51927	0.47309	0.16320

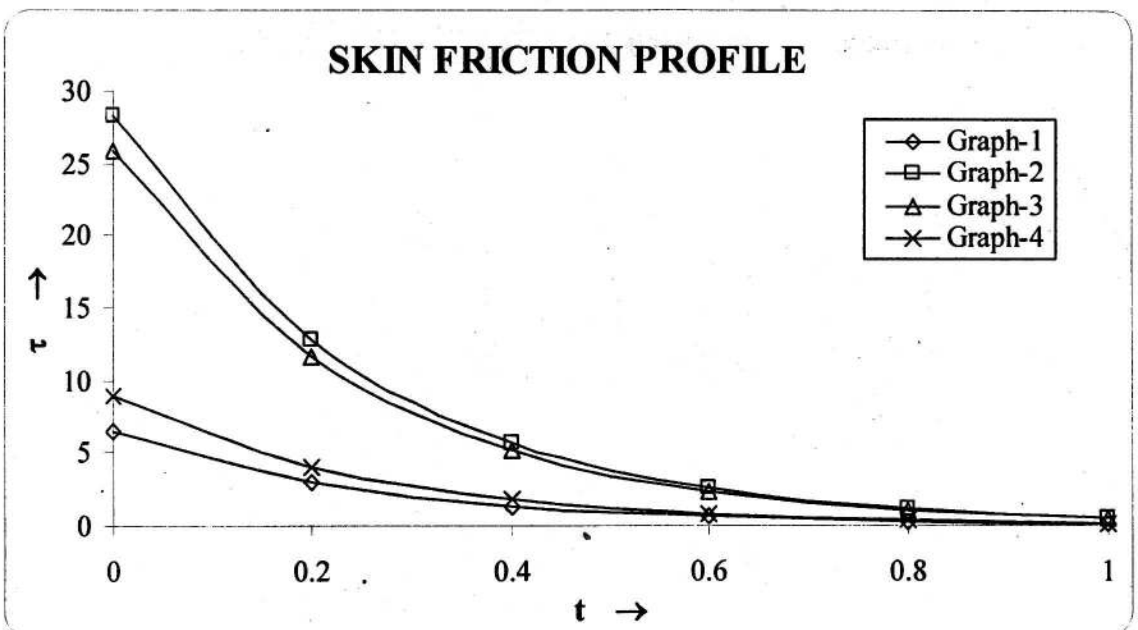

Fig.-4

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AN INVENTORY MODEL FOR HAZARDOUS ITEMS OF TWO-PARAMETER EXPONENTIAL DISTRIBUTION WITH FINITE PRODUCTION RATE

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ABSTRACT

In the present paper, an inventory model is developed with a constant hazardous rate of two-parameter negative exponential distribution. The production and demand rates are constant. Shortages are not allowed. The average total cost is minimized to decide the production cycle and hence the optimum inventory level and the time of production. The result is explained by various numerical examples with graphs. It is clear that how the total cost minimizes in different situations.

Keywords: Finite production rate, hazardous items, inventory model, two-parameter exponential distribution.

Mathematics Subject Classification (2000): 90B05.

1. INTRODUCTION

Inventory is defined as the stock of items to satisfy the future demands. Harris [5] developed the mathematical model to decide the number of products at ones. He also gave the concept of economic order quantity (EOQ). After him many mathematical models have been developed for controlling the inventory. In several exciting models, it is assumed that the products have infinite shelf time. But actually deterioration plays a vital role in inventory. Deterioration is defined as decay, spoilage, loss of utility of products etc. The process of deterioration is observed in volatile liquids, beverages, medicines, blood components, sweets, fruits and vegetables. There are many other products in the real world which deteriorate with a significant rate. So it should not be neglected in the decision process of production lot size.

In recent years, mathematical ideas have been used in different areas in real life problems, particularly for controlling inventory. When the items of the commodity are kept in stock as an inventory for fulfilling the future demand, there may be deterioration of items in the inventory system.

At the end of the storage period, deterioration is studied by Whitin [9] for the fashion goods. Ghare and Schrader [4] analyzed the problem of decaying inventories exponentially and developed an EOQ model with constant demand. Covert and Philip [3] extended Ghare and Schrader's model by considering a two parameter Weibull's distribution for variable rate of deterioration. Shah and Jaiswal [7] developed an order-level inventory model for a system with constant rate of deterioration. Aggarwal [1] modified Shah and Jaiswal's model in calculating the average inventory holding cost. Yang and Wee [6] developed an integrated multi-lot-size production inventory model for deteriorating item. Sharma et al. [8] developed a deterministic production inventory model for deteriorating products with exponentially declining demand and shortages. Baten and Kamil [2] studied the inventory management systems with two-parameter exponential distributed hazardous items in which production and demand rates are constant.

Some commodities were observed to shrink with time by a proportion which can be approximated by a negative exponential function of time. The probability density function of a two-parameter exponential distribution is given by

$$f(t;\mu,\eta) = \frac{1}{\eta}e^{-\frac{(t-\mu)}{\eta}}, t \geq \mu, \eta > 1,$$

where μ is the location parameter and η is the scale parameter.

The unreliability function is given by

$$(t;\mu,\eta) = 1 - e^{-\frac{(t-\mu)}{\eta}}.$$

The failure or hazard rate function of on-hand inventory is given by

$$H(t;\mu,\eta) = \frac{f(t;\mu,\eta)}{1-F(t;\mu,\eta)} = \frac{1}{\eta}, t \geq \mu, \eta > 1.$$

So the hazardous rate followed by the two-parameter exponential distribution is constant.

In this paper, the objective is to develop a mathematical model for obtaining an optimal production cycle time for hazardous items associated with two-parameter exponential distribution during the cycle time which minimizes the total cost per unit time of an inventory-production system. The conclusion is illustrated by numerical examples and graphs in various situations.

2. ASSUMPTIONS AND NOTATIONS

1. d is the rate of demand, which is known.

2. I(t) is the on hand inventory at any time t.

3. R is the finite production rate per unit time.
4. $H(t; \mu, \eta) = \frac{1}{\eta}$ is constant hazardous rate per unit time. It follows two-parameter exponential distribution.
5. C_p is the production cost of one item.
6. C_h is the inventory holding cost coefficient per unit time.
7. C_o is the operating cost per order.
8. C is the average total cost per unit time
9. T is the time of one cycle.
10. t_1 is the time of production.
11. I_1 is the maximum inventory level
12. Lead time is zero.
13. Shortages are not allowed.
14. The inventory system deals with only one item.
15. $d, R, p, C_h, C_o, C, I, t_1, I_1 > 0; I(t), t \geq 0; R > d$.

3. MATHEMATICAL MODEL

The differential equations describing the behavior of the system are given by

$$\frac{dI(t)}{dt} = R - d - \frac{1}{\eta} I(t) \quad \text{for} \quad 0 \leq t \leq t_1 \qquad \dots (3.1)$$

and

$$\frac{dI(t)}{dt} = -d - \frac{1}{\eta} I(t) \text{ for } t_1 \leq t \leq T \qquad \dots (3.2)$$

with boundary conditions

$$I(0) = 0, I(t_1) = I_1 \quad \text{and} \quad I(T) = 0 \qquad \dots (3.3)$$

The solution of above system is

$$I(t) = \begin{cases} \eta(R-d)\left(1 - e^{-\frac{t}{\eta}}\right), 0 \leq t \leq t_1 \\ \eta d\left(e^{\frac{(T-t)}{\eta}} - 1\right), \ t_1 \leq t \leq T \end{cases} \qquad \dots (3.4)$$

With

$$I_1 = \eta(R-d)\left(1 - e^{-\frac{t_1}{\eta}}\right) \qquad \dots (3.5)$$

and $$T = \eta \cdot \log\left[\frac{R}{d}\left(e^{\frac{t_1}{\eta}} - 1\right) + 1\right] \quad \ldots (3.6)$$

The situation of inventory is illustrated in figure (3.1).

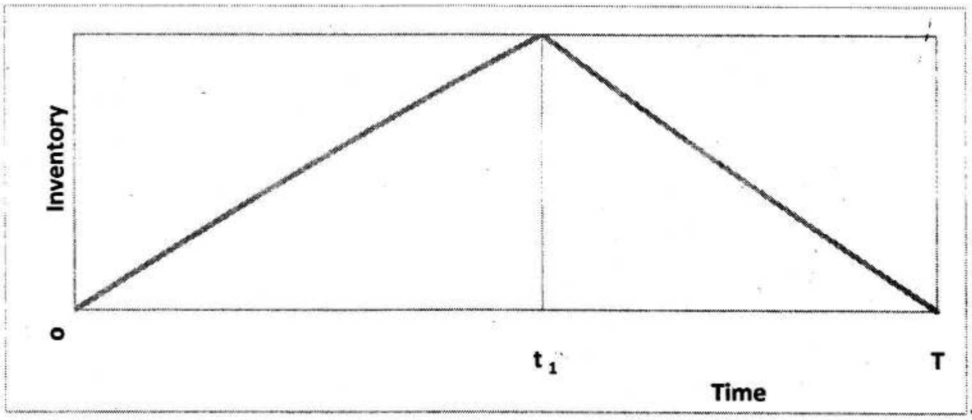

Figure (3.1)

Neglecting the terms containing η with degree greater than or equal to 2, we have

$$I(t) = \begin{cases} (R-d)\left(t - \frac{t^2}{2\eta}\right), & 0 \le t \le t_1 \\ d\left((T-t) + \frac{(T-t)^2}{2\eta}\right), & t_1 \le t \le T. \end{cases} \quad \ldots (3.7)$$

The inventory holding cost for one cycle is

$$HC = C_h \int_0^T I(t)dt$$

Using equation (3.7), we get

$$HC = \frac{C_h}{6\eta}[(R-d)(3\eta t_1^2 - t_1^3) + d\{3\eta(T-t_1)^2 + (T-t_1)^3\}]$$

The number of units produced in time $t_1 = Rt_1$

The production cost per production run is

$$PC = C_p R t_1$$

The set-up cost per cycle is

$$SC = C_o$$

The total cost for one cycle of time T is

$$TC = HC + PC + SC$$

Hence the average total cost per unit time is

$$C = \frac{TC}{T}$$

$$= \frac{C_h}{6\eta T}[(R-d)(3\eta t_1^2 - t_1^3) + d\{3\eta(T-t_1)^2 + (T-t_1)^3\}] + \frac{C_p R t_1}{T} + \frac{C_o}{T} \quad \dots (3.8)$$

By equations (3.6) and (3.8), it is clear that C is a function of only one variable t_1.

For C to be minimum, $\frac{dC}{dt_1} = 0$ and $\frac{d^2C}{dt_1^2} > 0$. By $\frac{dC}{dt_1} = 0$, we have

$$3\eta\left[C_h d\{(R-d)(2\eta - t_1)t_1 + 2\eta C_p R\}\left\{\frac{R}{d}\left(e^{\frac{t_1}{\eta}} - 1\right) + 1\right\}\right.$$
$$\left. + C_h d(R-d)(2\eta + T - t_1)(T - t_1)\right]\log\left\{\frac{R}{d}\left(e^{\frac{t_1}{\eta}} - 1\right) + 1\right\}$$
$$- R\left[C_h\{(R-d)(3\eta - t_1)t_1^2 + d(3\eta + T - t_1)(T - t_1)^2\} + 6\eta\left(C_p R t_1 + C_o\right)\right]e^{\frac{t_1}{\eta}}$$
$$= 0 \quad \dots (3.9)$$

Solving equation (3.9) by Newton-Raphson method, the value of t_1 is obtained numerically up to desired accuracy. That value of t_1, by which $\frac{d^2C}{dt_1^2} > 0$, gives the minimum value of C. Hence the optimum values of the cycle time T, inventory level I_1 and minimum average cost are obtained.

4. NUMERICAL EXAMPLES

1. Let us consider the values of parameters d = 6 units/week, R = 16 units/week, C_p = \$ 16/unit, C_o = \$240/set up, C_h = \$ 4 per unit per week, $H = \frac{1}{\eta} = 0.06$. Then the following results were obtained

from our inventory model: t_1 = 2.05 weeks, T = 4.99 weeks, I_1 = 19.3 units and minimum C = $ 191.43. The inventory level I(t) at any time t is shown in figure 4.1.

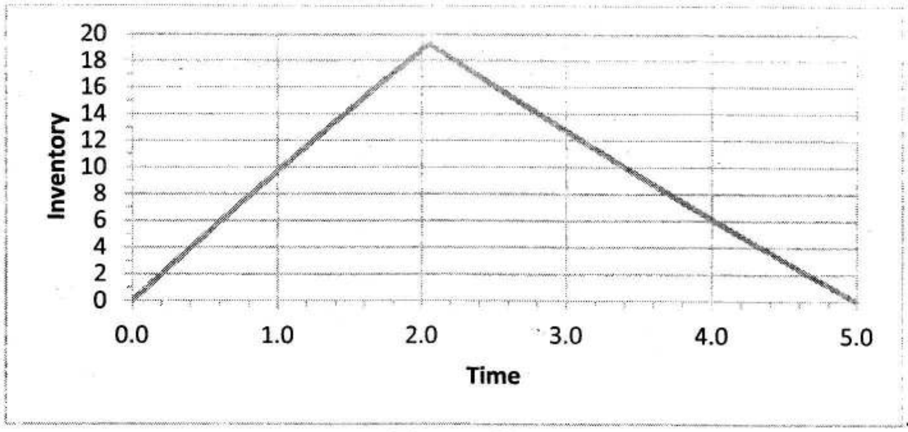

Figure 4.1

2. For d = 8 units/week and other values same as example 1, the results are t_1 = 2.66 weeks, T = 4.96 weeks, I_1 = 19.68 units and minimum C = $ 225.33.

3. For R = 20 units/week and other values same as example 1, the results are t_1 = 1.54 weeks, T = 4.65 weeks, I_1 = 20.57 units and minimum C = $ 197.67.

4. For C_p = $ 12/unit and other values same as example 1, the results are t_1 = 2.11 weeks, T = 5.12 weeks, I_1 = 19.8 units and minimum C = $ 165.11.

5. For C_o = $180/set up and other values same as example 1, the results are t_1 = 1.76 weeks, T = 4.33 weeks, I_1 = 16.69 units and minimum C = $ 178.56.

6. For C_h = $ 2 per unit per week and other values same as example 1, the results are t_1 = 2.72 weeks, T = 6.44 weeks, I_1 = 25.07 units and minimum C = $ 169.88.

7. For d = 8 units/week, R = 20 units/week and other values same as example 1, the results are t_1 = 1.92 weeks, T = 4.44 weeks, I_1 = 21.75 units and minimum C = $ 235.64.

8. For R = 20 units/week, C_o = $180/set up and other values same as example 1, the results are t_1 = 1.32 weeks, T = 4.04 weeks, I_1 = 17.76 units and minimum C = $ 183.88.

9. For d = 8 units/week, R = 20 units/week, C_o = $180/set up and other values same as ex. 1, the results are t_1 = 1.65 weeks, T = 3.85 weeks, I_1 = 18.82 units and minimum C = $ 221.15.

10. Taking different values of H and constant values of other parameters same as in ex. 1, the results are shown in the table 4.1 and figures 4.2, 4.3 and 4.4.

Table 4.1

C_o	C_h	C_p	R	d	H	t_1	T	C(T)	I_1
240	4	16	16	6	0.04	2.07	5.18	188.07	19.88
240	4	16	16	6	0.06	2.05	4.99	191.43	19.30
240	4	16	16	6	0.08	2.03	4.83	194.66	18.78
240	4	16	16	6	0.10	2.02	4.68	197.77	18.29
240	4	16	16	6	0.12	2.01	4.55	200.76	17.84
240	4	16	16	6	0.14	2.00	4.44	203.65	17.43
240	4	16	16	6	0.16	1.99	4.33	206.45	17.04
240	4	16	16	6	0.18	1.98	4.24	209.15	16.68
240	4	16	16	6	0.20	1.98	4.15	211.77	16.34

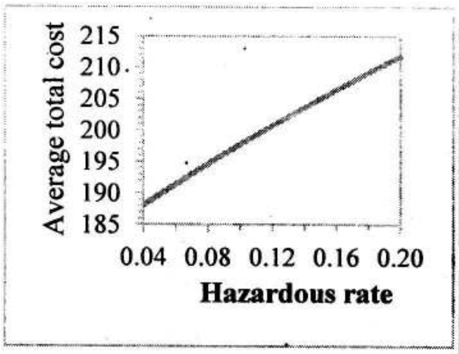

Figure 4.2

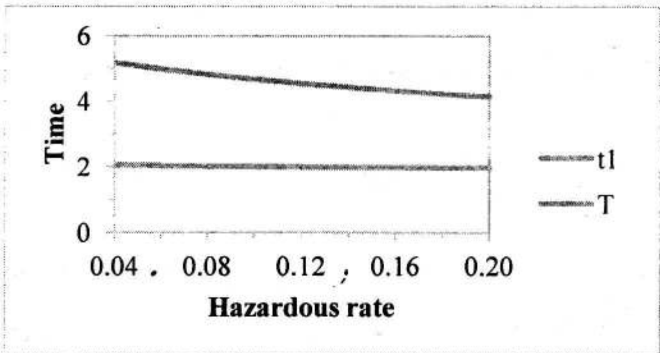

Figure 4.3

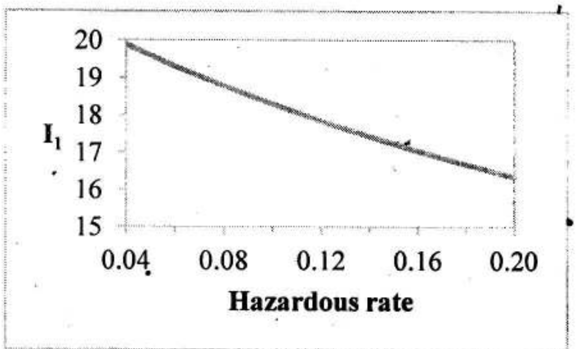

Figure 4.4

5. CONCLUSION

In this paper a mathematical model is developed with exponential distributed hazardous items which include finite production and constant demand rates. The average total cost per unit time is minimized to decide the production cycle time and hence the optimum inventory level. The average cost C increases when demand rate increases although it will be profitable. C also increases when production rate increases because time for deterioration increases but in this case operating cost will be less, so it will be a profitable case. It is obvious that C increases when operating, holding and production costs increase. When hazardous rate increases, then the average total cost increases, inventory level, production and cycle times decrease.

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AN EPIDEMIOLOGICAL MODEL WITH FERTILITY REDUCTION DUE TO THE INFECTION

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ABSTRACT

In this paper an epidemiological SIS model with standard incidence is presented. It has been observed that many of the diseases reduce the fertility of mother due to the infection. A fertility reduction factor is introduced. Model we present here includes vital dynamics influenced by disease and vertical transmission. A reproduction number has been obtained.

Key Words: Carrying Capacity, Vertical Transmission, Stability, Reproduction Number

[2000] Mathematics Subject Classifications: 92 D 30

1. INTRODUCTION

The stability and behavior of solutions of an infectious disease transmission model depends not only on the epidemiological formulation but also on the demographic process which has been incorporated in the model. A demographic model with density dependent restricted population growth is given by the logistic equation

$$\frac{dN}{dt} = r(1-\frac{N}{K})N,$$

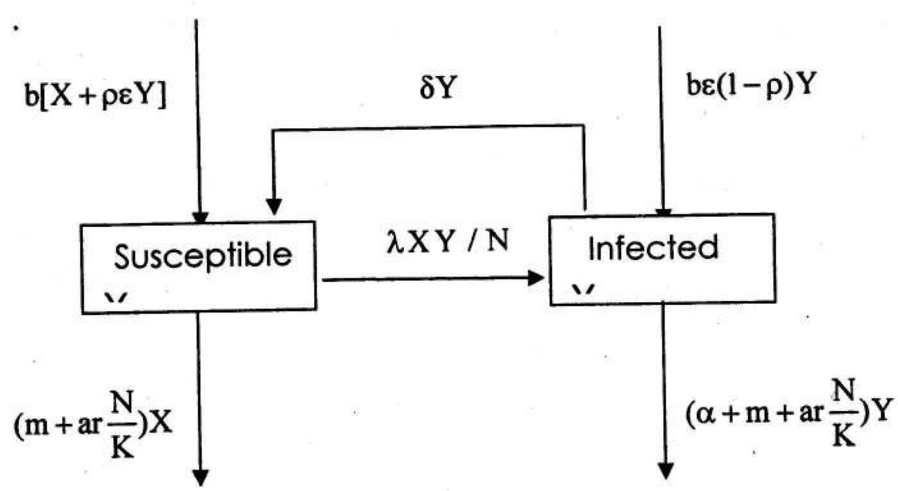
where N is the total population, r is the growth rate and K is the carrying capacity of the environment.

Anderson and May [1] proposed a variety of models for infectious disease with varying population sizes. Various aspects were contributed through different models by Mena Lorca [8], Eldestein [3] and Hethcote [4]. A brief description of the dynamics of vertically transmitted disease was discussed in Busenberg [2].

Using generalized logistic rate the vertical transmission models on infectious diseases were explained by Khandelwal and Singh [6, 7]. A vertical transmission model with normal death rate and with fertility reduction was suggested by Zinshi Zhou [8]. Here we have analyzed an SIS model considered by Zinshi Zhou [8] with standard incidence and generalized death rate $(m + ar\frac{N}{K})$ where 'a' is convex combination constant.

2. FORMULATION OF THE MODEL

The transfer diagram of the model is



In this compartmental model the total population N(t) has been divided in two compartments, first is X(t) of susceptible and second is Y(t) of infectious. Here r is population growth rate=b-m>0, where b and m are the natural birth and death rates respectively. K is the carrying capacity of the population. λ is the contact rate. The fertility of the susceptible is b and of infective is $b\varepsilon$ with ε in [0, 1], so that the ε is the fertility reduction factor due to infection. Assume that the new born of infective women are susceptible with the probability ρ and are infective with $(1-\rho)$ probability. Thus $(1-\rho)$ is the probability of the vertical transmission from mother to her newborn baby before, during or just after the birth. δ is recovery rate constant and α is the rate of death due to the infection.

The transfer diagram leads to the following differential equations:

$$\frac{dX}{dt} = b[X + \rho\varepsilon Y] - (m + ar\frac{N}{K})X - \frac{\lambda XY}{N} + \delta Y \qquad \dots (2.1)$$

$$\frac{dY}{dt} = \frac{\lambda XY}{N} + b\varepsilon(1-\rho)Y - (\delta + \alpha + m + ar\frac{N}{K})Y \qquad \dots (2.2)$$

$$\frac{dN}{dt} = [r(1 - a\frac{N}{K})]N - \{\alpha + b(1-\varepsilon)\}\, Y \qquad \dots (2.3)$$

Since X+Y=N, only two equation of the above system are independent. Reformulating equations (2.2) and (2.3) in terms of the fractions $S = \frac{X}{N}, I = \frac{Y}{N}$, $S + I = 1$, and put $\alpha + b(1-\varepsilon) = \alpha_1$ to get

$$\frac{dI}{dt} = I[(\alpha_1 - \lambda)I + (\lambda - \delta - \alpha_1 - b\varepsilon\rho)] \qquad \dots (2.4)$$

$$\frac{dN}{dt} = N[r - ar\frac{N}{K} - \alpha_1 I] \qquad \dots (2.5)$$

Setting the time derivatives of I and N equal to zero we get

$$I[(\alpha_1 - \lambda)I + (\lambda - \delta - \alpha_1 - b\varepsilon\rho)] = 0 \qquad \dots (2.6)$$

$$N[r - ar\frac{N}{K} - \alpha_1 I] = 0 \qquad \dots (2.7)$$

From equation (2.6) we get

either $I = 0$ or $I = \dfrac{\lambda - \delta - \alpha_1 - b\varepsilon\rho}{\lambda - \alpha_1}$

At disease free stage, from equation (2.7) we have

either $N = 0$ or $N = \dfrac{K}{a}$

So we get the diseases free equilibrium points

$$E_1(I_1, Y_1) = (0,0) \text{ and } E_2(X_2, Y_2) = (0, \frac{K}{a})$$

There is an endemic equilibrium in (I, N) plane,

For the endemic equilibrium point in equation (2.7), we put

$$I = I_3 = \frac{\lambda - \delta - \alpha_1 - b\varepsilon\rho}{\lambda - \alpha_1} \quad \text{to get} \qquad \dots (2.8)$$

$$N_3 = \frac{K}{a}[1 - \frac{\alpha_1}{r_1} I_3]$$

Solving we get,

$$N_3 = \frac{K}{ra}[\frac{(r - \alpha_1)(\lambda - \alpha_1) + \alpha_1(\delta + b\varepsilon\rho)}{(\lambda - \alpha_1)}] \qquad \dots (2.9)$$

Thus we have the endemic equilibrium point

$$E_3(I_3, N_3) \text{ where } I_3 = \frac{\lambda - \delta - \alpha_1 - b\varepsilon\rho}{\lambda - \alpha_1} \text{ and } N_3 = \frac{K}{a}[1 - \frac{\alpha_1}{r_1} I_3]$$

3. STABILITY ANALYSIS

For the study of the stability of the equilibrium points the variation matrix of the above system is

$$J = \begin{pmatrix} 2I(\alpha_1 - \lambda) + (\lambda - \delta - \alpha_1 - b\varepsilon\rho) & 0 \\ -\alpha_1 N & r - 2ar\frac{N}{K} - \alpha_1 I \end{pmatrix}$$

The value of the J at the point E_1 (0, 0) is

$$J_1 = \begin{pmatrix} \lambda - \delta - \alpha_1 - b\varepsilon\rho & 0 \\ 0 & r \end{pmatrix}$$

The Eigen values are

$$\psi_1 = \lambda - \delta - \alpha_1 - b\varepsilon\rho \text{ and } \psi_2 = r$$

Clearly one Eigen value ψ_2 is positive, so the equilibrium point E_1 is an unstable point.

At the equilibrium point E_2 (0, K/a) the value of Jacobian is

$$J_2 = \begin{pmatrix} \lambda - \delta - \alpha_1 - b\varepsilon\rho & 0 \\ \alpha_1 \dfrac{K}{a} & -r \end{pmatrix}$$

Here the Eigen values at this point are

$$\psi_1 = \lambda - \delta - \alpha_1 - b\varepsilon\rho \text{ and } \psi_2 = -r$$

Clearly one Eigen value $\psi_2 = -r$ is negative and so the point will be stable if the other Eigen value ψ_1 is negative i.e. if $\lambda - \delta - \alpha_1 - b\varepsilon\rho < 0$ is satisfied. Put the value of α_1 to get the condition as

$$\lambda - [\alpha + b - b\varepsilon + \delta + b\varepsilon\rho) < 1$$

This turns in

$$\frac{\lambda + b\varepsilon(1-\rho)}{\alpha + b + \delta} < 1$$

Thus we get

$$R_0 = \frac{\lambda + b\varepsilon(1-\rho)}{\alpha + b + \delta}$$ .This is called reproduction number.

Hence the diseases free equilibrium point E_2 is stable if $R_0 < 1$.

At the endemic equilibrium point E_3, the variation matrix is

$$J_2 = \begin{pmatrix} -(\lambda - \delta - \alpha_1 - b\varepsilon\rho) & 0 \\ -\alpha_1 \dfrac{K}{ra}[\dfrac{(r-\alpha_1)(\lambda-\alpha_1) + \alpha_1(\delta + b\varepsilon\rho)}{(\lambda - \alpha_1)}] & -[(r-\alpha_1) + \dfrac{\alpha_1(\delta + b\varepsilon\rho)}{(\lambda-\alpha_1)}] \end{pmatrix}$$

The Eigen values of the above variation matrix are

$$\psi_1 = -(\lambda - \delta - \alpha_1 - b\varepsilon\rho) > 0 \text{ and } \psi_2 = -[(r-\alpha_1) + \frac{\alpha_1(\delta + b\varepsilon\rho)}{(\lambda - \alpha_1)}]$$

To get the signs of the above Eigen values negative we must have

$$(\lambda - \delta - \alpha_1 - b\varepsilon\rho) > 0 \text{ and } (r - \alpha_1) > 0 \quad \text{where} \quad \alpha + b(1 - \dot{\varepsilon}) = \alpha_1$$

Hence we conclude that for the stability of the endemic equilibrium point E_3 we must have following conditions

$$R_0 = \frac{\lambda + b\varepsilon(1-\rho)}{\alpha + b + \delta} > 1 \text{ and } r > \alpha + b\,(1-\varepsilon).$$

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I, Dr. B.L. Meena, hereby declare that the particulars given above are true to the best of my knowledge and belief.

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Dr. B. L. Meena

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The Sequence