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# REPLENISHMENT POLICY FOR DECAYING INVENTORY SYSTEM WITH VARIABLE DEMANDS AND PARTIAL BACKLOGGING

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## ABSTRACT

*In this study, two deteriorating inventory models are illustrated with time decreasing and linearly stock depending demands. In both the models, the partial backlogging is applied. The deteriorating parameter is taken constant. By using numerical example, both models are validated. And sensitivity of cost function is performed with respect to the variation in other parameters.*

**KEYWORDS:** deterministic inventory systems, deterioration, time dependent partial backlogging.

## 1. INTRODUCTION

Deteriorating inventory models have been widely studied in recent years. Silver and Meal [22] gave a heuristic solution procedure for the inventory model with time varying demand. Donaldson [9] exhibited a very complicated solution procedure taking demand to be linear. Ritchie [16, 17 and 18] developed a simple procedure for adjusting the economic order quantity model for linearly increasing and decreasing demand.

Dev and patel [7] derived a lot size model for constant deterioration of items with time proportional demand. Sachan [19] allowed shortages in Dev and Patel [7]'s model. Covert and Philip [6] developed inventory model using a two parameter Weibull distribution for deterioration of units. Philip [14] formulated inventory model when deterioration start after sometime and used a three parameter Weibull distribution for deterioration of units. Mishra [12] extended Covert and Philip [6]'s model for finite rate of replenishment. Raafat [15], Shah and Shah [20] and Goyal and Giri [10] cite upto date review on deteriorating inventory models. Abad [1,2] derived a pricing and ordering policy for a variable rate of deterioration and partially backlogging. Chung et al [5] took into account the backorder cost and lost sale. Ouyang et al. [13] gave an inventory model for deteriorating

items with exponentially declining demand and partial backlogging. Shah and Shukla [21] gave a deteriorating inventory model for waiting time partial backlogging. An order level inventory model for items deteriorating at a constant rate was discussed by Aggarwal [3] in calculating the average inventory holding cost. Bahari-Kashani [4] discussed a heuristic model with time-proportional demand. Deb and Chaudhuri [8] studied a model with a finite rate of production and a time proportional deterioration rate, allowing backlogging. Mehta and Shah [11] presented an EOQ model with exponentially decreasing demand pattern.

The present work provides the extensions of the work done by Shah and Shukla [21]. Here, two decaying inventory models are illustrated. In both the models, demand rate is decreasing type function. Shortages are backlogged partially. For validation of the models, numerical examples are given. And sensitivity analysis is also carried out to understand the effect of model parameters on total cost function. Results of each sensitivity table are summarized just after of each sensitivity table.

## 2. ASSUMPTIONS AND NOTATIONS

**a)** $O$ = the ordering cost (OC) per order.

**b)** $C_P$ = the purchase cost (PC) per unit.

**c)** $h$ = the inventory holding cost (IHC) per unit per time unit.

**d)** $C_b$ = the backordered cost (BC) per unit short per time unit.

**e)** $C_L$ = the cost of lost sales (LS) per unit.

**f)** $t_1$ = the time at which the inventory level reaches zero, $t_1 \geq 0$.

**g)** $t_2$ = the length of period during which shortages are allowed.

**h)** $T\,(=t_1+t_2)$ the length of cycle time.

**i)** $D(t)$ = demand rate.

**j)** $\theta$ = deterioration rate.

**k)** $\delta$ = backlogging parameter.

**l)** $I_M$ = the maximum level during $[0, T]$.

**m)** $I_B$ = the maximum backordered units during stock out period.

**n)** $Q\,(=I_M+I_B)$ the order quantity during a cycle of length $T$.

**o)** $I(t)$ = the inventory level at time t.

**p)** $I_1(t)$ = the level of positive inventory at times $t$, $0 \leq t \leq t_1$.

**q)** $I_2(t)$ = the level of negative inventory at time $t$, $t_1 \leq t \leq t_1+t_2$.

r) $G(t_1, t_2)$ = the total cost per time unit.

s) $B(t) = \dfrac{1}{1+\delta(T-t)}$ the backlogging rate; $\delta > 0$ denotes the backlogging parameter, $t_1 \le t \le T$.

## 3. GRAPHICAL MODEL

Under above assumption, the on – hand inventory level at any instant of time is exhibited in figure 3.1.

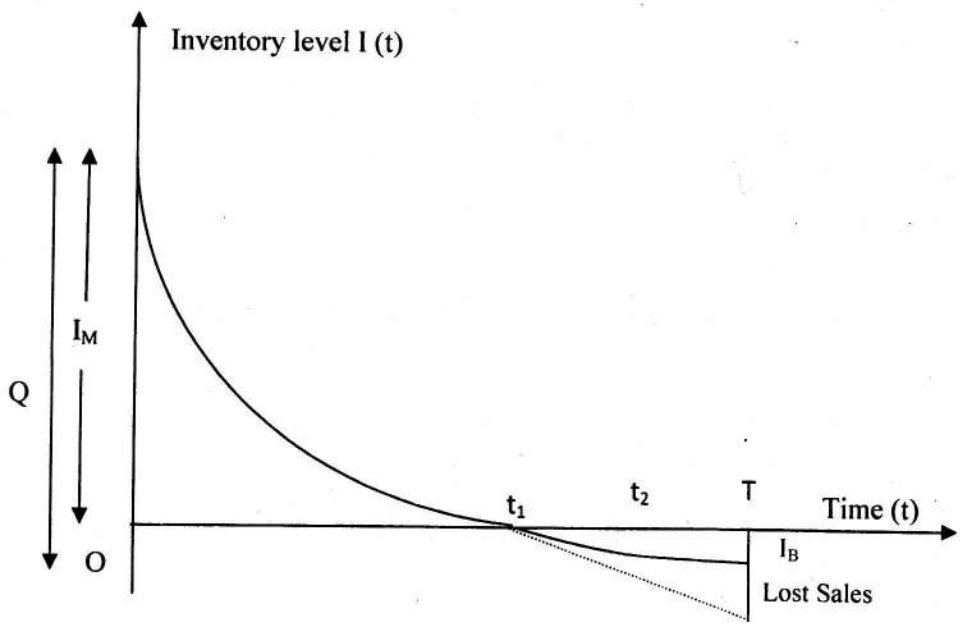


**Figure – 3.1**

## 4. MATHEMATICAL MODELS

### 4.1. *MODEL* I : Decreasing Time Dependent Demand :

Here, demand rate D (t) = $ae^{-bt}$

During the period $[0, t_1]$, the inventory depletes due to the cumulative effects of demand and deterioration. Hence, the inventory level at any instant of time during $[0, t_1]$ is described by the differential equation:

$$\frac{dI_1(t)}{dt} = -ae^{-bt} - \theta I_1(t) \qquad \ldots (4.1.1)$$

From time $t_1$ onwards shortages occur and inventory level reaches to zero. During the interval $[t_1, t_1+t_2]$, the inventory level depends on demand and a fraction of the demand is backlogged. The state of inventory during $[t_1, t_1+t_2]$, can be represented by the differential equation,

$$\frac{dI_2(t)}{dt} = -\frac{ae^{-bt}}{1+\delta(T-t)} \qquad \ldots(4.1.2)$$

With $I_1(t_1)=0$ and $I_2(t_1)=0$.

Solution of differential Equation (4.1.1) is

$$I_1(t)e^{\theta t} = -\frac{a}{(\theta-b)}e^{(\theta-b)t} + \frac{a}{(\theta-b)}e^{(\theta-b)t_1} = \frac{a}{(\theta-b)}\left[e^{(\theta(t_1-t)-bt_1)} - e^{-bt}\right] \qquad \ldots(4.1.3)$$

Solution of equation (4.1.2) is

$$I_2(t) = \frac{a}{\delta}\log(1+\delta(T-t)) - \frac{ab}{\delta^2}[(1+\delta T)\log(1+\delta(T-t)) - (1+\delta(T-t))] + c$$

Using initial condition $I_2(t_1)=0$, we get

$$I_2(t) = \frac{a}{\delta}\left\{\left(1-\frac{b}{\delta}(1+\delta T)\right)(\log(1+\delta(T-t)) - \log(1+\delta t_2)) - b(t-t_1)\right\} \qquad \ldots(4.1.4)$$

The maximum positive inventory level is

$$I_M = I_1(0) = \frac{a}{(\theta-b)}\left(e^{(\theta t_1 - bt_1)} - 1\right) = \frac{a}{(\theta-b)}\left(e^{(\theta-b)t_1} - 1\right) \qquad \ldots(4.1.5)$$

The maximum backordered units are

$$I_B = -I_2(t_1+t_2) = \frac{a}{\delta}\left[\log(1+\delta t_2) - \frac{b}{\delta}\left((1+\delta T)\log(1+\delta t_2) - \delta t_2\right)\right] \qquad \ldots(4.1.6)$$

Hence, the order size during [0, T] is

$$Q = I_M + I_B$$

$$= \left[\frac{a}{(\theta-b)}\left(e^{(\theta-b)t_1} - 1\right) + \frac{a}{\delta}\left[\log(1+\delta t_2) - \frac{b}{\delta}[(1+\delta T)\log(1+\delta t_2) - \delta t_2]\right]\right] \qquad \ldots(4.1.7)$$

The total variable cost is comprised of the sum of the ordinary cost, holding cost, backordered cost, lost sales cost and purchase cost. They are grouped together after evaluating the above said costs individually.

**A)** Order cost per cycle OC = O

**B)** Inventory holding cost per cycle

$$IHC = \int_0^{t_1} I_1(t)dt = h\int_0^{t_1} \frac{a}{(\theta - b)}\left\{e^{(\theta(t_1-t)-bt_1)} - e^{-bt}\right\}dt$$

$$= \frac{ha}{b\theta(\theta-b)}\left[(\theta-b)e^{-bt_1} + be^{(\theta-b)t_1} - \theta\right]$$

**C)** Backordered cost per cycle

$$BC = C_b \int_{t_1}^{t_1+t_2} -I_2(t)dt$$

$$= -C_b \int_{t_1}^{t_1+t_2} \frac{a}{\delta}\left\{\left(1-\frac{b}{\delta}(1+\delta T)\right)\left(\log(1+\delta(T-t)) - \log(1+\delta t_2)\right) - b(t-t_1)\right\}dt$$

$$= C_b \frac{a}{\delta}\left\{\left(\frac{b}{\delta}(1+\delta T)-1\right)\left(\frac{1}{\delta}\left[(1+\delta T)\log(1+\delta t_2) + \delta t_1 + 1\right]\right) + \frac{bt_2^2}{2}\right\}$$

**D)** Cost due to lost sales per cycle

$$LS = C_L \int_{t_1}^{t_1+t_2} ae^{-bt}\left(1 - \frac{1}{1+\delta(T-t)}\right)dt = aC_L \int_{t_1}^{t_1+t_2}\left(e^{-bt} - \frac{e^{-bt}}{1+\delta(T-t)}\right)dt$$

**E)** Purchase cost PC = $C_P$Q

$$PC = C_P a\left[\begin{array}{l} \frac{1}{(\theta-b)}\left(e^{(\theta-b)t_1}-1\right) + \frac{1}{\delta}\log(1+\delta t_2) \\ + \frac{1}{\delta}\left[\log(1+\delta t_2) - \frac{b}{\delta}\left[(1+\delta T)\log(1+\delta t_2) - \delta t_2\right]\right] \end{array}\right]$$

Therefore, the total cost per time unit is $G(t_1,t_2) = \frac{1}{(t_1+t_2)}\left[OC + IHC + BC + LS + PC\right]$

$$
= \frac{1}{(t_1+t_2)}\left\{\begin{array}{l}
O+\dfrac{ha}{b\theta(\theta-b)}\left[(\theta-b)e^{-bt_1}+be^{(\theta-b)t_1}-\theta\right]+C_b\dfrac{a}{\delta}\left\{\begin{array}{l}\left(\dfrac{b}{\delta}(1+\delta T)-1\right)\\ \left(\dfrac{1}{\delta}\left[\log(1+\delta t_2)+\delta t_1+1\right]\right)\\ +\dfrac{bt_2^2}{2}\end{array}\right\}\\
+\left[aC_L\left[\dfrac{e^{-bt_1}}{b}\left(1-e^{-bt_2}\right)+\dfrac{1}{\delta}\left[\log(1+\delta t_2)+\dfrac{b}{\delta}\left[\delta t_2-(1+\delta T)\log(1+\delta t_2)-1\right]\right]\right]\right]\\
+C_p a\left[\begin{array}{l}\dfrac{1}{(\theta-b)}\left(e^{(\theta-b)t_1}-1\right)+\dfrac{1}{\delta}\log(1+\delta t_2)\\ +\dfrac{1}{\delta}\left[\log(1+\delta t_2)-\dfrac{b}{\delta}\left[(1+\delta T)\log(1+\delta t_2)-\delta t_2\right]\right]\end{array}\right]
\end{array}\right\} \quad \ldots (4.1.8)
$$

$$
\frac{\partial G}{\partial t_1}=\left\{\begin{array}{l}
\dfrac{1}{(t_1+t_2)}\left\{\begin{array}{l}\dfrac{ha}{b\theta(\theta-b)}\left(b(\theta-b)e^{-bt_1}\left(e^{\theta t_1}-1\right)-1\right)\\ +C_b\dfrac{a}{\delta}\left[\left(\dfrac{b}{\delta}(1+\delta T)-1\right)+\dfrac{b}{\delta}\left(\begin{array}{l}\log(1+\delta t_2)\\ +\delta t_1+1\end{array}\right)\right]\\ +aC_L\left[e^{-b(t_1+t_2)}-e^{-bt_1}-\dfrac{b}{\delta}\log(1+\delta t_2)\right]\\ +C_p a\left[\begin{array}{l}e^{(\theta-b)t_1}\\ -\dfrac{b}{\delta}\log(1+\delta t_2)\end{array}\right]\end{array}\right\}\\
-\dfrac{1}{(t_1+t_2)^2}\left\{\begin{array}{l}O+\dfrac{ha}{b\theta(\theta-b)}\left[(\theta-b)e^{-bt_1}+be^{(\theta-b)t_1}-\theta\right]\\ +C_b\dfrac{a}{\delta}\left\{\left(\dfrac{b}{\delta}(1+\delta T)-1\right)\left(\dfrac{1}{\delta}\left[\log(1+\delta t_2)+\delta t_1+1\right]\right)+\dfrac{bt_2^2}{2}\right\}\\ +\left[aC_L\left[\dfrac{e^{-bt_1}}{b}\left(1-e^{-bt_2}\right)+\dfrac{1}{\delta}\left[\begin{array}{l}\log(1+\delta t_2)\\ +\dfrac{b}{\delta}\left[\begin{array}{l}\delta t_2-(1+\delta T)\\ \log(1+\delta t_2)-1\end{array}\right]\end{array}\right]\right]\right]\\ +C_p a\left[\begin{array}{l}\dfrac{1}{(\theta-b)}\left(e^{(\theta-b)t_1}-1\right)+\dfrac{1}{\delta}\log(1+\delta t_2)\\ +\dfrac{1}{\delta}\left[\log(1+\delta t_2)-\dfrac{b}{\delta}\left[(1+\delta T)\log(1+\delta t_2)-\delta t_2\right]\right]\end{array}\right]\end{array}\right\}
\end{array}\right\}=0 \quad \ldots (4.1.9)
$$

$$\frac{\partial G}{\partial t_2}=\left\{\begin{array}{l}\frac{1}{(t_1+t_2)}\left[\begin{array}{l}C_b\frac{a}{\delta}\left\{\begin{array}{l}\frac{b}{\delta^2}\left(\log(1+\delta t_2)+\delta t_1+1\right)\\+\left(\frac{b}{\delta}(1+\delta T)-1\right)\left(\frac{1}{(1+\delta t_2)}\right)+bt_2\end{array}\right\}\\+aC_L\left\{e^{-b(t_1+t_2)}+\left[\frac{1}{(1+\delta t_2)}+\frac{b}{\delta}\left(1-\log(1+\delta t_2)-\frac{(1+\delta T)}{(1+\delta t_2)}\right)\right]\right\}\\+C_P\frac{a}{\delta}\left\{\frac{\delta}{(1+\delta t_2)}-b\left[\log(1+\delta t_2)+\frac{(1+\delta T)}{(1+\delta t_2)}-1\right]\right\}\end{array}\right]\\-\frac{1}{(t_1+t_2)^2}\left\{\begin{array}{l}O+\frac{ha}{b\theta(\theta-b)}\left[(\theta-b)e^{-bt_1}+be^{(\theta-b)t_1}-\theta\right]\\+C_b\frac{a}{\delta}\left\{\begin{array}{l}\left(\frac{b}{\delta}(1+\delta T)-1\right)\\\left(\frac{1}{\delta}\left[\log(1+\delta t_2)+\delta t_1+1\right]\right)\\+\frac{bt_2^2}{2}\end{array}\right\}\\+\left[aC_L\left[\frac{e^{-bt_1}}{b}\left(1-e^{-bt_2}\right)+\frac{1}{\delta}\left[\log(1+\delta t_2)+\frac{b}{\delta}\left[\delta t_2-(1+\delta T)\log(1+\delta t_2)-1\right]\right]\right]\right]\\+C_Pa\left[\begin{array}{l}\frac{1}{(\theta-b)}\left(e^{(\theta-b)t_1}-1\right)+\frac{1}{\delta}\log(1+\delta t_2)\\+\frac{1}{\delta}\left[\log(1+\delta t_2)-\frac{b}{\delta}\left[(1+\delta T)\log(1+\delta t_2)-\delta t_2\right]\right]\end{array}\right]\end{array}\right\}\end{array}\right\}=0$$

. . . (4.1.10)

Provided that

$$\left(\frac{\partial^2 G}{\partial t_1^2}\right)\left(\frac{\partial^2 G}{\partial t_2^2}\right)-\left(\frac{\partial^2 G}{\partial t_1\partial t_2}\right)^2>0$$

Here, equation (4.1.9) and equation (4.1.10) are highly non-linear and not easy to solve.

4.2. ***MODEL* II : Stock Dependent Demand :**

Here, Demand rate is D (t) = a+bI (t), where a>0 and 0<b<1

During the period $[0, t_1]$, the inventory depletes due to the cumulative effects of demand and deterioration. Hence, the inventory level at any instant of time during $[0, t_1]$ is described by the differential equation:

$$\frac{dI_1(t)}{dt} = -(a + bI(t)) - \theta I_1(t);\ 0 \le t \le t_1 \qquad \ldots (4.1.11)$$

From time $t_1$ onwards, shortages occur and inventory level reaches to zero. During the interval $[t_1, t_1 + t_2]$, the inventory level depends on demand and a fraction of the demand is backlogged. The state of inventory during $[t_1, t_1 + t_2]$, can be represented by the differential equation,

$$\frac{dI_2(t)}{dt} = -\frac{a}{1 + \delta(t_1 + t_2 - t)};\ t_1 \le t \le t_1 + t_2 \qquad \ldots (4.1.12)$$

The solution of differential equation (4.1.11) is:

Using boundary condition $I_1(t_1) = 0$, we get

$$I_1(t)e^{(\theta+b)t} = \frac{-a}{(\theta+b)} + \frac{a}{(\theta+b)}e^{(\theta+b)t_1}$$

$$I_1(t) = \frac{a}{(\theta+b)}\left\{e^{(\theta+b)(t_1-t)} - 1\right\}; \qquad 0 \le t \le t_1 \qquad \ldots (4.1.13)$$

The solution of differential equation (4.1.12) is obtained by using boundary condition $I_2(t_1)=0$.

$$I_2(t) = \frac{a}{\delta}[\log(1 + \delta(t_1 + t_2 - t)) - \log(1 + \delta t_2)]; \qquad t_1 \le t \le t_1 + t_2 \qquad \ldots (4.1.14)$$

The maimum positive inventory is

$$\text{IM}= I_1(0) = \frac{a}{\theta}\left(e^{(\theta+b)t_1} - 1\right) \qquad \ldots (4.1.15)$$

The maximum backordered units are

$$\text{I}_\text{B} = -\, I(t_1 + t_2) = \frac{a}{\delta}\log\left(1 + \delta t_2\right) \qquad \ldots (4.1.16)$$

Hence, the order size during [0, T] is

$$Q = I_M + I_B = \frac{a}{(\theta+b)}\left(e^{(\theta+b)t_1} - 1\right) + \frac{a}{\delta}\log\left(1+\delta t_2\right) \qquad \ldots (4.1.17)$$

The total cost per cycle consists of the following cost components:

**A)** Ordering cost per cycle (OC) = O

**B)** Inventory holding cost per cycle (IHC)

$$IHC = h\int_0^{t_1} I_1(t)\,dt = h\int_0^{t_1}\frac{a}{(\theta+b)}\left(e^{(\theta+b)(t_1-t)} - 1\right)dt = \frac{ha}{(\theta+b)^2}\left[e^{(\theta+b)t_1} - 1 - (\theta+b)t_1\right]$$

**C)** Backordered cost per cycle (BC)

$$\text{BC} = C_b\int_{t_1}^{t_1+t_2} -I_2(t)dt = C_b\int_{t_1}^{t_1+t_2} -\frac{a}{\delta}\left[\log\left(1+\delta\left(t_1+t_2-t\right)\right) - \log\left(1+\delta t_2\right)\right]dt$$

$$\text{BC} = C_b\,\frac{a}{\delta^2}\left[\delta t_2 - \log\left(1 + \delta t_2\right)\right]$$

**D)** Cost due to lost sales per cycle (LS)

$$\text{LS}= C_L a\int_{t_1}^{t_1+t_2}\left(1 - \frac{1}{1+\delta(t_1+t_2-t)}\right)dt = C_L\,\frac{a}{\delta}\left(\delta t_2 - \log\left(1 + \delta t_2\right)\right)$$

**E)** Purchase cost (PC) = $C_P Q$

$$= C_P\left[\frac{a}{(\theta+b)}\left(e^{(\theta+b)t_1}-1\right)+\frac{a}{\delta}\log\left(1+\delta t_2\right)\right]$$

Therefore, the total cost under above condition per time unit is

$$G(t_1,t_2)=\frac{1}{(t_1+t_2)}\left[OC+IHC+BC+LS+PC\right]$$

$$=\frac{1}{(t_1+t_2)}\begin{bmatrix} O+\frac{ha}{(\theta+b)^2}\left[e^{(\theta+b)t_1}-1-(\theta+b)t_1\right]+C_h\frac{a}{\delta^2}\left[\delta t_2-\log(1+\delta t_2)\right] \\ +C_L\frac{a}{\delta}\left(\delta t_2-\log(1+\delta t_2)\right)+C_P\left[\frac{a}{(\theta+b)}\left(e^{(\theta+b)t_1}-1\right)+\frac{a}{\delta}\log(1+\delta t_2)\right]\end{bmatrix} \quad \ldots(4.1.18)$$

The necessary conditions for the total cost per time unit, to be minimized are

$$\frac{\partial G}{\partial t_1}=0 \quad \text{and} \quad \frac{\partial G}{\partial t_2}=0$$

$$\frac{\partial G}{\partial t_1}=\frac{ha\left(e^{(\theta+b)t_1}-1\right)+C_Pa(\theta+b)e^{(\theta+b)t_1}}{(t_1+t_2)}-\frac{O(\theta+b)^2+ha\left[e^{(\theta+b)t_1}-1-(\theta+b)t_1\right]}{(t_1+t_2)^2(\theta+b)^2}$$

$$-\frac{a(\delta C_L+C_b)\left(\delta t_2-\log(1+\delta t_2)\right)}{\delta^2(t_1+t_2)^2}-\frac{C_Pa\left[\delta\left(e^{(\theta+b)t_1}-1\right)+(\theta+b)\log(1+\delta t_2)\right]}{\delta(\theta+b)(t_1+t_2)^2}=0 \quad \ldots(4.1.19)$$

$$\frac{\partial G}{\partial t_2}=\frac{a(\delta C_L+C_b)t_2+C_Pa}{(1+\delta t_2)(t_1+t_2)}-\frac{O(\theta+b)^2+ha\left[e^{(\theta+b)t_1}-1-(\theta+b)t_1\right]}{(\theta+b)^2(t_1+t_2)^2}$$

$$-\frac{a\left(\delta t_2-\log(1+\delta t_2)\right)(\delta C_L+C_b)}{\delta^2(t_1+t_2)^2}-\frac{C_Pa\left[\delta\left(e^{(\theta+b)t_1}-1\right)+(\theta+b)\log(1+\delta t_2)\right]}{\delta(\theta+b)(t_1+t_2)^2}=0 \quad \ldots(4.1.20)$$

Provided that

$$\left(\frac{\partial^2 G}{\partial t_1^2}\right)\left(\frac{\partial^2 G}{\partial t_2^2}\right)-\left(\frac{\partial^2 G}{\partial t_1 \partial t_2}\right)^2 > 0$$

Equation (4.1.19) & equation (4.1.20) are highly non- linear equation.

## 5. NUMERICAL EXAMPLE

Consider an inventory system with following parametric values in proper units.

$$[O, C_P, h, C_b, C_L, a, b, \delta] = [250, 8, 0.05, 12, 15, 25, 0.5, 2]$$

## 6. SENSITIVITY ANALYSIS TABLES

### 6.1. *CASE* 1: Eexponentially Time Dependent Demand :

**TABLE – 1** Variations in Deterioration Rate 'θ'

| Theta (θ) | $t_1$ | $t_2$ | Q | G | Profit (P) |
|---|---|---|---|---|---|
| 0.0500 | 4.2800 | 0.2500 | 41.3435 | 368.8819 | 36.8882 |
| 0.1000 | 3.3700 | 0.3300 | 41.3588 | 326.2103 | 32.6210 |
| 0.1500 | 2.8200 | 0.4200 | 40.8014 | 305.7038 | 30.5704 |

It is observed that increase in deterioration rate increases shortages time period and decreases positive inventory time period, procurement quantity, total cost and profit per time unit of an inventory system.

**TABLE - 2 Variations in Backlogging Parameter '$\delta$'**

| 'δ' | Time ($t_1$) | Time ($t_2$) | Q | G | Profit |
|---|---|---|---|---|---|
| 5 | 4.3200 | 0.1700 | 43.8917 | 210.8868 | 21.0887 |
| 10 | 4.3400 | 0.1100 | 45.4477 | 171.0448 | 17.1045 |
| 15 | 4.3600 | 0.0600 | 46.4657 | 160.4711 | 16.0471 |

It is observed in above table that increase in backlogging parameter, decreases shortage time period, total cost and profit and increases positive time period, and procurement quantity per time unit of an inventory system.

**TABLE - 3 Variations in Demand Parameter '*a*'**

| '*a*' | $t_1$ | $t_2$ | Q | G | Profit |
|---|---|---|---|---|---|
| 25 | 3.0900 | 0.1500 | 40.0853 | 211.1731 | 21.1173 |
| 50 | 2.2300 | 0.0900 | 69.8624 | 351.1183 | 35.1118 |
| 75 | 1.5900 | 0.0600 | 85.9206 | 481.7668 | 48.1767 |

Increase in demand parameter '*a*', increases procurement quantity, total cost and profit per time unit of an inventory system.

**TABLE - 4 Variations in demand Parameter '*b*'**

| 'b' | $t_1$ | $t_2$ | Q | G | Profit |
|---|---|---|---|---|---|
| 0.5000 | 3.0900 | 0.1500 | 40.0853 | 211.1731 | 21.1173 |
| 0.7000 | 2.2300 | 0.0900 | 28.3307 | 247.8760 | 24.7876 |
| 0.9000 | 1.5900 | 0.0600 | 21.1961 | 292.5494 | 29.2549 |

Increase in demand parameter '*b*', decreases procurement quantity and increases total cost and profit per time unit of an inventory system.

**6.2 *CASE* 2 : When Demand is Stock-Dependent :**

**TABLE - 5 Vvariations in Theta 'θ'**

| Theta (θ) | $t_1$ | $t_2$ | Q | G | Profit (P) |
|---|---|---|---|---|---|
| 0.0500 | 4.2800 | 0.2500 | 296.8103 | 621.9083 | 62.1908 |
| 0.1000 | 3.3700 | 0.3300 | 220.5244 | 584.5103 | 58.4510 |
| 0.1500 | 2.8200 | 0.4200 | 176.9230 | 556.1178 | 55.6118 |

Increases in deterioration rate increases shortages time period and decreases positive inventory time period, procurement quantity, total cost and profit per time unit of an inventory system.

**TABLE - 6 Vvariations in Backlogging Parameter '$\delta$'**

| 'δ' | Time ($t_1$) | Time ($t_2$) | Q | G | Profit |
|---|---|---|---|---|---|
| 5 | 4.3200 | 0.1700 | 297.9692 | 628.9724 | 62.8972 |
| 10 | 4.3400 | 0.1100 | 298.3238 | 634.1513 | 63.4151 |
| 15 | 4.3600 | 0.0600 | 299.1144 | 638.3648 | 63.8365 |

Increase in backlogging parameter, decreases shortages time period, positive time period, increases procurement quantity , total cost and profit per time unit of an inventory system.

**TABLE - 7 Variations in Demand Parameter 'a'**

| 'a' | $t_1$ | $t_2$ | Q | G | Profit |
|---|---|---|---|---|---|
| 100 | 3.0900 | 0.1500 | 805.0731 | 2212.7 | 221.2701 |
| 150 | 2.2300 | 0.0900 | 793.8193 | 3030.0 | 303.0020 |
| 200 | 1.5900 | 0.0600 | 649.8319 | 3487.3 | 348.7289 |

Increase in demand parameter 'a', decreases procurement quantity and increases total cost and profit per time unit of an inventory system.

**TABLE - 8 Variations in Demand Parameter 'b'**

| 'b' | $t_1$ | $t_2$ | Q | G | Profit |
|---|---|---|---|---|---|
| 0.5000 | 3.0900 | 0.1500 | 201.2683 | 611.0457 | 61.1046 |
| 0.7000 | 2.2300 | 0.0900 | 126.0993 | 563.8434 | 56.3843 |
| 0.9000 | 1.5900 | 0.0600 | 83.2924 | 569.8555 | 56.9855 |

Increase in demand parameter '*b*', decreases procurement quantity, total cost and profit per time unit of an inventory system significantly.

## 6. CONCLUSION

Results of sensitivity analysis are summarized after the end of each sensitivity table. These results can be used for forming maximum profit. Since deterioration is not always constant, deterioration can be switch to time proportional or assuming any well known probability distribution like weibull distribution. Demand rate can be taken ramp type function of time (a special type stabilizing demand). Other factors (inflation, discounts, etc.) may be incorporated for making this study fruitful in current business scenarios.

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# SINGLE INTEGRAL REPRESENTATIONS FOR EXTON'S HYPERGEOMETRIC FUNCTION $X_9$

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## ABSTRACT

*The aim of this paper to evaluate new single integral representations of Euler kind for the Exton's hypergeometric function $X_9$. The results are derived by making use of some rather elementary techniques together with certain known integrals. The results are interesting and may be useful.*

**Keywords and phrases:** Exton's Triple Hypergeometric Functions, Laplace Integral Representation, Euler Integral.

## 1. INTRODUCTION

Exton [1] has encountered twenty Laplace integral representations for the Exton's functions viz. $X_1$ to $X_{20}$. One of them is given below:

$$X_9(a,b;c;x,y,z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty e^{-s-t} s^{a-1} t^{b-1} \cdot {}_0F_1(-;c;xs^2 + yst + zt^2)\,ds\,dt \qquad \ldots (1.1)$$

$\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$

Exton [1] introduced twenty triple hypergeometric series namely $X_1$ to $X_{20}$. We recall here the definitions of

$X_9$, $X_{10}$ and $X_{11}$.

$$X_9(a,b;c;x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}x^m y^n z^p}{(c)_{m+n+p}m!\ n!\ p!} \qquad \dots (1.2)$$

$$X_{10}(a,b;c,d;x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}x^m y^n z^p}{(c)_{m+n}(d)_p\ m!\ n!\ p!} \qquad \dots (1.3)$$

$$X_{11}(a,b;c,d;x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}x^m y^n z^p}{(c)_{m+n}(d)_n\ m!\ n!\ p!} \qquad \dots (1.4)$$

The region of convergence of each series (1.2) to (1.4) are given by Srivastava and Karlsson [4, pp 105,106, entry 55(a), 56(a), 57(a)]

$$r < \frac{1}{4}\Lambda t < \frac{1}{4}\Lambda s < \frac{1}{2} + \frac{1}{2}\sqrt{[(1-4r)(1-4t)]} \qquad \dots (1.5)$$

$$r < \frac{1}{4}\Lambda t < \frac{1}{4}\Lambda s < (1-2\sqrt{t})\left[\frac{1}{2} + \frac{1}{2}\sqrt{(1-4r)}\right] \qquad \dots (1.6)$$

$$r < \frac{1}{4}\Lambda t < \frac{1}{4}\Lambda s < \left[1-2\sqrt{(rw_1)}\right]\left[1-2\sqrt{(t-tw_1)}\right]$$

$$P(w) = 2 - 4w + \sqrt{w/r} - \sqrt{(1-w)/t}$$

$$w_1 : \textit{the root of } P(w) = 0 \textit{ in } \begin{cases} (0,\ r/(r+t)), & r \le t \\ (r/(r+t), 1), & r > t \end{cases} \qquad \dots (1.7)$$

where $|x| < r, |y| < r, |z| < r$. The positive quantities $r$, $s$ and $t$ are associated radii of convergence.

## 2. PRELIMINARY RESULTS

The following integral [3] is used in the present investigation

$$\int_a^b (x-a)^{u-1}(b-x)^{v-1}(x-c)^{-(u+v)}\,dx=(b-c)^{-u}(b-a)^{u+v-1}(a-c)^{-v}\,\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \qquad \dots (2.1)$$

provided that a<c<b, Re(u)>0, Re(v) >0

An interesting integral due to Mac Robert [2] is given below

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[1+\lambda x+\mu(1-x)]^{-(\alpha+\beta)}\,dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)(1+\lambda)^{\alpha}(1+\mu)^{\beta}} \qquad \dots (2.2)$$

$\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0$ and the constants $\lambda$ and $\mu$ are such that none of the expression $1+\lambda,\ 1+\mu$ and $[1+\lambda x+\mu(1-x)]$ , where $0\le x\le 1$ are zero.

## 3. THE MAIN INTEGRAL REPRESENTATIONS

$$X_9(a,b;c;x,y,z)=\frac{(\beta-\alpha)^{1-\lambda}(\beta-\gamma)^{a}(\alpha-\gamma)^{\lambda-a}\Gamma(\lambda)}{\Gamma(\lambda-a)\Gamma(a)}.$$

$$\times\left[\int_a^b (\xi-\alpha)^{a-1}(\beta-\xi)^{\lambda-a-1}(\xi-\gamma)^{-\lambda}\times X_9(\lambda,b;c;\sigma_1 x,\sigma_2 y,z)\,d\xi\right] \dots (3.1)$$

[ $\alpha<\gamma<\beta,\ \mathrm{Re}(\lambda)>0, \mathrm{Re}(a)>0$, Exton function $X_9$ and its region of convergence are defined in equations (1.2) and (1.5) respectively]

Where $\sigma_1=\left[\dfrac{(\xi-\alpha)(\beta-\gamma)(\xi-\gamma)}{(\beta-\alpha)\,(\xi-\gamma)}\right]^2$ and $\sigma_2=\dfrac{(\xi-\alpha)(\beta-\gamma)(\xi-\gamma)}{(\beta-\alpha)\,(\xi-\gamma)}$

$X_9(a,b;c;x,y,z)=$

$$\frac{\Gamma(\alpha)\,(1+\lambda)^{a}}{2\Gamma(a)\Gamma(\alpha-a)}\int_0^{\pi/2}\frac{(\sin^2\xi)^{a-\frac{1}{2}}(\cos^2\xi)^{\alpha-a-\frac{1}{2}}}{(1+\lambda\sin^2\xi)^{\alpha}}X_9(\alpha,b;c;\rho_1 x,\rho_2 y,z)\,d\xi \qquad \dots (3.2)$$

[ $R(\alpha)>\mathrm{Re}(a)>0$, Exton function $X_9$ and its region of convergence are defined in equations (1.2) and (1.5)

respectively.]

Where $\rho_1 = \left( \dfrac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi} \right)^2$ and $\rho_2 = \dfrac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi}$

$$X_9(a,b;c;x,y,z) = \frac{(1+\lambda)^d (1+\mu)^{c-d}\Gamma(c)}{\Gamma d \;\; \Gamma(c-d)}$$

$$\times\left[\int_0^1 t^{d-1}\,(1-t)^{c-d-1}[1+\lambda t+\mu(1-t)]^{-c}\, X_{10}(a,b;d,c-d;\sigma_1 x,\sigma_1 y,\sigma_2 z)\,dt\right] \quad \ldots (3.3)$$

[ $\mathrm{Re}(c) > \mathrm{Re}(d) > 0; \lambda > -1,\ \mu > -1$, Exton function $X_{10}$ and its region of convergence are defined in equations (1.3) and (1.6) respectively.]

$$\sigma_1 = \frac{(1+\lambda)\,t}{1+\lambda t+\mu(1-t)} \quad \text{and} \quad \sigma_2 = \frac{(1+\mu)(1-t)}{1+\lambda t+\mu(1-t)}$$

$$X_9(a,b;c;x,y,z) = \frac{2(1+\lambda)^c (1+\mu)^{d-c}\Gamma(d)}{\Gamma c\;\Gamma(d-c)}$$

$$\times\left[\int_0^{\pi/2} (\sin^2\xi)^{c-\frac{1}{2}} (\cos^2\xi)^{d-c-\frac{1}{2}} (1+\lambda\sin^2\xi+\mu\cos^2\xi)^{-d}\left(X_{11}(a,b;c,d-c;\rho_1 x,\rho_2 y,\rho_1 z)\,d\xi\right)\right] \quad \ldots (3.4)$$

[ $\mathrm{Re}(d) > \mathrm{Re}(c) > 0; \lambda > -1,\ \mu > -1$, Exton function $X_{11}$ and its region of convergence are defined in equations (1.4) and (1.7) respectively.]

$$\rho_1 = \frac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi+\mu\cos^2\xi} \quad \text{and} \quad \rho_2 = \frac{(1+\mu)\cos^2\xi}{1+\lambda\sin^2\xi+\mu\cos^2\xi}$$

## 4. PROOF OF RESULTS

It is noted that the integral representation from (3.1) to (3.4) can be proved by expressing the involved Exton's function in series form [1.2 to 1.4], then changing the order of integration and summation and finally applying the well known integral formulas and hypergeometric identities.

**Proof of (3.1)**

By applying (1.2) to the right hand side of (3.1) and then changing the order of integration and summation, we have

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda-a)\Gamma(a)}.\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\lambda)_{2m+n}(b)_{n+2p}x^m y^n z^p(\beta-\alpha)^{-(\lambda+2m+n-1)}(\beta-\gamma)^{a+2m+n}(\alpha-\gamma)^{\lambda-a}}{(c)_{m+n+p}m!\,n!\,p!}$$

$$\times\left[\int_a^b(\xi-\alpha)^{a+2m+n-1}(\beta-\xi)^{\lambda-a-1}(\xi-\gamma)^{-(\lambda+2m+n)}d\xi\right. \qquad \ldots(4.1)$$

on using (2.1) in (4.1), we obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}x^m y^n z^p}{(c)_{m+n+p}m!\,n!\,p!} \qquad \ldots(4.2)$$

Which on applying (1.2) in (4.2), we finally get the desired result. The other results (3.2) to (3.4) can also be derived by similar method.

## 5. CONCLUSION

We conclude it by remarking that many other interesting integral representations can be obtained for Exton's functions.

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# EFFECT OF MHD VISCO-ELASTIC FLUID (KUVSHINSKI) AND POROUS MEDIUM THROUGH A CIRCULAR CYLINDER BOUNDED BY A PERMEABLE BED

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## ABSTRACT

*The analysis of the unsteady laminar visco-elastic (Kuvshinshki) conducting fluid through a circular cylinder bounded by permeable bed under the influence of exponentially decreasing pressure gradient and porous medium is presented. A uniform constant magnetic field is applied in azimuthal direction (i.e. in the direction of θ). The expressions for velocity are obtained and discussed with the help of Tables and Graphs.*

**Keywords :** Visco-elastic (Kuvshinski) fluid, MHD flow, Porous medium, A circular cylinder,

## INTRODUCTION

MHD flow of visco-elastic fluid through pipes plays significant role in different areas of science and technology such as Petroleum industry, Biomechanics, Drainage and Irrigation engineering and so on. The steady flow of a viscous incompressible fluid through various cross-sections have been studied extensively by many authors. Yuan and Finkelstein[19] have obtained the solution of laminar pipe

flow with injection and suction through a porous wall. Drake[3] and Khamrui[5] have studied steady viscous incompressible flow through circular and elliptic tubes under the influence of a periodic pressure. Laminar flow of a steady viscous incompressible fluid through a circular pipe under the influence of aligned magnetic field was discussed by Hughes and Young.[4] Sinha and Chowdary[14] have studied viscous incompressible flow between two co-axial rotating cylinders. Yuan[18] discussed the steady laminar viscous incompressible flow through the cylinder.

Batchelor[2] studied the startling flow in a circular pipe which is initially at rest, and set in motion by pressure gradient suddenly imposed and maintained by external means. Patabhirama[7] made a study on the flow through a circular pipe completely filled with porous material. The laminar flow of a unsteady viscous liquid through a cirucular cylinder under the influence of exponentially decreasing pressure gradient was studied by Sambasiva Rao and Rama Murthy[11]. Badar[1] studied numerically the development with time of the two dimensional flow of viscous incompressible fluid around a circular cylinder which suddenly starts rotating about its axis with constant speed. Kumari and Bansal[6] used Oseen's approximation to study the slow motion of a viscous incompressible electrically conduction fluid past a circular cylinder in the presence of aligned magnetic field and obtained the numerical values for the tangential drag for different values of magnetic interaction parameter and viscous Reynolds number. Sharma et al.[13] obtained the numerical solution of steady motion of second order fluid past a circular cylinder with suction or injection. Soundalgekar[15] obtained the finite difference solution for the laminar developing flow in an annulus between two rotating cylinders. Steady two dimensional flow of viscous incompressible fluid at low Reynolds number pas t a circular cylinder was studied by Raghava Rao[8] and Swachuk and Zamir[12] presented quasi-laminar solutions for the boundary layer on a circular cylinder in axial flow using a Kellerbox numerical scheme for velocity components rather than a stream function. The solutions extend

earlier results considerably and cover a wide range of cylinder radii from very small to very large. Recently, Subramanyam et at.[17] have studied the unsteady laminar viscous conducting fluid through a circular cylinder boundary by permeable bed under the influence of exponentially decreasing pressure gradient.

In this section, we study the problem of Subramanyam[17] with visco-elastic fluid (Kuvshinshki) through porous medium. Purpose of this study is to obtain the velocity distribution to compare with already existing paper Subramanyam et al[17].

## NOMENCLATURE

$\mu$–coefficient of visosity,, $J$ - current density vector, $(r, \theta, z)$ - cylinderical polar co-ordinates, $Q$ - Darcy velocity, $\alpha$- dimensionless parameter, $p$ - dimensional pressure parameter, $J_1$ - first order Bessel function of first kind, $\rho$ - fluid density, $P$ - fluid pressure, $K$ - permeability of the bed, $\sigma_1 = \frac{a}{\sqrt{K}}$ permeability parameter, $M$ - magnetic parameter (Hartmann number), $H = (0, H_0, 0)$ - magnetic field vector, $\mu_e$- magnetic permeability, $I_1$ - modified Bessel function of first order, $I_0$- modified Bessel function of zeroth order, $\alpha_1$- slip parameter, $V_B$ - slip velocity, $t$ - time, $q = (V_r, V_\theta, V_z)$ - velocity vector, $J_0$ - zeroth order Bessel function of first kind, $\lambda_1$ - visco-elastic coefficients, $K_0$ - permeability of the porous medium, $\nu$ - kinematic viscosity, $K_1$- permeability parameter of the medium, $\beta_1$ - visco-elastic parameter.

## FORMULATION OF THE PROBLEM

Consider the laminar flow of unsteady visco - elastic (Kuvshinshki) electrically conducting fluid through a circular cylinder boundary by permeable bed under the influence of exponentially decreasing pressure gradient and porous medium. the axis of the cylinder is taken as $z$-axis. Cylindrical polar coordinates $(r, \theta, z)$ are used. The velocity of the fluid is taken in $z$ direction. A uniform magnetic field $H_0$

is applied in θ direction. We make the following assumptions :

1. Flow is laminar and fully developed.
2. The direction of the flow is in the z direction i.e. in the direction of the axis of the cylinder. So $V_r = V_\theta = 0$ and $V_z$ is a function of r and t only.
3. Electric field is neglected Rossow[9]. Induced magnetic field is also neglected Sparrow and Cess[16].

Under the above assumptions, the governing equations are :

$$-\frac{\partial p}{\partial r} = 0 \qquad \ldots (1)$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \qquad \ldots (2)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial V_z}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right)$$

$$-\left(\frac{\sigma}{\rho}\mu_e^2 H_o^2 + \frac{\nu}{K_o}\right)\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)V_z \qquad \ldots (3)$$

Boundary conditions are

$V_z$ is finite at $r = 0$, $V_z = V_B$, $\dfrac{dV_z}{dr} = \dfrac{\alpha_1 a}{\sqrt{K}}(V_B - Q)$ at $r = a$, $\qquad \ldots (4)$

where $Q = -\dfrac{K}{\mu}\left(\dfrac{\partial p}{\partial z} + \sigma \mu_e^2 H_o^2 Q\right)$

## NON-DIMENSIONAL PARAMETERS

We define the following non-dimensional quantities

$$V_z^* = \frac{V_z}{V_0}, \qquad r^* = \frac{r}{a}, \qquad z^* = \frac{z}{a}, \qquad p^* = \frac{pa}{\mu V_0}$$

$$t^* = \frac{t\nu}{a^2}, \qquad Q^* = \frac{Q}{V_0}, \qquad V_B{}^* = \frac{V_B}{V_0}$$

Using the above quantities and after removing the asterisks, the equation (1) and (4) become

$$-\frac{\partial p}{\partial r} = 0 \qquad \ldots (5)$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 \qquad \ldots (6)$$

$$\left(1 + \frac{\nu\lambda_1}{a^2}\frac{\partial}{\partial t}\right)\frac{\partial V_z}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right)$$

$$-\left(M^2 + \frac{1}{K_1}\right)\left(1 + \frac{\nu\lambda_1}{a^2}\frac{\partial}{\partial t}\right)V_z, \qquad \ldots (7)$$

where $M^2 = \dfrac{\sigma a^2 \mu_e^2 H_o^2}{\rho\nu}$ (Magnetic Parameter), $K_1 = \dfrac{K_o}{a^2}$ (Permeability Parameter)

Boundary conditions are

$V_z$ is finite at $r = 0$,

$$V_z = V_B, \qquad \frac{dV_z}{dr} = \alpha_1 a \qquad (V_B - Q) \text{ at } r = 1, \qquad \ldots (8)$$

where $Q = -\frac{K}{\mu}\left(\frac{\mu}{a^2}\frac{\partial p}{\partial z} + \sigma\mu_e^2 H_o^2 Q\right)$

## SOLUTION OF THE PROBLEM

From equation (5) and (6), it is observed that the pressure is constant in $r$ and $\theta$ directions. Taking the pressure to be an exponential functions of $t$ given by

$$-\frac{\partial p}{\partial z} = P e^{-\alpha t} \qquad \ldots (9)$$

Here $P$ is constant and equals pressure gradient in fully developed motion when $t = 0$.

Let us consider

$$V_z = f(r)\, e^{-\alpha t} \qquad \ldots (10)$$

With the help of equations (9) and (10), the equation (7) becomes

$$\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} + S^2 f(r) = -P, \qquad \ldots (11)$$

where

$$\beta_1 = 1 - \frac{\alpha \nu \lambda_1}{a^2}$$

$$S^2 = \beta_1 \left\{\alpha - \left(M^2 + \frac{1}{K_1}\right)\right\}$$

Now, Boundary conditions become

$V_z$ is finite at $r = 0$,

$$f(r) = V_B e^{-\alpha t}, \quad \frac{df(r)}{dr} = \alpha_1 a \qquad (V_B - Q)e^{\alpha t} \text{ at } r = 1 \qquad \ldots (12)$$

**CASE - I :** Let $\alpha - \left(M^2 + \frac{1}{K_1}\right) > 0$

On solving equation (11) using boundary condition (12), we get

$$f(r) = \frac{P}{S^2}\left[\frac{J_o(Sr)}{J_o(S)} - 1\right] + V_B\left[\frac{J_o(Sr)}{J_o(S)}\right]e^{\alpha t} \qquad \ldots (13)$$

The velocity distribution is

$$V_z = f(r)\, e^{-\alpha t} = \frac{P}{S^2}\left[\frac{J_o(Sr)}{J_o(S)} - 1\right] e^{-\alpha t} + V_B\left[\frac{J_o(Sr)}{J_o(S)}\right] \qquad \ldots (14)$$

**CASE - II :** Let $\alpha - \left(M^2 + \frac{1}{K_1}\right) = 0$

The equation (11) becomes

$$\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} + P = 0 \qquad \ldots (15)$$

On solving equation (15) using boundary condition (12), we get

$$f(r) = \frac{P}{4}\left(1 - r^2\right) + V_B\, e^{\alpha t} \qquad \ldots (16)$$

In this case, the velocity distribution is

$$V_z = f(r)\; e^{-\alpha t} = \frac{P}{4}\left(1 - r^2\right)\; e^{-\alpha t} + V_B \qquad \ldots (17)$$

**CASE - III :** Let $\alpha - \left(M^2 + \frac{1}{K_1}\right) < 0$

The equation (11) becomes

$$\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} - S_1^2 f(r) = -P, \qquad \ldots (18)$$

where

$$S_1^2 = \beta_1 \left\{ \left( M^2 + \frac{1}{K_1} \right) - \alpha \right\}$$

On solving equation (18) using boundary condition (12), we get

$$f(r) = \frac{P}{S_1^2}\left[\frac{I_o(S_1 r)}{I_o(S_1)} - 1\right] + V_B\left[\frac{I_o(S_1 r)}{I_o(S_1)}\right]e^{\alpha t} \qquad \ldots (19)$$

The velocity distribution is

$$V_z = f(r)e^{-\alpha t} = \frac{P}{S_1^2}\left[\frac{I_o(S_1 r)}{I_o(S_1)} - 1\right]e^{-\alpha t} + V_B\left[\frac{I_o(S_1 r)}{I_o(S_1)}\right] \qquad \ldots (20)$$

## PARTICULAR CASE

When $\beta_1$ (Visco-elastic parameter) is one and $K_1$ tends to infinite, this problem reduces the problem of Subramanyam et at [17].

## RESULT AND DISCUSSION

The Velocity Profile is tabulated in Table-I and plotted in Fig.-I having Graph-I to $V$ at $\alpha = 0.2$, $t = 0.5$, $V_B = 0.001$ and following different values of $M$, $P$, $\beta_1$ and $K_1$.

| | $M$ | $P$ | $\beta_1$ | $K_1$ |
|---|---|---|---|---|
| For Graph-*I* | 0.2 | 0.077 | 1 | 10 |
| For Graph-*II* | 0.3 | 0.077 | 1 | 10 |
| For Graph-*III* | 0.2 | 0.1 | 1 | 10 |
| For Graph-*IV* | 0.2 | 0.077 | 4 | 10 |
| For Graph-*V* | 0.2 | 0.077 | 1 | 100 |

From the Graph-I to $V$ of Fig.-I it is observed that the velocity decreases with the increase in $r$. It is also noticed that the velocity increases with the increase in $P$, $\beta_1$ and $K_1$, but it decreases with the increase in $M$.

## CONCLUSION

The velocity of fluid increases with the increase in $\beta_1$ (Viscoelastic Parameter).

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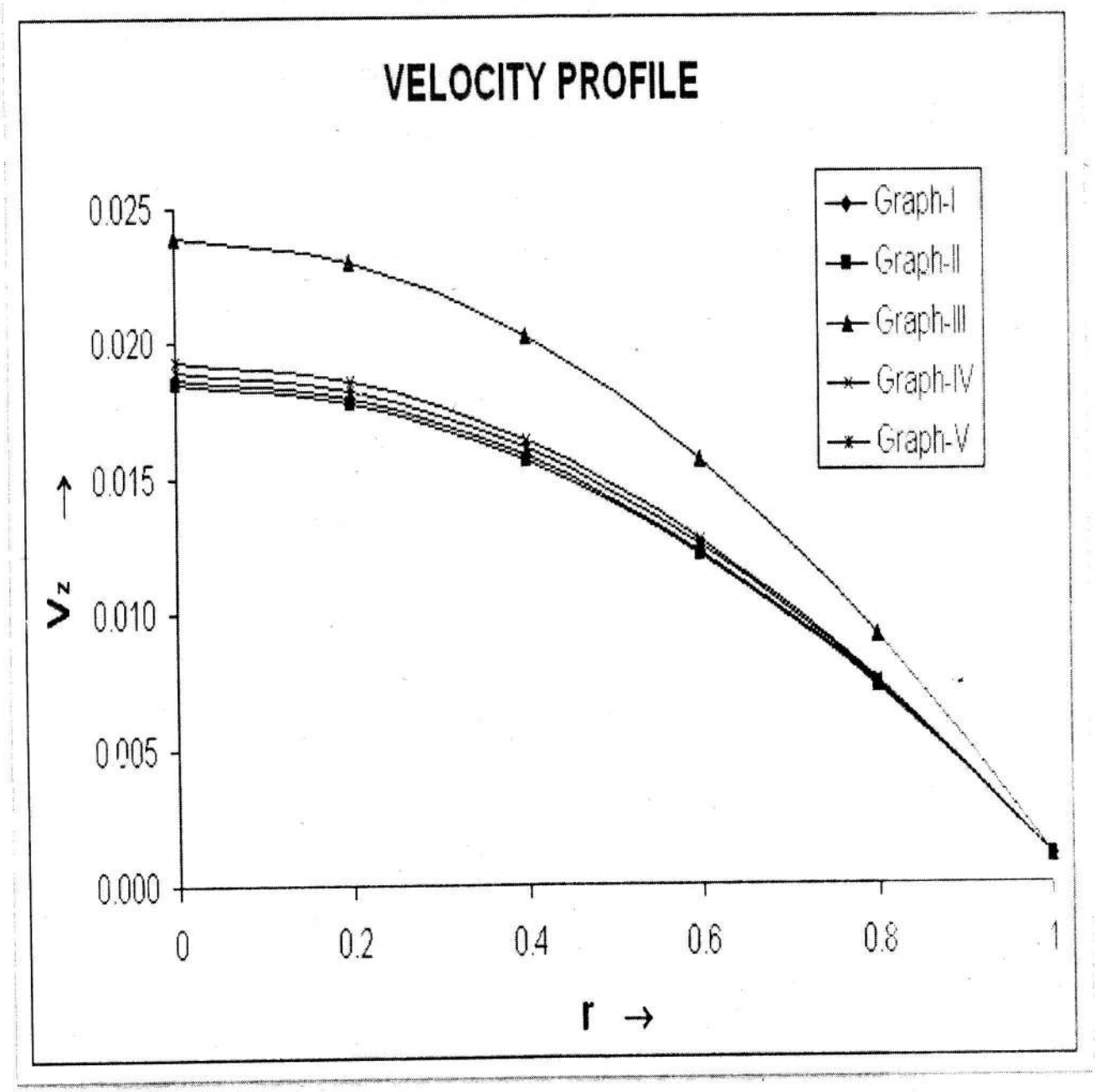
VELOCITY PROFILE
Graph-I
Graph-II
Graph-III
Graph-IV
Graph-V
0.025
0.020
0.015
0.010
0.005
0.000
Vz
0
0.2
0.4
0.6
0.8
1
r

**Fig.-I**

**Table-I :** Values of velocity at $\alpha = 0.2$, $t = 0.5$, $V_B = 0.001$ and different values of $M$, $P$, $\beta_1$ and $K_1$.

| *r* | Graph - *I* | Graph - *II* | Graph - *III* | Graph - *IV* | Graph – *V* |
|---|---|---|---|---|---|
| **0** | 0.018631 | 0.018453 | 0.023893 | 0.019299 | 0.018960 |
| **0.2** | 0.017924 | 0.017755 | 0.022974 | 0.018557 | 0.018235 |
| **0.4** | 0.015801 | 0.015659 | 0.020219 | 0.016334 | 0.016064 |
| **0.6** | 0.012269 | 0.012168 | 0.015632 | 0.012648 | 0.012455 |
| **0.8** | 0.007332 | 0.007281 | 0.009222 | 0.007524 | 0.007427 |
| **1** | 0.001000 | 0.001000 | 0.001000 | 0.001000 | 0.001000 |

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# KANTOWSKI-SACHS PERFECT FLUID COSMOLOGICAL MODELS WITH VARIABLE *G* AND Λ

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**ABSTRACT**

*We have investigated the Kantowski-Sachs cosmological models generated by perfect fluid with variable gravitational constant G and the cosmological constant Λ , with the condition that the rate of expansion (θ) is propotional to the shear scalar (σ), which leads to the condition on metric potentials as $A = m B^n$, where A and B are functions of time t, while m and n are constants. To obtain the deterministic model we also assume the relation $p = \omega\rho$, where p is the pressure, ρ is energy density and $0 \leq \omega \leq 1$. Various geometrical and physical aspects of the model are also discussed.*

Key words: Kantowski-Sachs space-time, perfect fluid, variable G and Λ, cosmology

## 1 INTRODUCTION

The Kantowski-Sachs space-times [16] are important as they include simple examples of spatially homogeneous and spherically symmetrical space-times that are anisotropic [31]. It is interesting to investigate Kantowski-Sachs model because it has a different topology from the one's based on the Bianchi metrics, with a 3-Geometry given by the topological product of a sphere and a straight line, $S^2 \times R$. Mathematically it can be described as a spatially homogeneous and anisotropic model that does not permit a simply transitive group of motions [15].

The Kantowski-Sachs model is supposed to be a reasonable representation of the universe in early épochs and it is simple enough so that many of its features can be studied exactly [19]. Hence it have been extensively studied by a number of scholars for example, Collins, Letelier, Gron, Lorenz, Sahani et al., Torrence et al., Weber and many more [10,14,18,28,29,31,32]. Collins[10] have investigated global structure of the Kantowski-Sachs cosmological models with a zero cosmological constant and shown that the perfect fluid Kantowski-Sachs models are geodetically incomplete both to the future and to the past, and that the energy density of the fluid becomes infinite at both of these singularities. Letelier [18] have studied Bianchi Type-I and Kantowski-Sachs type cosmological models generated by a cloud of strings with particles attached to them for the equation of state (energy density) $\propto$ (tension density). In 1984,Weber [32] analyzed the Kantowski-Sachs model, in the presence of a non zero cosmological constant $\Lambda$ and he found two new types of singularity points, one an infinite singularity and another an infinite barrel. Gron [14] investigated an empty Kantowski-Sachs model and concluded that the model emerges from a pancake singularity and develops towards anisotropic de-Sitter universe. Matravers [21] presented solutions to the Einstein's field equations with Kantowski-Sachs symmetry and string dust source. Singh et al. [30] investigated Bianchi Type-I, II, III, V, $VI_0$ and Kantowski-Sachs models in scalar tensor theory. For Kantowski-Sachs models they presented vacuum ($\rho = p = 0$) and Zeldovich fluid ($\rho = p$)solutions, where $\rho$ is energy density and p is pressure. Henriques [15] compared the standard model of cosmology with the Kantowski-Sachs metric by applying cosmological tests of cosmology. Obukhov et al.[22] have investigated the brane Kantowski-Sachs universes in presence of cosmological constant $\Lambda$, when bulk space is five-dimentional anti-de Sitter with the condition $\rho = \rho_0 V^{-\gamma}$, where $\rho_0 > 0$ is a constant and V is spatial volume. Adhav et al. [2] have studied Kantowski-Sachs models in general relativity when the perfect fluid is coupled with mass less scalar field .They concluded that their model is expanding, anisotropic and free from singularities which decelerates.

Although one can set the cosmological constant $\Lambda$ to zero by hand, this amounts to a highly artificial fine-tuning. However, this problem can easily be solved by assuming a dynamically decaying $\Lambda$ due to its coupling with the matter field of the universe. The idea is that $\Lambda$, which was very large in the early universe, relaxed to its present small value in the course of the expansion of the universe by creating massive or massless particles. Bali and Singh [6] have presented bulk viscous Bianchi type I cosmological models with time dependent cosmological term- $\Lambda$. Bulk viscous Bianchi type I barotropic fluid cosmological model with varying $\Lambda$ and function relation on Hubble parameter is investigated by Bali and Singh [7]. Several others have also advocated a variable $\Lambda$ to account for this fact [17, 26, 27].

Variation of Newton's gravitational parameter G was originally suggested by Dirac on the basis of his large number hypothesis [11]. As G couples geometry to matter, it is reasonable to consider G = G(t) in an

evolving universe when one considers $\Lambda = \Lambda(t)$ . A modification linking the variation of $G$ with that of $\Lambda$ has been considered within the framework of general relativity by several researchers [3,4,8,9].

Some Robertson-Walker models admitting a contracted Ricci collineation along the fluid flow vector having time-varying $G$ and $\Lambda$ are investigated by Abdussattar et al.[1]. Bali et al. [8] have investigated the Bianchi Type-V bulk viscous barotropic fluid cosmological models with variable G and $\Lambda$, assuming the conditions on metric potintials as $\frac{\dot{A}}{A} = \frac{\dot{B}}{B} = \frac{\dot{C}}{C} = \frac{m}{t^n}$, where $A, B, C$ are functions of t only, while m and n are constants. Baysal et al.[9] obtained several set of explicit solutions in the five - dimentional Kaluza-Klein type perfect fluid cosmological models with variable $G$ and $\Lambda$. They found that the cosmological parameter $\Lambda$ varies inversely with the square of time. Arbab [5] have studied viscous dark energy model with variable $G$ and $\Lambda$ for a flat Friedmann-Lematre-Robertson-Walker metric in which the cosmological parameter is proportional to some power of energy density.

In the present article we have investigated the Kantowski-Sachs cosmological models generated by perfect fluid with variable gravitational constant G and the cosmological constant $\Lambda$, with the condition that the rate of expansion ($\theta$) is proportional to the shear scalar ($\sigma$), i.e. $\theta \propto \sigma$ which leads to the condition on metric potentials as $A = mB^n$ , where A and B are functions of time t, while m and n are constants. To obtain the deterministic model we also assume the relation $p = \omega\rho$, where p is the pressure, $\rho$ is energy density and $0 \leq \omega \leq 1$. Various geometrical and physical aspects of the model are also discussed. The paper is organized as follows: The basic equations of the model are given in Section 2 and solutions of the field equations are presented in Section 3. A special case for $n = -3/2$ is discussed in Section 4. Conclusion and results are provided in Section 5.

## 2 THE METRIC AND FIELD EQUATIONS

We consider the Kantowski-Sachs space-time metric in the form

$$ds^2 = dt^2 - A^2 dr^2 - B^2 (d\theta^2 + \sin^2\theta\, d\phi^2), \qquad \ldots (1)$$

where the metric potentials A and B are functions of cosmic time t only.

The Einstein's field equations with time dependent G and $\Lambda$ are

$$R_i^j - \frac{1}{2} R\, g_i^j = -8\pi\, G(t) T_i^j + \Lambda(t) g_i^j \,. \qquad \ldots (2)$$

The stress energy tensor $T_i^j$ corresponding to perfect fluid is given by

$$T_i^j = (\rho + p) v_i v^j + p\, g_i^j, \qquad \ldots (3)$$

Satisfying the equation of state

$$p = \omega\rho,\ 0 \le \omega \le 1, \qquad \ldots (4)$$

where perfect fluid satisfying the relation

$$g_i^j v_i v^j = -1 \qquad \ldots (5)$$

where $v^i$ is the four velocity vector of the fluid. ρ and $p$ are energy density and pressure respectively.

The average scale factor R(t) and average volume V(t) are defined as

$$R^3(t) = V(t) = AB^2. \qquad \ldots (6)$$

The Hubble's parameter $H$ in the anisotropic cosmological model is given as

$$H = \frac{(\log V_4)}{3} = \frac{1}{3}\left[\frac{A_4}{A} + \frac{2B_4}{B}\right]. \qquad \ldots (7)$$

The scalar of expansion θ and the shear scalar σ for the line element (1) are given by

$$\theta = \left[\frac{A_4}{A} + \frac{2B_4}{B}\right], \qquad \ldots (8)$$

$$\sigma^2 = \frac{2}{3}\left[\frac{A_4}{A} - \frac{B_4}{B}\right]^2. \qquad \ldots (9)$$

The field equations (2) for the line element (1) have been set up as

$$\frac{2B_{44}}{B} + \frac{B_4^2}{B^2} + \frac{1}{B^2} = 8\pi G p - \Lambda, \qquad \ldots (10)$$

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}=8\pi G\,p-\Lambda, \qquad \ldots (11)$$

$$\frac{2A_4B_4}{AB}+\frac{B_4^2}{B^2}+\frac{1}{B^2}=-8\pi G\,p-\Lambda, \qquad \ldots (12)$$

The suffix '4' stands for ordinary time-derivative of the concerned quantity.

An additional equation for time changes of $G$ and $\Lambda$ is obtained by the divergence of Einstein tensor ,i.e., $(R_i^j-\frac{1}{2}Rg_i^j)_{;j}=0$, which leads to $(8\pi GT_i^j-\Lambda g_i^j)_{;j}=0$, then

$$8\pi\rho G_4+\Lambda_4+8\pi G\left[\rho_4+(\rho+p)\left(\frac{A_4}{A}+\frac{B_4}{B}\right)\right]=0. \qquad \ldots (13)$$

The equation (13) demonstrates that $\Lambda$ is a constant in the absence of matter ($T_{ij}=0$) implying that matter is essential for a time varying $\Lambda$. In the field equations (2), $\Lambda$ accounts for vacuum energy with its energy density $\rho_v$ and isotropic pressure $p_v$.

$$p_v=-\rho_v=-\frac{\Lambda}{8\pi G}. \qquad \ldots (14)$$

The usual conservation law of energy momentum tensor together $(T_i^j)_{;j}=0$ with equations (4) and (6) gives

$$\rho_4+3(\omega+1)\rho\left(\frac{R_4}{R}\right)=0, \qquad \ldots (15)$$

which indicates that $G$ and $\Lambda$ are coupled fields.

Using equation (15) in (13) we get

$$8\pi\rho G_4+\Lambda_4=0, \qquad \ldots (16)$$

implying that $\Lambda$ is a constant whenever G is constant. Equation (15) on integration gives

$$\rho = \frac{k}{R^{3(\omega+1)}}, \qquad \ldots (17)$$

where $k > 0$ is a constant of integration.

## 3 SOLUTIONS OF THE FIELD EQUATIONS

The system of Equations (4), (10) to (12) and (16) supply only five equations in six unknowns $A$, $B$, $p$, $\rho$, $G$ and $\Lambda$. To obtain a determinate solution we require one more condition. For finding exact solutions of Einstein equations we rely upon some physically reasonable restrictions. Without any loss of generality we can assume that the rate of expansion ($\theta$) is proportional to the shear scalar ($\sigma$), i.e. $\theta \propto \sigma$, which leads to the following relation between metric potentials

$$A = mB^n \qquad \ldots (18)$$

where $m$ and $n$ are constants.

Equations (10) and (11) together with Equation (18) lead to

$$\frac{B_{44}}{B} + (1+n)\frac{B_4^2}{B^2} + \frac{1}{(1-n)B^2} = 0 \qquad \ldots (19)$$

which leads to

$$\frac{d(f^2)}{dB} + \frac{2(1+n)}{B^2}(f^2) = \frac{2}{(1-n)B}, \qquad \ldots (20)$$

where $B_4 = f(B)$.

From equation (20), we have

$$\left(\frac{dB}{dt}\right)^2 = \left[\frac{1}{(n^2-1)} + \frac{a}{B^{2(n+1)}}\right], \quad n \neq \pm 1, \qquad \ldots (21)$$

where $a$ is the constant of integration.

After using the transformation $B = T$ the metric (1) reduces to the form

$$ds^2 = \frac{dT^2}{\left[\frac{1}{(n^2-1)} + \frac{a}{T^{2(n+1)}}\right]} - m^2 T^{2n} dr^2 - T^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \qquad \ldots (22)$$

The cosmic time $t$ is defined by

$$t = \int \frac{dT}{\sqrt{\frac{1}{(n^2-1)} + \frac{a}{T^{2(n+1)}}}}. \qquad \ldots (23)$$

Volume expansion $\theta$ and shear $\sigma$ for the model (22) are given by

$$\theta = 3H = \frac{(n+2)}{T}\sqrt{\frac{1}{(n^2-1)} + \frac{a}{T^{2(n+1)}}}, \qquad \ldots (24)$$

$$\sigma^2 = \frac{2(n-1)^2}{3T^2}\left[\frac{1}{(n^2-1)} + \frac{a}{T^{2(n+1)}}\right] \qquad \ldots (25)$$

Thus we see that $\left(\frac{\sigma}{\theta}\right) = \frac{(n-1)}{(n+2)}\sqrt{\frac{2}{3}}$,

Therefore, the model does not approach isotropy.

We observe that the values of $\rho$, $G$, and $\Lambda$ are

$$\rho = \frac{k}{m^{(\omega+1)}} \frac{1}{T^{(n+2)(1+\omega)}}, \qquad \ldots (26)$$

$$G = \frac{m^{(1+\omega)}}{4\pi k (1+\omega)}\left[\frac{nT^{[n(1+\omega)+2\omega]}}{(1-n^2)} - \frac{a(2n+1)}{T^{(n+2)(1-\omega)}}\right], \qquad \ldots (27)$$

$$\Lambda = \left[ \frac{a\,(2n+1)(1-\omega) - T^{2n+2}}{(1+\omega)\,T^{2n+4}} + \frac{n\,[n\,(1+\omega) + 2\omega]}{(1+\omega)\,(n^2-1)\,T^2} \right]. \qquad \ldots (28)$$

The deceleration parameter ($q$) is given by

$$q = -\frac{RR_{44}}{R_4^{\,2}} = \frac{(n-1)}{(n+2)} \left[ \frac{2a(n+1)\,(n+2) - T^{2n+2}}{a(n^2-1) + T^{2n+2}} \right], \qquad \ldots (29)$$

In the model we observe that the spatial volume $V$ is zero at $T = 0$ and scalar of expansion θ is infinite at $T = 0$, which shows that the universe starts evolving with zero volume and infinite rate of expansion at $T = 0$.The deceleration parameter $q$ for the model is 2 at $T = 0$ and as $T$ increases , $q$ becomes zero at $T = [2a(n+1)(n+2)]^{\frac{1}{(2n+2)}}$, and it becomes -1 at $T = \left[ \frac{3a(n+2)(1-n^2)}{2} \right]^{\frac{1}{(2n+2)}}$. Hence after $T = \left[ \frac{3a(n+2)(1-n^2)}{2} \right]^{\frac{1}{(2n+2)}}$ ,where $0 < n < 1$, the model represents an accelerating universe. Also at late times the cosmological term Λ tends to a genuine cosmological constant . In order to satisfy reality conditions, we require $k > 0$. As T increases, spatial volume $V$ increases, but the rate of expansion slows down. Spatial volume becomes infinitely large at $T \to \infty$ . The model represents a shearing and expanding universe with a big bang start.

If we analyze our solutions for ω, three cases arise.

### 3.1 Case (i)

For $\omega = 0$ (dust phase)

$$p = 0, \qquad \ldots (30)$$

$$\rho = \frac{k}{mT^{(n+2)}}, \qquad \ldots (31)$$

$$G = \frac{m}{4\pi k}\left[\frac{nT^n}{(1-n^2)} + \frac{a\,(2n+1)}{T^{(n+2)}}\right], \qquad \ldots(32)$$

$$\Lambda = \frac{a(2n+1)}{T^{(2n+4)}} - \frac{n^2}{(n^2-1)T^2}. \qquad \ldots(33)$$

The energy density varies inversely with power of time when $n > -2$ and proportionally with power of time when $n < -2$. It is clearly seen that when $T \to 0$, then $\rho \to 0$ if $n < -2$, otherwise $\rho \to \infty$. For $n = -2$, $\rho$ is constant and for $n = -(1/2)$, $\Lambda = \frac{1}{3T^2}, G = \frac{\delta}{T}$, where $\delta = \frac{-m}{6\pi k}$. It shows that $\Lambda$ varies inversely with square of time and $G$ varies inversely with time. In this case $\Lambda \to \infty$ when $T \to 0$, also $\Lambda \to 0$ when $T \to \infty$. The sign of $G$ is determined by $\delta$.

### 3.2 Case (ii)

For $\omega = -1$ (dark energy phase)

$$p = -k, \qquad \ldots(34)$$

$$\rho = k, \qquad \ldots(35)$$

In this case $G \to \infty$, $\Lambda \to \infty$. Hence we obtain the static universe $(p + \rho = 0)$ with dark energy in which G and $\Lambda$ both diverge.

### 3.3 Case (iii)

For $\omega = 1$ ( stiff matter or Zeldovich Fluid i.e. $p = \rho$)

$$p = \rho = \frac{k}{m^2 T^{2(n+2)}}, \qquad \ldots(36)$$

$$G = \frac{m^2}{8\pi k}\left[a\,(2n+1) + \frac{nT^{2n+2}}{(1-n^2)}\right], \qquad \ldots(37)$$

$$\Lambda = \frac{n}{(1-n)T^2}. \qquad \ldots(38)$$

It is interesting to note that if $T \to 0$ then $\Lambda \to \infty$, and when $T \to \infty$ then $\Lambda \to 0$. Here if we consider $0 < n < 1$, then $\Lambda$ remains positive in the range. The behaviour of the gravitational constant $G$ is also affected by the choice of $n$.

(i) if $T \to 0$ then $G \to \infty$, and if $T \to \infty$ then $G \to$ constant, for $n < (-1)$.

(ii) if $T \to 0$ then $G \to$ constant, and if $T \to \infty$ then $G \to \infty$, for $n > (-1)$.

If we take $n = -1/2$, $G = \delta_1 T$, where $\delta_1 = \frac{-m^2}{12\pi k}$ and $\Lambda = -\frac{1}{3T^2}$. Clearly G varies proportionally with time.

### 3.4 Case (iv)

For $\omega = 1/3$ (radiation dominated phase)

$$p = \frac{\rho}{3}, \qquad \ldots (39)$$

$$\rho = \frac{k}{m^{\frac{4}{3}} T^{\frac{4(n+2)}{3}}}, \qquad \ldots (40)$$

$$G = \frac{3m^{\frac{4}{3}}}{16\pi k}\left[\frac{nT^{\frac{2(2n+1)}{3}}}{(1-n^2)} + \frac{a(2n+1)}{T^{\frac{2(n+2)}{3}}}\right], \qquad \ldots (41)$$

$$\Lambda = \frac{a\,(2n+1)}{2\,T^{(2n+4)}} - \frac{n\,(2n+1)}{2(n^2-1)\,T^2} \qquad \ldots (42)$$

In this case if $T \to 0$ then $\rho \to 0$ for $n < (-2)$, otherwise $\rho \to \infty$. If $n = -1/2$, $G = \frac{-m^{4/3}}{8\pi k}, \rho = \frac{k}{m^{4/3}T^2}, \Lambda = 0$. it shows that in the absence of $\Lambda$, the gravitational constant $G$ becomes constant.

To visualize the exact behaviour of our universe we investigate the model taking such a value of n so that the non-linear differential equation (21) can exactly be solved.

## 4 MODEL I: SOLUTION FOR $n$ = - 3/2

When $n$ = -3/2, the equation (21) reduces to

$$\frac{dB}{\sqrt{4+5aB}}=\frac{dt}{\sqrt{5}}, \qquad \ldots (43)$$

Which on integration leads to

$$B=\left[a(t+\gamma)^2-\frac{4}{5a}\right], \qquad \ldots (44)$$

where $\gamma$ is the constant of integration. Hence using equation (18), we get

$$A=m\left[a(t+\gamma)^2-\frac{4}{5a}\right]^{-3/2} \qquad \ldots (45)$$

Thus the metric (1) after using equations (44) and (45) leads to

$$ds^2=\frac{5}{4(4+5aT)}dT^2-\frac{m^2}{T^3}dR^2-T^2(d\Theta^2+\sin^2\Theta d\Phi^2), \qquad \ldots (46)$$

where

$$\left[a(t+\gamma)^2-\frac{4}{5a}\right]=T, r=R, \theta=\Theta \; and \; \phi=\Phi \qquad \ldots (47)$$

The cosmic time '$t$' is given by

$$t=\frac{2\sqrt{(4+5aT)}}{a\sqrt{5}}+\mu, \qquad \ldots (48)$$

where $\mu$ is the constant of integration.

where $\mu=-4/a\sqrt{5}, then T\to 0 \; implies \; t=0.$

When $T \to -4/5a, then\ t \to \mu(cons\tan t)$.

Hence the model has a finite time span in cosmic time scale.

Volume expansion $\theta$ and shear $\sigma$ for the model (46) are given by

$$\theta = 3H = \frac{1}{\sqrt{5T}}\sqrt{5a+\frac{4}{T}}, \qquad \ldots (49)$$

$$\sigma^2 = \frac{10}{3T}\left[5a + \frac{4}{T}\right], \qquad \ldots (50)$$

Thus we see that $\sigma/\theta = 5\sqrt{2/3}$. Therefore, the model does not approach isotropy.

We observe that the values of $\rho, G\ and\ \Lambda$ are

$$\rho = \frac{k}{m^{\omega+1}}\frac{1}{T^{(\omega+1)/2}}, \qquad \ldots (51)$$

$$G = \frac{m^{\omega+1}}{\pi k\,(\omega+1)\,T^{(1-\omega)/2}}\left[2a + \frac{1}{T^2}\right], \qquad \ldots (52)$$

$$\Lambda = \frac{1}{(\omega+1)T}\left[8a(\omega-1) + \frac{(19\omega-1)}{5T}\right]. \qquad \ldots (53)$$

The deceleration parameter ($q$) is given by

$$q = -\left[\frac{650a-36}{39(5aT+4)}\right]. \qquad \ldots (54)$$

If we analyze our solutions for $\omega$ , again three cases arise.

## 4.1 Case (i)

For $\omega = 1$ (dust phase)

$$p = 0, \qquad \ldots (55)$$

$$\rho = \frac{k}{m\sqrt{T}}, \qquad \ldots (56)$$

$$G = \frac{m}{\pi\, k\, \sqrt{T}} \left[2a + \frac{1}{T}\right], \qquad \ldots (57)$$

$$\Lambda = -\frac{1}{T}\left[8a + \frac{1}{5T}\right], \qquad \ldots (58)$$

Cleary $\rho$ varies inversely with the square root of time. Therefore $\rho \to \infty$ when $T \to 0$ and $\rho \to 0$ when $T \to \infty$. G and $\Lambda$ becomes infinite for $T \to 0$ ; G and $\Lambda$ approaches for $T \to \infty$.

### 4.2 Case (ii)

For $\omega = -1$ (dark energy phase)

$$p = -k, \qquad \ldots (59)$$

$$\rho = k, \qquad \ldots (60)$$

In this case $G \to \infty, \Lambda \to \infty$. Hence we obtain the static universe with dark energy in which $G$ and $\Lambda$ both diverge.

### 4.3 Case (iii)

For $\omega = 1$ (stiff matter or Zeldovich fluid i.e. $p = \rho$)

$$p = \rho = \frac{k}{m^2\, T}, \qquad \ldots (61)$$

$$G = \frac{m^2}{2\,\pi\, k}\left[2a + \frac{1}{T}\right], \qquad \ldots (62)$$

$$\Lambda = \frac{9}{5T^2}. \tag{63}$$

In this case $\Lambda$ always remains positive and varies inversely with the square of time. Energy density and $G$ vary inversely with time. When $T = 0, G \to \infty$ and when $T \to \infty$, $G \to$ constant.

### 4.4 Case (iv)

For $\omega = 1/3$ (radiation dominated phase)

$$p = \frac{\rho}{3}, \qquad \ldots (64)$$

$$\rho = \frac{k}{m^{4/3}\, T^{2/3}}, \qquad \ldots (65)$$

$$G = \frac{3m^{4/3}}{4\pi\, kT^{1/3}}\left[2a + \frac{1}{T}\right], \qquad \ldots (66)$$

$$\Lambda = \left[\frac{4}{5T^2} - \frac{1}{2T}\right]. \qquad \ldots (67)$$

Here the energy density $\rho$ is inversely proportional to some power of time, therefore $\rho \to \infty$ for $T \to 0$ and $\rho \to 0$ for $T \to \infty$. It is worth noting that when $T = 0, G \to \infty$ and when $T \to \infty$, $G \to$ zero. In this case $\Lambda$ remains positive for T < 8/5.

## 5 CONCLUSION

In this paper we have presented some exact solutions to Einstein's field equations with variable $G$ and $\Lambda$ for Kantowski-Sachs space-time in such a way which conserves the energy momentum tensor of matter.

The behaviour of the universe in our models will be determined by the cosmological term $\Lambda$ and the gravitational constant $G$ which depends on time. It is worth mentioning that both the parameters $G$ and $\Lambda$ are coupled.

The matter density $\rho \to \infty$ at $T = 0$ for $n > 2$. The model (22) starts with a big-bang at $T = 0$ where $(a / n+1) > 0$, $n \neq \pm 1$, and the expansion in the model decreases as time increases for $n > 2$. $\rho$ and $\theta$ vanish for large values of T. The expansion in the model stops at $T = \infty$. The models (22) represents a realistic model. since $\lim_{T \to \infty} \left( \frac{\sigma}{\theta} \right) \neq 0$, the model does not approach isotropy for large values of $T$ in general. There is a point type singularity in the model at $T = 0$ when n > 0. If $n < 0$ the model has cigar type singularity at $T = 0$ [MacCallum, 1971]. $q < 0$ represents an accelerating universe and $q > 0$ represents a decelerating universe.

The model (22) starts with a big-bang at $T = 0$ where $a > 0$, and the expansion in the model decreases as time increases. Here we have $a > 0$, therefore the model (22) has a cigar type singularity at $T = \frac{1}{a}\sqrt{\frac{4}{5}}$ [ MacCallum, 1971]. For dust phase, when $n = -1 / 2$, we have positive cosmological constant together with an increasing $G$ for negative m, and a decreasing $G$ for positive m. For the dark energy phase, our model reduces to a static universe in which $G$ and $\Lambda$ both diverge.

The recent cosmological observations [10, 11, 21, 22, 23, 27] suggest the existence of a positive cosmological constant $\Lambda$ with the magnitude $\Lambda(Gh/c^3) \approx 10^{-123}$. In our model we have found that whenever $0 < n < 1$ the cosmological term $\Lambda$ being very large at initial times relaxes to a genuine cosmological constant at the late times, which is in accordance with the above mentioned cosmological observations. It is worth mentioning that the model possesses a gravitational constant $G$ which is initially zero and tends to a large value when time increases $0 < n < 1$. It is also remarkable that during the radiation dominant phase, when $n = -1 / 2$, we have a universe with negative gravitational constant which does not depend on time and the cosmological constant $\Lambda$ becomes zero.

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# CERTAIN MAPPING PROPERTIES OF A NEW CLASS OF RATIONAL FUNCTIONS

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## ABSTRACT

*In the present investigation our main objective is to find mapping properties of a new class of rational functions denoted by* $F(A,B;z)$ *by putting various conditions on* $A$ *and* $B$ *and its connection with the classes* $\mathcal{P}(2\lambda)$ *and* $\mathcal{U}(\lambda)$.

## 1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , \qquad \ldots (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbf{C} \text{ and } |z| < 1\}$ and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that are univalent in $\mathbb{U}$. A function $f(z)$ in $\mathcal{A}$ is said to be in class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \text{ for } z \in \mathbb{U}.$$

In particular, $\mathcal{S}^*(0) = \in \mathcal{S}^*$, consisting functions of starlike of order zero. Let $\mathcal{K}$ denote the class of all functions $f \in \mathcal{A}$ that are convex. It is well known that, $f$ is convex if and only if $zf'(z)$ is starlike. For more details about these classes see [1].

A function $f \in \mathcal{A}$ is said to be in $\mathcal{U}(\lambda)$ if

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 \right| \leqq \lambda \ (z \in \mathbb{U}) \qquad \ldots (1.2)$$

for some $\lambda \geqq 0$. We set $\mathcal{U}(1) = \mathcal{U}$. In [2], the authors studied a subclass $\mathcal{P}(2\lambda)$ of $\mathcal{U}(\lambda)$, consisting of functions $f$ for which

$$\left| \left(\frac{z}{f(z)}\right)'' \right| \leqq 2\lambda \quad (z \in \mathbb{U}) \qquad \ldots (1.3)$$

Moreover, a close connection between the classes $\mathcal{P}(2\lambda)$ and $\mathcal{U}(\lambda)$ is given $\mathcal{P}(2\lambda) \subset \mathcal{U}(\lambda)$, (see [2, 3]).

## 2. PRELIMINARY LEMMAS

In the present paper, we shall investigate the properties of a two-parameter family of functions and its connection with the classes $\mathcal{P}(2\lambda)$ and $\mathcal{U}(\lambda)$.

**Lemma 1.** *Let*

$$F(A,B;z) = \frac{z}{1 + Az + Bz^2} \ (z \in \mathbb{U},\ A,\ B \in \boldsymbol{R}). \qquad \ldots (2.1)$$

*If* $|B| \leq 1$, then $F(A,B;z)$ *is univalent in* $\mathbb{U} \setminus \{z : 1 + Az + Bz^2 = 0\}$.

**Proof.** We have

$$F(A,B;z_1) = F(A,B;z_2) \Leftrightarrow (z_1 - z_2)(1 - Bz_1 z_2) = 0. \qquad \ldots (2.2)$$

If $|B| \leq 1$, then it is easy to see that $1 - Bz_1 z_2 \neq 0$ for any $z_1, z_2 \in \mathbb{U}$. Therefore by (2.2) the function $F(A,B;z)$ is univalent in $\mathbb{U}$.

**Lemma 2.** *Let* $F(A,B;z)$ *defined by* (2.1)

(1) *If* $B=0$ *and* $|A|\leqq 1$, *then* $F(A,B;z)\in\mathcal{S}$

(2) *If* $B\in(0,1]$ *and* $A^2=4B$, *then* $F(A,B;z)\in\mathcal{S}$

(3) *If* $B\in(0,1]$ *and* $A^2<4B$, *then* $F(A,B;z)\in\mathcal{S}$

(4) *If* $B\in[-1,1]$ *and* $A=0$, *then* $F(A,B;z)\in\mathcal{S}$.

**Proof.** It is easy to see that under all mentioned conditions

$$F(A,B;0)=0 \text{ and } F'(A,B;0)=1.$$

Thus $F(A,B;z)$ satisfies normalization conditions. Using Lemma 1, we can also say that $F(A,B;z)$ is univalent in $\mathbb{U}$ if $|B|\leqq 1$ and $1+Az+Bz^2\neq 0$. Now we shall show that $F(A,B;z)$ is analytic under the above mentioned conditions.

Poles of (2.1) are given by

$$Bz^2+Az+1=0. \qquad \ldots(2.3)$$

If $B=0$ and $|A|\leqq 1$ then $1+Az\neq 0$ in $\mathbb{U}$. Thus $F(A,B;z)\in\mathcal{S}$. Now consider the case when $B\neq 0$. Let $z_1$ and $z_2$ are roots of (2.3), then

$$z_1z_2=\frac{1}{B}\ ;\ z_1+z_2=\frac{-A}{B} \quad \text{and} \quad z=\frac{-A\pm\sqrt{A^2-4B}}{2B}. \qquad \ldots(2.4)$$

If $|B|\leqq 1$ and $B\neq 0$, then from (2.4)

$$|z_1z_2|\geqq 1. \qquad \ldots(2.5)$$

(2.5) implies that at most one root of (2.3) lies inside $|z|=1$ and other lies outside it or both roots lies on $|z|=1$ or both roots lies outside $|z|=1$.

If $A^2=4B$, then from (2.4) both roots are same and hence either both roots lies on $|z|=1$ or both roots lies outside $|z|=1$. Therefore $F(A,B;z)\in\mathcal{S}$.

If $A^2<4B$, and $B\in(0,1]$ then

$$|z_1| = \left|\frac{-A + i\sqrt{4B - A^2}}{2B}\right| = \frac{1}{\left|\sqrt{B}\right|} \quad \text{and} \quad |z_2| = \left|\frac{-A - i\sqrt{4B - A^2}}{2B}\right| = \frac{1}{\left|\sqrt{B}\right|}, \qquad \ldots (2.6)$$

Thus both roots are at same distance from origin and hence either both roots lies on $|z|=1$ or both roots lies outside $|z|=1$. This implies that $F(A,B;z) \in \mathcal{S}$.

If $A = 0$, and $|B| \leqq 1$ then $1 + Bz^2 \neq 0$ in $\mathbb{U}$. Thus $F(A,B;z) \in \mathcal{S}$.

The following are some well know example of $F(A,B;z) \in \mathcal{S}$.

(1) If $A = \pm 1$, $B = 1$, so that $A^2 < 4B$ and hence from Lemma 2, $f(z) = \dfrac{z}{1 \pm z + z^2} \in \mathcal{S}$.

(2) If $A = \pm 2$, $B = 1$, so that $A^2 = 4B$ and hence from Lemma 2, $f(z) = \dfrac{z}{(1 \pm z)^2} \in \mathcal{S}$.

(3) If $A = 0$, $B = \pm 1$ then from Lemma 2, $f(z) = \dfrac{z}{1 \pm z^2} \in \mathcal{S}$.

## 3. MAPPING PROPERTIES OF $F(A,B;z)$

In this section we consider the mapping properties of $F(A,B;z)$ under various conditions on $A$ and $B$. For this we consider $F(A,B;z)$ on boundary of unit circle. Putting $z = e^{i\theta}, \theta \in [0, 2\pi)$ in (2.1), we have

$$F(A,B;e^{i\theta}) = u(\theta) + i\,v(\theta) = \frac{e^{i\theta} + A + Be^{-i\theta}}{|1 + Ae^{i\theta} + Be^{2i\theta}|^2}.$$

After simplification we get

$$u(\theta) = \frac{A + (1+B)cos\theta}{1 + A^2 + B^2 + 2Acos\theta + 2Bcos2\theta + 2ABcos\theta} \qquad \ldots (3.1)$$

and

$$v(\theta) = \frac{(1-B)sin\theta}{1 + A^2 + B^2 + 2Acos\theta + 2Bcos2\theta + 2ABcos\theta}. \qquad \ldots (3.2)$$

**Theorem 1.** *Let* $A = 0$ and $B| \leqq 1$, then

$$-\frac{1}{1-A} < \Re\,[F(A,0;z)] < \frac{1}{1+A} \quad (\text{for } |A| < 1), \qquad \ldots (3.3)$$

$$\Re\,[F(A,0;z)] < 1/2 \;(\; for \quad A=1) \qquad \ldots (3.4)$$

and

$$\Re\,[F(A,0;z)] > -1/2 \;(\; for \quad A=-1). \qquad \ldots (3.5)$$

So, $F(A,B;z) \in \mathcal{S}^*$.

**Proof.** For $B=0$ and $|A|<1$, $F(A,0;z)$ transforms interior of unit disk onto interior of the following circle in $w$-plane.

$$\left(u+\frac{A}{1-A^2}\right)^2 + v^2 = \frac{1}{(1-A^2)^2}. \qquad (3.6)$$

Which in turns proves (3.3). Also image domain is starlike with respect to origin. Hence $F(A,0;z) \in \mathcal{S}^*$. Using (3.1) it is easy to verify that for $A=1$, $F(A,0;z)$ maps $|z|<1$ onto $\Re\,[F(A,0;z)] < 1/2$ and hence starlike with respect to origin.

Similarly for $A=-1$, $F(A,0;z)$ maps $|z|<1$ onto $\Re\,[F(A,0;z)] > -1/2$ and hence starlike with respect to origin.

**Theorem 2.** *Let $A=0$ and $|B| \leqq 1$, then*

$$-\frac{1}{1+B} < \Re\,[F(0,B;z)] < \frac{1}{1+B} \quad (\text{for } |B|<1), \qquad \ldots (3.7)$$

*for B = −1, F (0, −1; z) maps the unit disk univalently onto entire complex plane minus two slit on imaginary axis one along [1/2, ∞) and other is (−∞, −1/2] and For B = 1 F (0, 1; z) maps the unit disk univalently onto entire complex plane minus two slit one along [1/2, ∞) and other is (−∞, −1/2]. Also, $F(0,B;z) \in \mathcal{S}^*$.*

**Proof.** For $A=0$ and $|B| \leqq 1$, from (2.1) and (2.2) we have

$$u(\theta) = \frac{(1+B)cos\theta}{1+B^2+2Bcos2\theta} \;; \qquad v(\theta) = \frac{(1+B)sin\theta}{1+B^2+2Bcos2\theta}. \qquad \ldots (3.8)$$

Using (3.8) we can conclude that resulting curve is symmetric with respect to both $u$ and $v$ axis and meet $u$-axis at $-1/1+B$ and $1/1+B$ and hence we get (3.7). Also, curve meets $v$-axis at $-1/1-B$ and $1/1-B$.

For the case $B=1$, it is clear from (3.8) that $v=0$ and $u=\sec(\theta/2), \theta \in [0,2\pi)\setminus\{\pi/2, 3\pi/2\}$ $\Rightarrow u \in \boldsymbol{R}-(-1/2,1/2)$. Thus $F(0,1;z)$ maps the unit disk univalently onto entire complex plane minus two slit

one along $[1/2,\infty)$ and other is $(-\infty,-1/2]$ on real axis.

For the case $B=-1$, it is clear from (3.8) that $u=0$ and $v=cosec\theta/2, \theta\in(0,2\pi)\setminus\{\pi\}$ $\Rightarrow v\in \boldsymbol{R}-(-1/2,1/2)$ Thus $F(0,-1;z)$ maps the unit disk univalently onto entire complex plane minus two slit one along $[1/2,\infty)$ and other is $(-\infty,-1/2]$ on imaginary axis. In all case image domain is starlike with respect to origin.

**Theorem 3.** *Let $A^2=4B$ and $|B|\leqq 1$, then for $A=-2$, $F(-2,1;z)$ maps the unit disk univalently onto entire complex plane minus one slit along $(-\infty,-1/4]$ and for $A=2, F(2,1;z)$ maps the unit disk univalently onto entire complex plane minus one slit one along $[1/4,\infty)$. Also, $F(0,B;z)\in\mathcal{S}^*$.*

**Proof.** Putting $z=e^{i\theta}$ in (2.5) and separating real and imaginary parts we get

$$u(\theta)=\frac{A+\left(1+A^2/4\right)cos\theta}{(1+Acos\theta+A^2/4)^2} \quad \text{and} \quad v(\theta)=\frac{\left(1-A^2/4\right)sin\theta}{(1+Acos\theta+A^2/4)^2} \qquad \ldots(3.9)$$

But for $A=\pm 2, v=0$ and $u=\dfrac{1}{2(cos\theta\pm 1)}$ $(provided\ cos\theta\pm 1\neq 0)$. Which in turns implies that $F(-2,1;z)$ maps the unit disk univalently onto the entire complex plane minus one slit along $(-\infty,-1/4]$ and for $A=2, F(2,1;z)$ maps the unit disk univalently onto entire complex plane minus one slit along $[1/4,\infty)$. In both cases F is starlike with respect to origin.

**Theorem 4.** *Let $A^2<4B$ and $B\in(0,1]$*, then $F(A,1;z)$ maps the unit disk univalently onto entire complex plane minus two slits on $u$-axis. One from $(-\infty,\dfrac{1}{A-2}]$ and other is from $[\dfrac{1}{A+2},\infty)$. Also, $F(A,B;z)\in\mathcal{S}^*$.

**Proof.** Putting $z=e^{i\theta}$ in (2.5), we have

$$F(A,B;e^{i\theta})=u(\theta)+i\,v(\theta)=\frac{e^{i\theta}}{B(e^{i\theta}-a)(e^{i\theta}-\overline{a})}=\frac{e^{i\theta}(e^{-i\theta}-\overline{a})(e^{-i\theta}-a)}{B|e^{i\theta}-a|^2|e^{i\theta}-\overline{a}|^2}, \qquad \ldots(3.10)$$

where

$$a=\frac{-A+i\sqrt{4B-A^2}}{2B} \quad \text{and} \quad \overline{a}=\frac{-A-i\sqrt{4B-A^2}}{2B}.$$

Separating real and imaginary parts we get

$$u(\theta)=\frac{cos\theta\left(1+\frac{1}{B}\right)+\frac{A}{B}}{K} \quad \text{and} \quad v(\theta)=\frac{cos\theta\left(\frac{1}{B}-1\right)}{K}, \qquad \ldots(3.11)$$

where

$$K=\left(1+\frac{1}{B}\right)^2+2\left(1+\frac{1}{B}\right)\frac{A}{B}cos\theta+\frac{A^2}{B^2}-\frac{4}{B}sin^2\theta. \qquad \ldots(3.12)$$

For the case $B=1$ so that $A\in(-2,2)$ we have $v(\theta)=0$ and $u(\theta)=\frac{1}{A+2cos\theta}\,(A+2cos\theta\neq 0)$. Which in turns implies that the image of $z=e^{i\theta}$ is two slit on $u$-axis. One from $(-\infty,\frac{1}{A-2}]$ and other is from $[\frac{1}{A+2},\infty)$. Thus the interior of the unit disk is mapped onto entire complex plane minus the above mentioned slits.

It is clear from geometry of image domain that $F(A,1;z)$ is starlike with respect to origin but not convex.

**Theorem 5.** *Let* $1+Az+Bz^2\neq 0$ *for* $z\in\mathbb{U}$, *then for* $\lambda\geqq 0$

*(a) If* $|B|\leqq\lambda$ then $F(A,B;z)\in\mathcal{U}(\lambda)$

*(b) If* $|B|\leqq\lambda$ then $F(A,B;z)\in\mathcal{P}(2\lambda)$.

**Proof.** Let $|B|\leqq\lambda$. To prove $F(z)=F(A,B;z)\in\mathcal{U}(\lambda)$, it is sufficient to show that

$$\left|F'(z)\left(\frac{z}{F(z)}\right)^2-1\right|\leqq\lambda \quad (z\in\mathbb{U}). \qquad \ldots(3.13)$$

Using (2.1), we have

$$\left|F'(z)\left(\frac{z}{F(z)}\right)^2-1\right|=|1-Bz^2-1|=|B|r^2\lhd|B|\leqq\lambda$$

As $|B|\leqq\lambda$. Result (b) can be found in a similar way.

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# OPTIMUM ORDERING INTERVAL FOR RANDOM DETERIORATION WITH SELLING PRICE AND STOCK DEPENDENT DEMAND RATE AND SHORTAGES

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## ABSTRACT

*We consider here the inventory model for optimum ordering interval with random deterioration. Demand is the function of selling price and stock level. Shortages are allowed and completely backlogged. The optimized results obtain by numerical example.*

**Keywords:** Random deterioration, Shortages, Selling price and Stock dependent demand

## INTRODUCTION

Many studies of inventory system indicate the effect of deterioration. Certain types of inventory become partially or completely useless for consumption in storage. The assumption is entirely false that the goods in inventory always preserves their physical characteristics. In general decay, damage, spoilage, evaporation, obsolescence, pilferage, loss of utility or loss of marginal value of the commodity results in decreasing usefulness from the original is termed as perishability or deterioration. Since longtime researchers were engaged in analyzing inventory model for deteriorating items such as volatile liquids, blood, fruits and vegetables etc. Deterioration is a function of time so many researchers considered it linear, quadratic, two parameter Weibull function and random etc.

Demand is a simple term which is the part of the common life but demand is also as important as deterioration. It has been observed that demand rate is usually affected by the time, selling price, stock level; selling and stock level both, ramp types, exponential. Stock dependent demand is always proportional to the

inventory level. In case of deterministic models, it is optional that the inventory level be zero but not in stock dependent demand rates model. In this type of demand pattern the quantity demand decreases as the inventory level decreases and finally result is lower sales and profits. To keep sales high the inventory level should be higher which results turn in higher holding and procurement cost. Super markets and big bazaar is the live examples of these types of demand. Gupta and Jauhari [1] developed an optimum ordering for constant demand rate of inventory (which is discrete in time). Shah and Shah [2] considered a lot size model for exponential decay under price increments. An optimal and lot size under certain conditions of perishability and partial back ordering was discussed by Abad [3].An optimal ordering policy for deteriorating items was discussed for permissible delay in payments by Jamal, Sarkar and Wang [4].Sharma and Kumar [5] developed an optimum ordering interval with known demand for items with variable rate of deterioration when shortages were allowed. Shinn and Hwang [6] discussed an optimal pricing and ordering policies for retailers under order size dependent delay in payments.

Ordering and pricing policy was also discussed by Lakshminarayana [7] for second sale. Stock and selling price dependent demand during inflation and supplier credits based EOQ model was developed by Gor and Shah [8]. An optimal policy for deteriorating inventory model with finite rate of replenishment rate, price dependent demand rate and cycle length dependent price was studied by Sabahno [9]. An optimum ordering interval with selling price dependent demand rate for items with random deterioration and shortages was discussed by Sharma, Kumar and Sharma [10].

In this paper an attempt has been made to develop a mathematical model for obtaining an optimum ordering interval where demand is not only a function of selling price but also a function of stock level. Deterioration is random for items with shortages which is completely backlogged .All these major factors affect the total inventory cost. This analysis is measured by numerical example and by the variations in different parameters.

## ASSUMPTION AND NOTATIONS

The mathematical model is developed on the basis of the following assumptions :

1. Shortages are allowed and completely backlogged.
2. Lead time is zero.
3. There is no repair or replenishment of the deteriorated units during the cycle.
4. $T$ is the scheduling period.

The following notations are used :

$d(s,t) = \alpha - \beta s + \delta I(t)$ : Demand is selling price and stock dependent

$I_{WD}(t)$ : Without decay on hand inventory.

$H_c$ : Holding cost at Rs./unit/unit time

$O_c$ : Ordering cost

$C_s$ : Shortage cost

$\theta(\alpha,t) = \theta_o(\alpha)t$ : Random deterioration rate. This is special form of two parameter Weibull function, where $\alpha$ is the random variable which ranges over a space ( $\Gamma$ ) and a probability density random variable function $p(\alpha)$ is defined such that $\int p(\alpha)d\alpha = 1$

$I(t)$ : On hand inventory at any time $t$.

$Z(t)$ : Stock loss due to decay at time $t$.

$C_p$ : Purchasing cost at Rs./unit

## MATHEMATICAL MODEL

The differential equations for the instantaneous state over $(0,T)$ are given by

$$\frac{dI(t)}{dt} + \theta(\alpha,t)I(t) = -d(s,t) \qquad \ldots(1)$$

The inventory without deterioration is given by

$$\frac{dI_{WD}(t)}{dt} = -(\alpha - \beta s + \delta I(t)) \qquad \ldots(2)$$

With the boundary condition $I(0)$, the solution of differential equations are

$$I(t) = I(0)\left(1 - \delta t - \theta_0(\alpha)\frac{t^2}{2}\right) - (\alpha - \beta s)\left(t - \frac{\delta t^2}{2} - \frac{\theta_0(\alpha)t^3}{3} - \frac{5}{12}\delta\theta_0(\alpha)t^4\right) \qquad \ldots(3)$$

With the boundary condition $I(0)$, the solution of differential equation is

$$I_{WD}(t) = I(0)(1 - \delta t) - (\alpha - \beta s)\left(t - \frac{\delta t^2}{2}\right) \qquad \ldots(4)$$

Stock loss is given by

$$Z(t) = I_{WD}(t) - I(t) \qquad \ldots(5)$$

$$= I(0)\theta_0(\alpha)\frac{t^2}{2} - (\alpha - \beta s)\left(\theta_0(\alpha)\frac{t^3}{3} + \frac{5}{12}\delta\theta_0(\alpha)t^4\right) \quad \ldots(6)$$

Depletion of stock in the interval $(0, T_1)$ is

$$Z(T_1) = I(0)\theta_0(\alpha)\frac{T_1^2}{2} - (\alpha - \beta s)\left(\theta_0(\alpha)\frac{T_1^3}{3} + \frac{5}{12}\delta\theta_0(\alpha)T_1^4\right) \quad \ldots(7)$$

Backlogged demand within the cycle is given by

$$B(T_1) = (\alpha - \beta s)(T - T_1) \quad \ldots(8)$$

Order quantity

$$\theta_{T_1} = I(0)\theta_0(\alpha)\frac{T_1^2}{2} + (\alpha - \beta s)\left(T_1 - \theta_0(\alpha)\frac{T_1^3}{3} - \frac{5}{12}\delta\theta_0(\alpha)T_1^4\right) \quad \ldots(9)$$

$$I(0) = (\alpha - \beta s)\left(T_1 + \theta_0(\alpha)\frac{T_1^3}{6} - \frac{5}{12}\delta\theta_0(\alpha)T_1^4\right) \quad \ldots(10)$$

Therefore inventory in hand is obtained by putting the value of $I(0)$ in equation

$$I(t) = (\alpha - \beta s)\left(T_1 + \frac{\theta_0(\alpha)T_1^3}{6} - \frac{5}{12}\delta\theta_0(\alpha)T_1^4 - \delta T_1 t - \frac{\theta_0(\alpha)\delta T_1^3 t}{6} - \frac{\theta_0(\alpha)T_1 t^2}{2}\right)$$
$$-(\alpha - \beta s)\left(t - \frac{\delta t^2}{2} - \frac{\theta_0(\alpha)t^3}{3} - \frac{5}{12}\delta\theta_0(\alpha)t^4\right) \quad \ldots(11)$$

Total cost per cycle is

$$TC = \frac{1}{T}\left[O_c + C_P I(0) + H\int_0^{T_1} I(t)dt + C_s\int_0^{T-T_1}(\alpha - \beta s + \delta I(t))t\,dt\right]$$

By putting $\frac{T_1}{T} = \eta$ be the fraction of cycle in which there is no excess demand.

Now optimize value of cost is obtained by differentiating with respect to '$T$' and '$\eta$', $\frac{\partial(TC)}{\partial T} = 0$ and $\frac{\partial(TC)}{\partial \eta} = 0$ and the second order differentiation is positive for the minimum cost.

## NUMERICAL EXAMPLE

In this section, a numerical example is tested to show accuracy of the model and the solution procedure. The following parameter values in proper units are:

| | | |
|---|---|---|
| $O_c$ = 1000Rs. | $C_P$ = 25Rs. /unit | $\alpha = 5$ |
| $\beta = 2$ | $s = 0.1$ | $\theta_o(\alpha) = 0.005$ |
| $\eta = 0.5$ | $\delta = 0.05$ | $C_s$ = 2 Rs./unit |
| $H_c$ = 5Rs./unit/time | | |

The optimum price and optimal order quantity for the optimal cycles are:

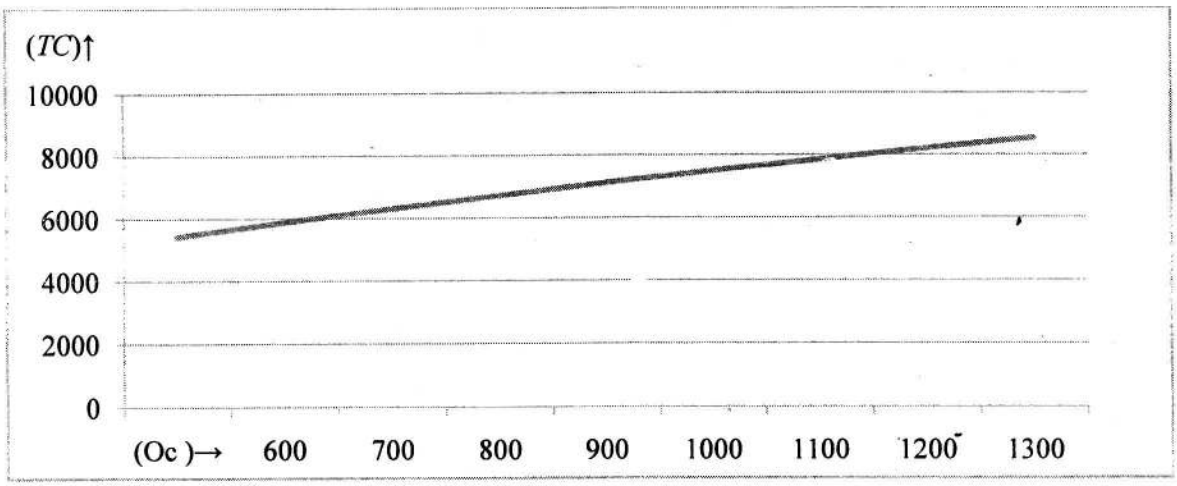
$$T^* = 0.1430, \quad \text{total Cost}^* = 7113.60 \text{ Rs.}$$

The model is solved by MATLAB 7.1 to obtain the necessary results.

Change in total cost with the respect to different parameters are :

**Table-1** Variation in total cost ($TC$) with ordering cost ($O_c$)

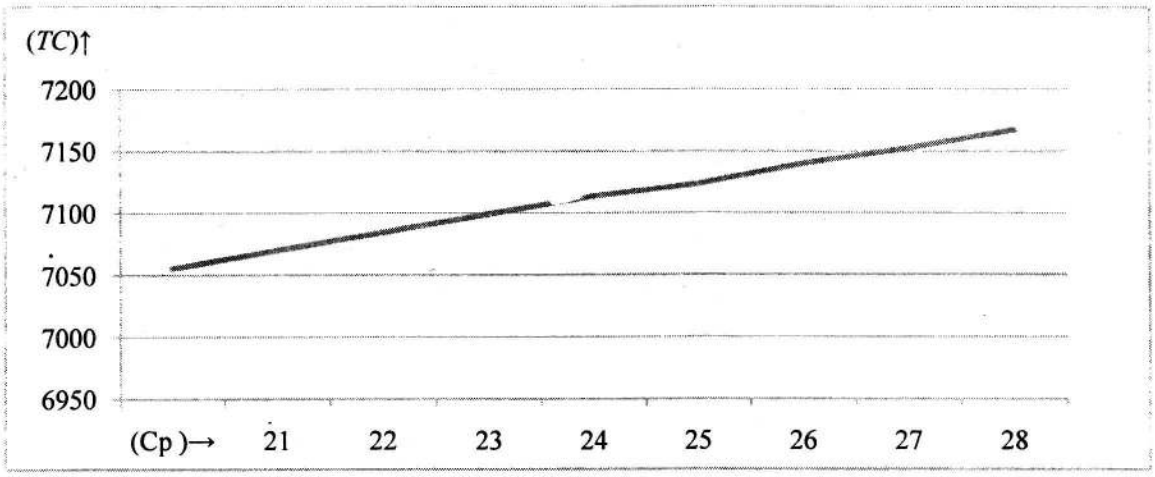
| Ordering Cost ($O_c$) | Time ($T$) | Total Cost ($TC$) |
|---|---|---|
| **600** | **0.1131** | **5425.01** |
| **700** | **0.1215** | **5881.82** |
| **800** | **0.1292** | **6312.49** |
| **900** | **0.1364** | **6718.81** |
| **1000** | **0.1430** | **7113.60** |
| **1100** | **0.1493** | **7488.34** |
| **1200** | **0.1553** | **7847.63** |
| **1300** | **0.1609** | **8200.22** |
| **1400** | **0.1663** | **8539.29** |




Observation- Total cost ($TC$) increases with time ($T$) and ordering cost ($O_c$).

**Table-2** Variation in total cost ($TC$) with purchasing cost ($C_p$)

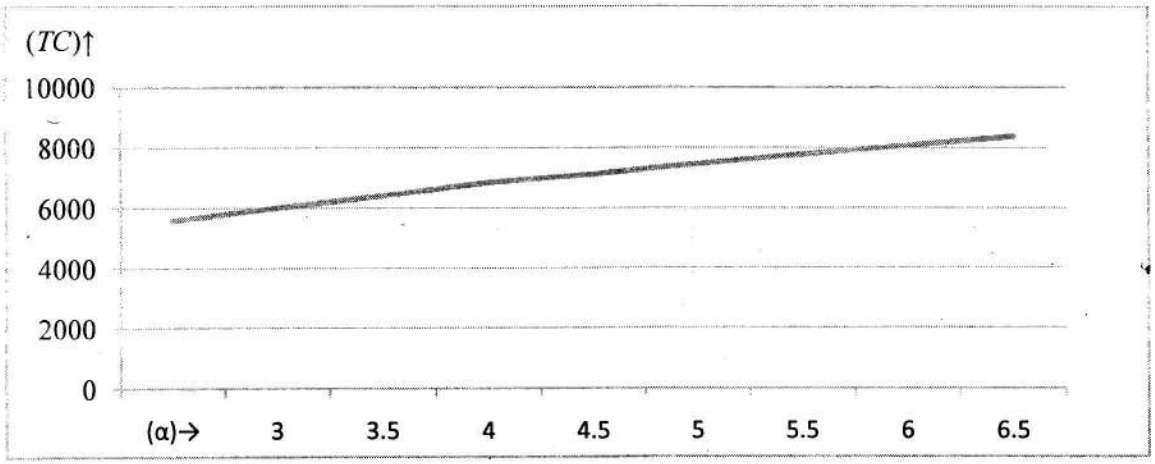
| **Purchasing Cost** ($C_p$) | **Time** ($T$) | **Total Cost** ($TC$) |
|---|---|---|
| **21** | **0.1438** | **7055.50** |
| **22** | **0.1436** | **7069.99** |
| **23** | **0.1434** | **7084.50** |
| **24** | **0.1432** | **7099.04** |
| **25** | **0.1430** | **7113.60** |
| **26** | **0.1429** | **7123.30** |
| **27** | **0.1427** | **7139.90** |
| **28** | **0.1425** | **7152.54** |
| **29** | **0.1423** | **7167.20** |




Observation- Total cost ($TC$) increases with time ($T$) and purchasing cost ($C_p$).

**Table-3** Variation in total cost ($TC$) with parameter ($\alpha$)

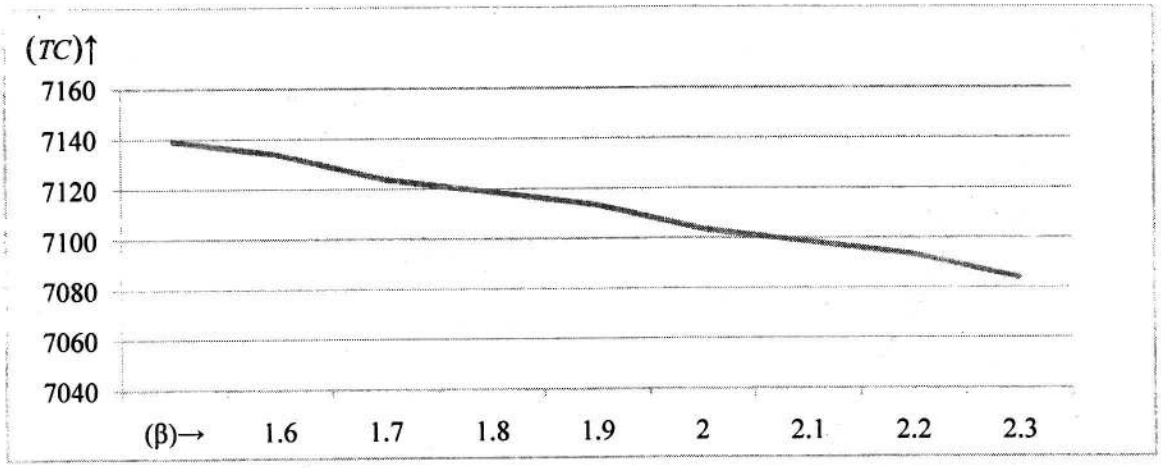
| Parameter ($\alpha$) | Time ($T$) | Total Cost ($TC$) |
|---|---|---|
| **3.0** | **0.1818** | **5570.99** |
| **3.5** | **0.1697** | **5996.64** |
| **4.0** | **0.1589** | **6388.79** |
| **4.5** | **0.1503** | **6836.54** |
| **5.0** | **0.1430** | **7113.60** |
| **5.5** | **0.1367** | **7448.42** |
| **6.0** | **0.1312** | **7767.61** |
| **6.5** | **0.1264** | **8069.58** |
| **7.0** | **0.1222** | **8367.44** |




Observation- Total cost ($TC$) increases with time ($T$) and parametr ($\alpha$)

**Table-4** Variation in total cost ($TC$) with parameter ($\beta$)

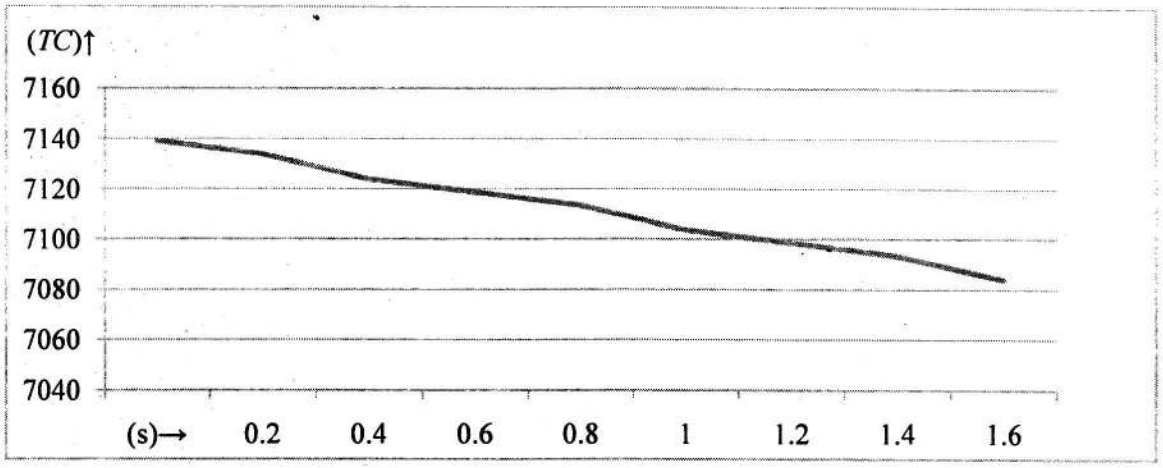
| Parameter ($\beta$) | Time ($T$) | Total Cost ($TC$) |
|---|---|---|
| **1.6** | **0.1425** | **7139.14** |
| **1.7** | **0.1426** | **7133.97** |
| **1.8** | **0.1428** | **7123.90** |
| **1.9** | **0.1429** | **7118.75** |
| **2.0** | **0.1430** | **7113.60** |
| **2.1** | **0.1432** | **7103.59** |
| **2.2** | **0.1433** | **7098.46** |
| **2.3** | **0.1434** | **7093.34** |
| **2.4** | **0.1436** | **7083.88** |




Observation- Total cost ($TC$) decreases with time ($T$) and parametr ($\beta$)

**Table- 5** Variation in total cost ($TC$) with parameter ($s$)

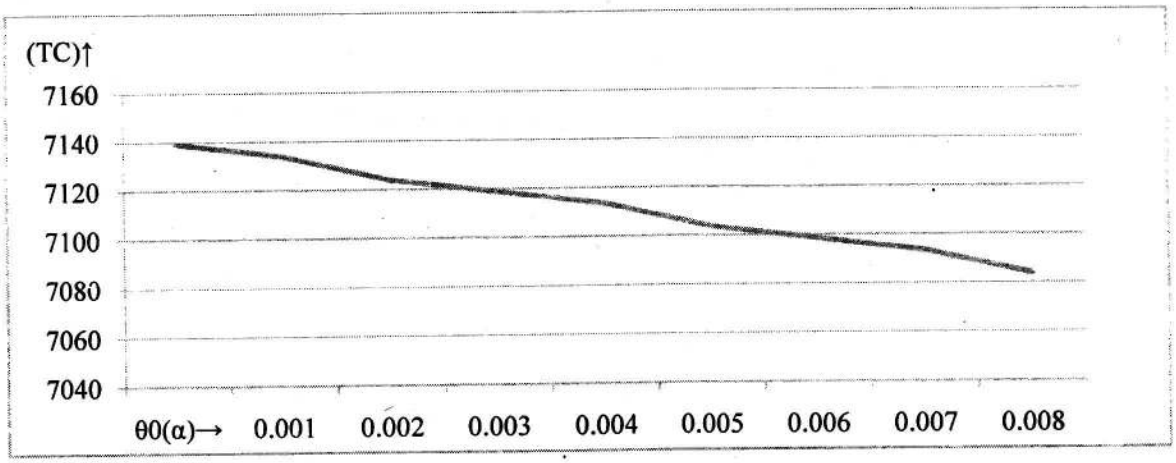
| Parameter ($s$) | Time ($T$) | Total Cost ($TC$) |
|---|---|---|
| 0.2 | 0.1409 | 7221.84 |
| 0.4 | 0.1414 | 7195.74 |
| 0.6 | 0.1420 | 7164.86 |
| 0.8 | 0.1425 | 7139.14 |
| 1.0 | 0.1430 | 7113.60 |
| 1.2 | 0.1436 | 7083.38 |
| 1.4 | 0.1441 | 7058.22 |
| 1.6 | 0.1447 | 7028.44 |
| 1.8 | 0.1452 | 7003.64 |




Observation- Total cost (*TC*) decreases with time (*T*) and selling price (*s*)

**Table-6** Variation in total cost (*TC*) with deterioration factor ( $\theta_0(\alpha)$)

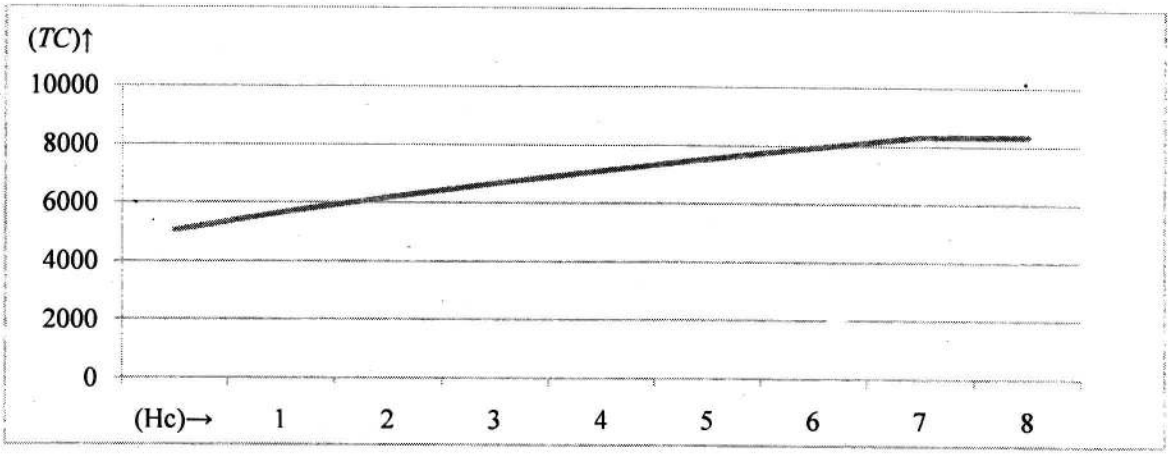
| **Deterioration Factor** ( $\theta_0(\alpha)$) | **Time** (*T*) | **Total Cost** (*TC*) |
|---|---|---|
| **0.001** | **0.1519** | **6703.91** |
| **0.002** | **0.1494** | **6814.06** |
| **0.003** | **0.1420** | **6918.71** |
| **0.004** | **0.1425** | **7017.16** |
| **0.005** | **0.1430** | **7113.60** |
| **0.006** | **0.1412** | **7202.74** |
| **0.007** | **0.1395** | **7289.04** |
| **0.008** | **0.1379** | **7372.27** |
| **0.009** | **0.1364** | **7451.95** |




Observation- Total cost ($TC$) decreases with deterioration factor ($\theta_0(\alpha)$)

**Table-7** Variation in total cost ($TC$) with Holding cost($H_c$)

| **Holding cost ($H_c$)** | **Time ($T$)** | **Total Cost ($TC$)** |
|---|---|---|
| **1** | **0.2029** | **5048.97** |
| **2** | **0.1809** | **5648.35** |
| **3** | **0.1650** | **6181.10** |
| **4** | **0.1528** | **6665.05** |
| **5** | **0.1430** | **7113.60** |
| **6** | **0.1350** | **7528.05** |
| **7** | **0.1281** | **7927.09** |
| **8** | **0.1222** | **8304.03** |
| **9** | **0.1171** | **8660.48** |




Observation- Total cost ($TC$) decreases with parameter ($H_c$)

## CONCLUSION

In this model an optimum ordering interval inventory model is developed for deteriorating items. The principle features of the model are as follows-The demand rate is assumed to be a function of selling price and stock dependent when the selling price and the related factor increases then the total cost decreases. Shortages are allowed and completely backlogged in the present model. In practical situations, stock out is unavoidable due to various uncertainties and these uncertainties increases the total cost. Deterioration factor is taken into consideration in the present model as almost all items undergo either direct spoilage or physical decay which increases the total cost. As the changes in the time value of money and in the price, holding cost cannot remain constant over time and increases the total cost. Ordering cost, purchasing cost, $\delta$, $\eta$ also increases the total cost.

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# HOMOGENOUS STRING DUST COSMOLOGICAL MODEL IN BRANS-DICKE THEORY

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## ABSTRACT

*Homogeneous and isotropic inflationary string cosmological model is investigated in Brans-Dicke scalar tensor theory of gravitation. An special model is also obtained. For determinate solution, we have assumed that the trace of energy momentum tensor of string cloud vanishes and expansion $\theta$ is proportional to Eigen value $\sigma_1^1$ of shear tensor $\sigma_i^j$ .The physical and kinematical parameters of the model are also discussed.*

**Key Words :** Homogeneous, Inflationary, Brans-Dicke Theory.

**PACS :** 98.80.Cq, 04.20.-q.

## 1. INTRODUCTION

Alternative theories of gravitation have been studied extensively by a number of researchers in connection with their cosmological applications .Noteworthy, one of them is scalar-tensor theory of gravitation formulated by Brans-Dicke [1]. Brans-Dicke theory of gravitation is the well-known modified theory of Einstein's theory, involving a scalar function in addition to the familiar general relativistic metric tensor. In this theory, the scalar field behaves as the reciprocal of Gravitational Constant G, where $\phi$ behaves as the reciprocal of gravitational constant G, where $\phi$ satisfies a scalar wave equation whose source is all the matter in the universe. Several authors have studied Brans-Dicke cosmological models in four dimensions. Singh and Rai [2] investigated Brans-Dicke cosmological models with perfect fluid as a source in detail.

String theory unifies all the elementary interactions including gravity. The string theory has a great importance to describe an event at early stage of evolution of the universe in a lucid way. According to Kibble [3] and Zel'Dovich [4], the cosmic strings arise during the phase transition as the universe passes through its critical temperature after the big-bang explosion and give rise to perturbation leading to the formation of galaxies. The cosmic string have stress energy and couple to the gravitational field and gravitational field is related to scalar field $\phi$ .Therefore it is interesting to study scalar field effect, which arises from string by Brans-Dicke field equations.

Letelier [5, 6] and Stàchel [7] initiated the general treatment of strings. In recent years various authors viz. Vilenkin [8],Chakraborty and Chakraborty [9], Tikekar and Patel [10,11], Bali and Upadhyaya [12], Bali and Singh [13], Bali and Anjali [14] and Rathore et al. [15 ] have investigated string cosmological models in general relativity, whereas Reddy [16,17] and Bali et al. [18] have studied string cosmological models in alternative theories of gravitation in four dimensions. Rahaman et al. [19], Mohanty and Mahanta [20], Rathore and Mandawat [21, 22], investigated string cosmological models in alternative theories of gravitation in higher dimensions.

The present investigation is concerned with Homogeneous string cosmological model having inflationary character in Brans-Dicke scalar tensor theory of gravitation, when the trace of energy momentum tensor of string cloud vanishes i.e. $\rho + \lambda = 0$. Here $\rho$ is the rest energy density and $\lambda$ is the tension density. To get a determinate solution. we have assumed a physically plausible condition that the expansion ($\theta$) is proportional to Eigen value $\sigma_1^1$ of shear tensor $\sigma_i^j$ , which is physically plausible condition, where

$$\theta = \left( \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \right)$$

$$\sigma_1^1 = \frac{1}{3}\left( \frac{2A_4}{A} - \frac{B_4}{B} - \frac{C_4}{C} \right)$$

Thus $\theta \propto \sigma_1^1$ leads to

$$\frac{A_4}{A} = \frac{\ell + 3}{2\ell - 3}\left( \frac{B_4}{B} + \frac{C_4}{C} \right)$$

This leads to

$$A = \ell_1 (BC)^n$$

where $\ell$ is proportional constant , $\ell_1$ is constant of integration and $n = \dfrac{\ell+3}{2\ell-3}$ . Some physical and kinematical properties of the model are also discussed.

## 2. THE METRIC AND FIELD EQUATIONS

We consider the spatially homogeneous and anisotropic Bianchi type-I metric in the form.

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2 \qquad \ldots (1)$$

where A, B, C are functions of cosmic time'$t$' alone.

The Brans-Dicke scalar tensor field equations are given by

$$G_{ij} = -\ 8\pi\ \phi^{-1} T_{ij} - \omega\phi^{-2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2} g_{ij}\ \phi_{,k}\phi'^{\,k}\right) - \phi^{-1}\left(\phi_{i;j} - g_{ij}\ \Pi\phi\right) \qquad \ldots (2)$$

and

$$\Pi\phi = \phi'^{\,k}_{;k} = 8\pi\ \phi^{-1} T\ (3 + 2\omega)^{-1} \qquad \ldots (3)$$

where $G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$ is the Einstein tensor, $T_{ij}$ is the stress Energy of matter, $\phi$ is the scalar field and $\omega$ is the dimensionless coupling constant and comma and semicolon denote partial and covariant differentiation respectively.

The equation of motion

$$T^{ij}{}_{;j} = 0 \qquad \ldots (4)$$

are consequences of the field equation.

The energy momentum tensor for string dust source is given by

$$T^i_j = \rho\, u^i u_j - \lambda\, x^i\, x_j, \qquad \ldots (5)$$

where $u^i$ is the four velocity of the string cloud, $x^i$ is the normal space like four vector which represents the string direction i.e. direction of anisotropy and $\rho$ is the rest energy density of the cloud of strings with particles attached to them and $\lambda$ is tension density of the cloud of strings. The string source is along the $x$-axis, which is the axis of symmetry. Orthonormalisation of $u^i$ and $x^i$ is given as

$$u^i u_i = -x^i x_i = -1 \quad \text{and} \quad u^i x_i = 0 \qquad \ldots (6)$$

In the co-moving coordinate system, we have from (5)

$$T^1_1 = -\lambda,\ T^2_2 = T^3_3 = 0,\ T^4_4 = -\rho$$
$$T^i_j == 0 \ \text{ For } \ i \neq j \ \text{ and } \ T = (\rho + \lambda) \qquad \ldots (7)$$

We also consider

$$\rho = \rho_p + \lambda$$

where $\rho_p$ is the rest energy density of the particles.The energy densities for coupled system $\rho$ and $\rho_p$ are restricted to be positive and the tension density $\lambda$ may be positive or negative. Here the quantities $\rho$ , $\lambda$ and scalar field $\phi$ are functions of cosmic time only.

The Brans-Dicke field equations (2), (3) and (4) for the metric (1) lead to

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} = 8\pi\, \phi^{-1} \lambda - \frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^2 + \frac{\phi_{44}}{\phi} + \frac{\phi_4}{\phi}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) \qquad \ldots (8)$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4 C_4}{AC} = -\frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^2 + \frac{\phi_{44}}{\phi} + \frac{\phi_4}{\phi}\left(\frac{C_4}{C} + \frac{A_4}{A}\right) \qquad \ldots (9)$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} = -\frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^2 + \frac{\phi_{44}}{\phi} + \frac{\phi_4}{\phi}\left(\frac{A_4}{B} + \frac{B_4}{B}\right) \qquad \ldots (10)$$

$$\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{C_4A_4}{CA}=8\pi\phi^{-1}\rho+\frac{\omega}{2}\left(\frac{\phi_4}{\phi}\right)^4+\frac{\phi_4}{\phi}\left(\frac{A_4}{B}+\frac{B_4}{B}+\frac{C_4}{C}\right) \quad \ldots(11)$$

$$\phi_{44}+\phi_4\left[\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right]=(\rho+\lambda)\frac{8\pi\phi^{-1}}{(3+2\omega)} \quad \ldots(12)$$

$$\rho_4+(\rho+\lambda)\frac{A_4}{A}+\rho\left(\frac{B_4}{B}+\frac{C_4}{C}\right)=0 \quad \ldots(13)$$

where, the sub index '4' denotes differentiation with respect to t.

## 3. SOLUTIONS OF FIELD EQUATIONS

From equation (9) and (10), we have

$$\frac{BC_{44}-CB_{44}}{BC_4-CB_4}+\frac{A_4}{A}=\frac{\phi_4}{\phi} \quad \ldots(14)$$

Using condition

$$A=\ell_1(BC)^n \quad \ldots(15)$$

in (14), we have

$$\frac{BC_{44}-CB_{44}}{BC_4-CB_4}+n\left(\frac{B_4}{B}+\frac{C_4}{C}\right)=\frac{\phi_4}{\phi}$$

On integration, it leads to

$$B^2\left(\frac{C}{B}\right)_4=\frac{k_1\phi}{(BC)^n} \quad \ldots(16)$$

Let $BC=\mu$ and $C/B=\upsilon$

Therefore, we have $B^2 = \frac{\mu}{\upsilon}$ , $C^2 = \mu\upsilon$

Now, from equation (16), we have

$$\frac{\upsilon_4}{\upsilon} = \frac{K_1\phi}{\mu^{n+1}} \qquad \ldots(17)$$

where, $K_1$ is constant of integration.

From equation (12), with the help of (15)

$$\phi_4 = \frac{K_2}{\mu^{n+1}} \qquad \ldots(18)$$

where, $K_2$ is constant of integration.

From the equations (8) and (11), together with the condition $\rho + \lambda = 0$ and equation (18) we have

$$\frac{(BC_4 + CB_4)_4}{BC_4 + CB_4} + n\left(\frac{B_4}{B} + \frac{C_4}{C}\right) = \frac{\phi_4}{\phi}$$

On integration, it leads to

$$\phi = \frac{\mu_4\mu^n}{K_3} \qquad \ldots(19)$$

where, $K_3$ is the constant of integration.

Equation (18) and (19) leads to

$$\mu\mu_{44} + n\mu_4^{\ 2} = K\mu^{-2n} \qquad \ldots(20)$$

where $K = K_2K_3$

This leads to

$$ff' + \frac{n}{\mu} f^2 = K\mu^{-2n-1} \qquad \ldots (21)$$

where $\mu_4 = f(\mu)$ and $f = \frac{df}{d\mu}$

From equation (21), we have

$$f^2 = \frac{1}{\mu^{2n}}\left(2K \log \mu + K_4\right) \qquad \ldots (22)$$

where, $K_4$ is the constant of integration.

$$\mu_4 = \frac{1}{\mu^n}\sqrt{\left(2K \log \mu + K_4\right)} \qquad \ldots (23)$$

From equation (17), we get

$$\upsilon = K_5 (\mu)^{\frac{K_1}{K_2}} \qquad \ldots (24)$$

where, $K_5$ is the constant of integration.

The metric (1) reduces to

$$ds^2 = -dt^2 + \mu^{2n} dx^2 + \frac{\mu}{\upsilon} dy^2 + \mu\upsilon dz^2 \qquad \ldots (25)$$

Using the transformation

$$\mu = T, \qquad x = X, \qquad \left(\frac{1}{\sqrt{K_5}}\right) y = Y \qquad \text{and} \qquad \sqrt{K_5}\, z = Z$$

Metric (25) reduces to

$$ds^2 = -\frac{dT^2}{\frac{1}{T^{2n}}\left(2K\log T + K_4\right)} + T^{2n}dX^2 + T^{1-\frac{k_1}{k_3}}dY^2 + T^{1+\frac{k_1}{k_3}}dZ^2 \qquad \ldots(26)$$

## 4. SOME PHYSICAL AND GEOMETRICAL PROPERTIES

The energy density ($\rho$), String tension density ($\lambda$) and the scalar field ($\phi$) for the model (26) are given by

$$\rho = \frac{K_6}{T^{2n+1}} \qquad \ldots(27)$$

$$\lambda = -\frac{K_6}{T^{2n+1}} \qquad \ldots(28)$$

$$\phi = \frac{1}{K_3}\sqrt{2K\log T + K_4} \qquad \ldots(29)$$

The scalar of expansion ($\theta$), the non-vanishing components of shear ( $\sigma_i^j$ ), shear ($\sigma$) and spatial volume ($V$) and deceleration parameter ($q$) for model (26) are given by

$$\theta = \frac{(n+1)}{T^{n+1}}\sqrt{2K\log T + K_4} \qquad \ldots(30)$$

$$\sigma_1^1 = \frac{(2n-1)}{3}\frac{1}{T^{n+1}}\sqrt{2K\log T + K_4} \qquad \ldots(31)$$

$$\sigma_2^2 = \left[\left(\frac{1-2n}{6}\right) - \frac{1}{2}\frac{K_1}{K_2}\right]\frac{1}{T^{n+1}}\sqrt{2K\log T + K_4} \qquad \ldots(32)$$

$$\sigma_3^3 = \left[\left(\frac{1-2n}{6}\right) + \frac{1}{2}\frac{K_1}{K_3}\right]\frac{1}{T^{n+1}}\sqrt{2K\log T + K_4} \qquad \ldots(33)$$

$$\sigma = \left[\frac{1}{12}(2n-1)^2 + \frac{1}{2}\frac{K_1^2}{K_3^2}\right]^{\frac{1}{2}} \frac{1}{T^{n+1}}\sqrt{2K\log T + K_4} \qquad \ldots(34)$$

$$V = T^{n+1} \qquad \ldots(35)$$

$$q = -\left[\frac{3\{K - (2K\log T + K_4)\}}{(n+1)(2K\log T + K_4)}\right] - 1 \qquad \ldots(36)$$

The model (26) starts with big-bang at $T = 0$ and expansion in the model decreases as the time increases for $(n + 1) > 0$ and expansion increases as time increases for $(n + 1) < 0$. The model has a point type singularity (Mac Callum [23]) at $T = 0$ if $n > 0$ with $\frac{K_1}{K_3} \in (-1,1)$ and cigar type singularity at $T = 0$ when $n < 0$ with $\frac{K_1}{K_3} \in (-1,1)$ or $n > 0$ with $\frac{K_1}{K_3} \rangle 1$.

The rest Energy density $\rho \to \infty$ as $T \to 0$ and $\rho \to 0$ as $T \to \infty$ with $K_6 > 0$ and $n > -\frac{1}{2}$. The string tension density $\lambda \to 0$ as $T \to \infty$ and $\lambda \to -\infty$ as $T \to 0$ for $K_6 > 0$ and $n > -\frac{1}{2}$ .Therefore, rest energy density and string tension density of the model has initial singularity at $T = 0$. The rest energy density for the model (26) satisfies the reality condition $\rho > 0$ given by Ellis if $K_6 > 0$. The Brans-Dicke scalar field $\phi$ increases indefinitely as time $T \to \infty$ and it blows up at its initial stage. The spatial volume $V$ of the model given by (35) shows that the model follows power law inflation. The model isotropizes at all epoch for $n = \frac{1}{2}$ and $K_1 = 0$, but remain anisotropic throughout the evolution when $n \neq \frac{1}{2}$ or $K_1 \neq 0$ .

The cosmological model has singularities of different kinds in different conditions at initial epoch, while rest Energy density, tension density and Brans-Dicke scalar field are not free from initial singularities. The sign of deceleration parameter $q$ is different in different time zones. Therefore, it is concluded that the universe will witness an epoch when it transits from the phase of acceleration to deceleration of vice-versa. Hence, the cosmological model obtained, represents an inflationary, shearing, non-rotating and anisotropic string cosmological model in Brans-Dicke scalar-tensor theory of gravitation and behaves in accordance with the present observational data. Therefore, this cosmological model is a realistic model.

## 5. SPECIAL MODEL

Here we are considering the case for which $n = -1$. This case leads to $\ell = 0$, which restricts us to consider the assumption that expansion is proportional to the shear component $\sigma_1^1$. Therefore we have solved the field equation by taking $n = -1$ in a different manners.

Putting $n = -1$ in the equation (23), we have

$$\mu_4 = \mu\sqrt{2K\log\mu + K_4}$$

On integration equation (37) reduces to

$$\mu = \exp.\left[\frac{K}{2}(t + K_7)^2 - \frac{K_4}{2K}\right]$$

where , $K_7$ is the constant of integration.

And with the help of (24), we have

$$A = \exp.\left[-\frac{K}{2}(t + K_1)^2 + \frac{K_4}{2K}\right]$$

$$B = \frac{1}{(K_5)^{\frac{1}{2}}}\exp.\left[\frac{1}{2}\left(1 - \frac{K_1}{K_3}\right)\left\{\frac{K}{2}(t + K_7)^2 - \frac{K_4}{2K}\right\}\right]$$

$$C = (K_5)^{\frac{1}{2}}\exp.\left[\frac{1}{2}\left(1 + \frac{K_1}{K_3}\right)\left\{\frac{K}{2}(t + K_1)^2 - \frac{K_4}{2K}\right\}\right]$$

and using suitable transformation of coordinates, the metric (1) reduces to

$$ds = -dT^2 + e^{-KT^2}dX^2 + e^{\left\{\frac{K}{2}\left(1-\frac{K_1}{K3}\right)T^2\right\}}dY^2 + e^{\left\{\frac{K}{2}\left(1+\frac{K_1}{K_3}\right)T^2\right\}}dZ^2$$

This special model has no initial singularity and for small values of $T$, it reduces to a spherically symmetric cosmological model. It is neither expanding nor contracting model but the shear increases as $T$ increases and it remains always anisotropic. Therefore, it is not a realistic model.

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