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Dedicated

to

Great Indian Mathematical Prodigy

SRINIVASA RAMANUJAN

Born : 22 Dec., 1887 **Died : 26 April, 1920**

on the Occasion of

125^{th} birth Anniversary

declared as

National Mathematics Year - 2012

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SRINIVASA RAMANUJAN - THE MAN

M. K. SINGAL *

Srinivasa Ramanujan, the greatest mathematical genius produced in India in the modern times, was born on Thursday, the 22nd December 1887, at Kumbhakonam in Tamil Nadu. His mother Komalammal and father Srinivasa Aiyanger were devout Hindus. They were worshippers of Goddess Namagiri of Namakkal in Salem District. It is said that Ramanujan was considered by them as a gift of goddess Namagiri to whom they had offered prayers for this purpose. The parents were orthodox Vaishnavites and since Shri Ramanuja Acharya, one of the best propounders of Vishisht Advaita Philosophy, was also born on a Thursday, the child was named Ramanujan after him. His mother used to fondly call him Chinna Swami (which means "little Lord').

Ramanujan had his early education in Kumbhakonam, the Prayag of the South, for it is the only place in South where a Mahamakham (a function resembling the Kumbh Mela) is held once in 12 years.

Ramanujan had earned a name for mathematical ability even as a child. He had read Loney's trigonometry while he was still a student of the 8th class. Around this time he came across Carr's Synopsis of elementary results in pure and applied mathematics published in 1880 by an Englishman named Carr, a private teacher in London. The book contained the statements of about 6,000 (6165 to be precise) results in various branches of mathematics. This book had a profound influence on the subsequent development of Ramanujan because the book contained only the results and had no proofs. The proofs were worked out by him independently.

He passed his matriculation examination at Kumbhakonam at the age of 16 and secured Junior Subramanyam Scholarship for further study at the Government College at Kumbhakonam. He was so much engrossed in mathematics that he hardly paid any attention to any other subject with the result that he failed in English at the first examination and lost the scholarship. This made him leave Kumbhakonam, first for Vishakapatnam and later on for Madras, where he appeared at the first examination in arts in December 1906. As luck would have it, he failed and never attempted to pass this examination again. He spent the next few years studying mathematics independently. On the 14th July 1909 he got married to Janaki and it became necessary for him to search for some permanent employment for he had not only to support himself but also his wife. While

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searching for a job he met Diwan Bahadur R. Ramchandra Rao, the then Collector of Neliore, a small town, some 125 kilometers north from Madras. Diwan Ramchandra Rao was a great lover of mathematics and he was very much impressed by the two bulky note books shown to him by Ramanujan in which he used to write down his theorems. The best description of Ramanujan's first meeting with Diwan Bahadur is in his own words : "Several years ago a nephew of mine, perfectly innocent of mathematical knowledge said to me; :'Uncle, I have a visitor who talks of mathematics ; I do not understand him. Can you see if there is anything in his talk?" And in the plenitude of my mathematical wisdom, I condescended to permit Ramanujan to walk into my presence. A short uncouth figure, stout, unshaved, not over clean, with one conspicuous feature—shining eyes—walked in with a frayed note-book under his arm. He was miserably poor. He had run away from Kumbhakonam to get leisure in Madras to pursue his studies. He never craved for any distinction. He wanted leisure; in other words, that simple food should be provided for him without exertion on his part and that he should be allowed to dream on.

"He opened his book and began to explain some of his discoveries. I saw quite at once that there was something out of the way; but my knowledge did not permit me to judge whether he talked sense or nonsense. Suspending judgment, I asked him to come over again, and he did. And then he had gauged my ignorance and showed me some of his simpler results. These transcended existing books and I had no doubt that he was a remarkable man. Then, step by step, he led me to elliptic integrals and hypergeometric series and at last his theory of divergent series not yet announced to the world converted me. I asked him what he wanted. He said he wanted a pittance- to live on so that he might pursue his researches."

Ramchandra Rao was so much impressed by Ramanujan that he supported him for some time, but Ramanujan was not willing to live on somebody's help indefinitely. He also tried to obtain some scholarship but without success.

In 1912 he got a job of an officiating clerk in the Accountant General's office from 12th January to 21st February on a meager salary of ₹ 20 per month. On 1st March of the same year he accepted an appointment in the office of the Madras Port Trust on a salary of ₹ 30 per month.

While at the Port Trust he received encouragement from the Manager of the Trust, Narain Iyer and Chairman of the Trust, Sir Frencis Spring. There were several other people as well who were highly impressed by Ramanujan's capabilities. These included Professor Middlemax, Principal and Professor of Mathematics, Presidency College, Madras, Professor L.T. Griffith, Professor of Mathematics in Government Engineering College Madras, R. T. Bourne, Director of Public Instructions, W. Graham, the Accountant General of Madras, and Sir Gilbert Walker, Director General of Observatories in Simla who was on a visit to Madras. Sir Gilbert wrote a letter to Francis Didsbury, the then Registrar of the University of Madras recommending him to provide a place to Ramanujan in the University for a few years before he could be sent to England for higher studies. Walker's letter had the desired effect. The Madras University granted him a scholarship of ₹ 75 per month for two years. This was communicated to him in a letter dated the 9th April 1911. He was granted leave for two years by the Port Trust and with effect from the 1st May, 1913 Ramanujan joined the University at Madras as the first research scholar of the University (It is only befitting that the Government of India set up the Ramanujan Institute of Advanced Study in Mathematics in the early fifties and the Institute is now synonymous with the Department of Mathematics of Madras University.

On the suggestion of Mr. Sheshu Aiyer, Ramanujan wrote to Professor G.H. Hardy at Cambridge. His letter to Hardy dated January 16, 1913 ran as follows :

"Dear Sir,

I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £ 20 per annum. I am now about 23 years of age. I have had no university education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a University course but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'.

" I would request you to go through the enclosed papers, Being poor, if you are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated the lines on which I proceed. Being inexperienced I would very highly value any advice you give me.

Requesting to be excused for the trouble I give you,

I remain, Dear Sir, *Yours truly*

S. Ramanujan".

Along with this letter he enclosed about 120 theorems. Hardy took pains to go through the results sent by Ramanujan and congratulated him at his work. Hardy sent him an invitation to come to England. On the 17th March 1914 Ramanujan sailed for England. After 27 days of journey on the ship, he reached England where he was welcomed by Professor E.H. Neville whom he had already met at Madras. On the 18th of April 1914 he joined as a research scholar at an annual scholarship of £ 250.

In 1916 Ramanujan was awarded the B.A. degree of the University of Cambridge on the basis of his research work. While in England Ramanujan was honoured in more ways than one (though no honour could be too? big for a mathematician of Ramanujan's calibre!). In 1918 he was elected a Fellow of the Royal Society. He was the first mathematician whose name was accepted for Fellowship of the Royal Society at the first proposal. In the same year he was elected a Fellow of the Trinity College, Cambridge. He was the first Indian to have been so elected. Ramanujan lived in England for five years. These were the most fruitful years of his life. He collaborated with Hardy and Littlewood to produce some of the most outstanding work.

In 1917 Ramanujan fell ill and had to be admitted to a nursing home. The illness grew from bad to worse and it was decided in 1919 the he should be sent back to India where the warm climate might help in his recovery. But alas, that was not to be! He arrived in Madras on the 2nd April, 1919 and passed away a year later, on the 26th April 1920 at Chetpet near Madras. Even on his death bed he devoted himself completely to Mathematics and produced research work of the highest order, a glimpse of which he communicated in his last letter to Hardy, but most of it remained unnoticed till it was discovered accidentally by George Andrews in 1976.

During the last quarter of the 20th century, there have been many admirers of Ramanujan, of whom three American mathematicians-Professor George E. Andre of Pennsylvania State University, Professor Bruce C. Bernd t of the University of Illinois, and Professor Richard Askey of the University of Wisconsin, Madis on Wisconsin, stand out head and shoulders above the rest. Through their enthusiasm determination and herculean efforts, each in his own way, they have succeeded in giving a fresh impetus to work in the areas which were of interest to Ramanujan.

Professor George Andrews, a student of Prof. H. Rademacher, wrote his Ph.D. thesis on θ-functions. During the summer of 1976, Andrews while looking through the old papers of the late Prof. G.N. Watson, accidentally came across some 140 she etc in Ramanujan's own handwriting and containing some 600 formulae. This was the work that Ramanujan did in the last year of his life after returning to India. Andrews has since then been working on them and has written extensively on the material contained therein. But for Andrews, this important work might never have seen the light of the day.

Prof. Bruce Berndt started the work of editing Ramanujan's Note-books around 1977 a work which Hardy, Wilson and Watson thought of (and tried to some extent) but left for one reason or the other. During ten years Berndt has devoted him self solely to this stupendous task. His efforts have resulted in the publication of number of papers and a volume entitled 'Ramanujan's Note-books 1 published by Springer-Verlag. The subsequent volumes were ready and likely to be out soon at that time.

Richard Askey paid his tribute to Ramanujan in a very different way. It was Askey's enthusiasm and effort which was solely responsible for requesting Paul Garlund, sculptor-in-residence at Gustayus Adolphus

College at Saint Peter, Minnesota USA to prepare a bronze bust of Ramanujan. The busts commissioned are at Mrs. Ramanujan's residence, Madras; Raman Research Institute, Bangalore ; Department of Defence, Government of India, New Delhi ; Tata Institute of Fundamental Research Bombay: Trinity College. Cambridge; Askeys, Madison (Wisconsin) US/ Chandrasehars, Chicago, USA.

The bust now at Mrs. Ramanujan's residence was presented to her on the 11th May 1984 at a ceremony held at the Ramanujan Institute for Advanced Study in Mathematics, Madras.

The bust now at Raman Research Institute was presented to the Indian Institute of Science by Prof. S. Chandrasekhar and Mrs. Lalitha Chandrasekhar at a ceremony held at Raman Research Institute on the 6th February 1985. The following massage from Richard Askey read out on the occassion gives the story of the bust:

"In the spring of 1976, Andrews went to Europe for a meeting and stopped in Cambridge to see what old manuscripts he could find. One find was not a manuscript but 140 pages of formulas in Ramanujan's handwriting.

The story of the thread from these sheets to the bust is simple. Andrews has done a lot of very deep work trying to understand what Ramanujan discovered. Eventually '*The New York Times*' heard about it and interviewed him. '*The Hindu*' followed with a more extensive interview and also published an interview with Ramanujan's widow, Janaki Ammal. She lamented the fact that a statue of Ramanujan had never been made, although one had been promised. Andrews sent me copies of these interviews, and after a few months my subconscious mind finally got through to my conscious mind and it was clear that a bust should be made. Since Janaki Ammal was 80, time was important, so it was upto individuals rather than government or societies, since institutions move slowly. My first reason for wanting a bust was simple; if Ramanujan's widow wanted one she should have it. That was the least we could do to show our appreciation of Ramanujan to some one who had been a great help to him, Later I realized there was a second reason, which Janaki Ammal must have realized all along. She knew Ramanujan, and while she did not understand his mathematics, she knew he was one of the few whose work will last. As long as people do mathematics, some of Ramanujan's work will be appreciated. Fame is a strange thing and is often fleeting. An interview on a television programme is now the accepted form of honour. In Ramanujan's case a permanent memorial is appropriate. One which can be appreciated by those who do not understand his mathematics should be added to the memorial Ramanujan made for himself with his work.

I am pleased to have played a role in this, and would like to thank the more than one hundred mathematicians and scientists who contributed money for the bust which was presented to Janaki Ammal. The bust being dedicated today was donated by a couple who are now friends, Subramanyan and Lalitha Chandrasekhar. When I asked Chandra about the appropriateness of a bust of Ramanujan, he immediately replied that it was a good idea and they would do all they could to help. They did. Finally. I want to thank the sculptor, Paul Garlund".

The bust which is at Cambridge was unveiled on 27 May 1986 in the library of the Department of Pure Mathematics and Statistics in Mill Lane, Cambridge, by Professor Richard Askey, who as a part of his address read out a message from Mrs. Ramanujan and an appreciation composed by the distinguished physicist. Professor S. Chandrasekhar, of Ramanujan's influence on the development of science in India. The ceremony was preceded by a lecture on 'Ramanujan today' given by Professor R.A.Rankin.

Our main sources for Ramanujan's research contributions are the following :

(i) The three quarterly reports submitted by him to Madras University.

(ii) His 37 published papers, and the 58 questions and solutions contributed by him to the Journal of the Indian Mathematical Society. These are to be found in the Collected Papers published by the Cambridge University Press. Out of these 37 papers, only 5 were published in the Journal of the Indian Mathematical Society before Ramanujan left for England. Seven of the remaining 32 papers were written in collaboration with Hardy. Ofcouse, all the 32 research papers were edited by Hardy.

(iii) **Ramanujan's Note-Books**. A fascimile edition of Ramanujan's Note-Book; was brought out by the Tata Institute of Fundamental Research in 1957 at the suggestion of Sir K. S. Krishnan, Professor T. Vijayararaghavan anc Professor P. L. Bhatnagar to Professor K. Chandrasekharan during the (Indian Mathematical Society Conference held at Delhi in December 1953 and with financial support from Sir Dorabji Tata Trust.

(iv) **The Lost Note-Book.** Discovered by Professor George E. Andrews in 1976 among the papers of Professor G. N. Watson, and published in 1987 by Narosa Publishing House, New Delhi.

The following tribute to Ramanujan by Bruce C. Berndt is worth recording :

"Because of the unique circumstances shaping Ramanujan's career, inevitable questions arise about his greatness.

Here are three brief assessments of Ramanujan and his work:

Paul Erdos has passed on to us Hardy's personal ratings of mathematicians. Suppose that we rate mathematicians on the basis of pure talent on a scale from 0 to 100. Hardy gave himself a score of 25, Littlewood 30, Hilbert 80 and Ramanujan 100.

Neville began a broadcast in Hindustani in 1941 with the declaration "Srinivasa Ramanujan was a mathematician so great that his name transcends jealousies, the one superlatively great mathematician whom India has produced in the last thousand years.

In notes left by Wilson, he tells us that George Polya was captivated by Ramanujan's formulae. One day in 1925 while Polya was visiting Oxford, he borrowed from Hardy his copy of Ramanujan's Note-books. A couple of days later, Polya returned them in almost a state of panic explaining that however long he kept them, he would have to keep attempting to verify the formulae therein and never again would have time to establish an original result of his own.

To be sure, India has produced other great mathematicians, and Hardy's views may be moderately biased. But even though the pronouncements of Neville and Hardy are overstated, the excess is insignificant, for Ramanujan reached a pinnacle scaled by few."

So long as our planet continues to exist in the Universe, and so long as civilization exists on our planet, Ramanujan will be remembered not only because of the outstanding research contributions made by him to Number Theory and Analysis, not just because his work has kept first-rate mathematicians busy for nearly ninty years even after his death, not merely because his work has had a tremendous influence on modern mathematics and has opened up new vistas for research, but also because he was able to do so without any formal training, without any means of support, and more so because he continued to produce work of the highest order even in the face of death.

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KOBER OPERATORS IN THE MATRIX CASE FROM A STATISTICAL POINT OF VIEW

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ABSTRACT

We look at Kober fractional integral operators in the matrix case in this article as the densities of a product and a ratio of two real positive definite matrix-variate random variables X_1 *and* X_2, *which are statistically independently distributed. We look at the density of the product* $U_1 = X_2^{1/2} X_1 X_2^{1/2}$ *and the density of the ratio* $U_2 = X_2^{1/2} X_1^{-1} X_2^{1/2}$. *We examine matrix-variate Kober fractional integral operators of the first and second kinds from a statistical perspective. We derive the densities of products and ratios where one variable has a matrix-variate type-1 beta density and the other variable has an arbitrary density. Generalizations by using pathway models, by appending matrix variate hypergeometric series etc are considered. Matrix-variate Saigo operator and other operators are also defined and properties studied.*

1. INTRODUCTION

All the matrices appearing in this article are $p \times p$ real symmetric and positive definite. The following standard notations will be used. $X > O$ means the $p \times p$ real matrix is symmetric, $X = X'$, and further, it is positive definite. $|A|$ means the determinant of A, tr (A) = trace of A, dX will stand for the wedge product of differentials in any matrix X. If X is $p \times q$ with $X = (x_{ij})$ then

$$dX = dx_{11} \wedge dx_{12} \wedge \ldots \wedge dx_{pq} \text{ for a general matrix} \quad \ldots(1.1)$$

$$= \prod_{i \geq j} \wedge dx_{ij} = \prod_{j \geq i} \wedge dx_{ij} \text{ when } X = X' \text{ or when } X \text{ is symmetric} \quad \ldots(1.2.)$$

Also $\int_X f(X)\,dX$ will mean the integrals over all X (need not be symmetric or even square) of a real-valued scalar function $f(X)$ of matrix argument X. In the same format

$$\int_A^B f(Y)\,dY = \int_{O<A<Y<B} f(Y)\,dY$$

will mean the integration of a real-valued scalar function of the real positive definite $p \times p$ matrix Y over the space of positive definite matrices such that $A > O,\ Y > O,\ Y - A > O,\ B - Y > O$ where A and B are $p \times p$ constant matrices. The notation will then imply that if $O < X < I$ then all eigenvalues of X are in the open interval (0, 1). We will need some Jacobians of matrix transformations in this paper. For further results on Jacobians and for many applications, see Mathai (1997).

$$Y = BXB',\ \ X = X',\ \ |B| \neq 0 \Rightarrow dY = |B|^{p+1}\,dX \qquad \text{...(1.3)}$$

$$Y = X^{-1} \Rightarrow dY = \begin{cases} |X|^{-2p}\,dX \text{ for a general } Y \\ |X|^{-(p+1)}\,dX \text{ for } X = X' \end{cases} \qquad \text{...(1.4)}$$

We will denote the unique positive definite square root of a positive definite matrix X by $X^{1/2}$. The following standard property will be used very often in this article. For $p \times p$ nonsingular matrices A and B

$$|I \pm AB| = |I \pm BA| = |A|\left|A^{-1} \pm B\right| = |B|\left|B^{-1} \pm A\right| \qquad \text{(when nonsingular)}$$

$$|I \pm AB| = \left|I \pm A^{1/2}BA^{1/2}\right| = \left|1 \pm B^{1/2}AB^{1/2}\right| \qquad \text{(when positive definite)...(1.5)}$$

The real matrix-variate gamma function, denoted by $\Gamma_p(\alpha)$, is defined as follows which has an integral representation when $\boldsymbol{R}(\alpha) > \frac{p-1}{2}$.

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma\left(\alpha - \tfrac{1}{2}\right)\ldots\Gamma\left(\alpha - \tfrac{p-1}{2}\right),\ \ \boldsymbol{R}(\alpha) > \tfrac{p-1}{2}$$

$$= \int_{X>O} |X|^{\alpha - \frac{p+1}{2}} e^{-tr(X)}dX,\ \ \boldsymbol{R}(\alpha) > \tfrac{p-1}{2} \qquad \text{...(1.6)}$$

The real matrix-variate type-1 beta density for the $p \times p$ positive definite matrix X_1, with parameters α and β and denoted by $f_1(X_1)$, is the following :

$$f_1(X_1) = \frac{\Gamma_p(\alpha+\beta)}{\Gamma_p(\alpha)\Gamma_p(\beta)}|X_1|^{\alpha - \frac{p+1}{2}}|I - X_1|^{\beta - \frac{p+1}{2}}, O < X_1 < I$$

$$f_2(X_1)=\frac{\Gamma_p(\alpha+\beta)}{\Gamma_p(\alpha)\Gamma_p(\beta)}|X_2|^{\beta-\frac{p+1}{2}}|I-X_2|^{\alpha-\frac{p+1}{2}}dX_2,\ \ O<X_2<I$$

For $R(\alpha)>\frac{p-1}{2},\ R(\beta)>\frac{p-1}{2}$ and $f(X_1)=0,\ f_2(X_2)=0$ elsewhere. Type-1 and Type-2 beta integrals and beta functions are defined and denoted as follows for $R(\alpha)>\frac{p-1}{2},\ R(\beta)>\frac{p-1}{2}$:

$$B_p(\alpha,\beta)=\frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)}=\int_{O<X_1<I}|X_1|^{\alpha-\frac{p+1}{2}}|I-X_1|^{\beta-\frac{p+1}{2}}dX_1 \qquad \text{(type - I)}$$

$$=\int_{O<X_2<I}|X_2|^{\beta-\frac{p+1}{2}}|I-X_2|^{\alpha-\frac{p+1}{2}}dX_2 \qquad \text{(type - I)}$$

$$=\int_{X_3>O}|X_3|^{\alpha-\frac{p+1}{2}}|I+X_3|^{-(\alpha+\beta)}dX_3 \qquad \text{(type - 2)}$$

$$=\int_{X_4>O}|X_4|^{\beta-\frac{p+1}{2}}|I+X_4|^{-(\alpha+\beta)}dX_4 \qquad \text{(type - 2)...(1.7)}$$

2. KOBER OPERATOR OF THE SECOND KIND FOR THE REAL MATRIX-VARIATE CASE

The following definition, given by this author earlier, will be used here.

Definition 2.1. Kober operator of the second kind for the real matrix-variate case is defined and denoted as follows:

$$K_X^{\rho,\alpha}f(X)=\frac{|X|^{\rho}}{\Gamma_p(\alpha)}\int_{T>X}|T-X|^{\alpha-\frac{p+1}{2}}|T|^{-\rho-\alpha}f(T)dT,\ \ R(a)>\frac{p-1}{2} \qquad ...(2.1)$$

Consider two $p\times p$ real matrix-variate random variables X_1 and X_2, independently distributed, where X_1 has a real matrix-variate type-1 beta density $f_1(X_1)$ with parameters $\left(\rho+\frac{p+1}{2},\alpha\right)$, that is,

$$f_1(X_1)=\frac{\Gamma_p\left(\rho+\alpha+\frac{p+1}{2}\right)}{\Gamma_p\left(\rho+\frac{p+1}{2}\right)\Gamma_p(\alpha)}|X_1|^{\rho}|1-X_1|^{\alpha-\frac{p+1}{2}},\ \ O<X_1<I$$

for $R(\rho) > -1$, $R(\alpha) > \frac{p-1}{2}$ and $f_1(X_1) = 0$ elsewhere. Let X_2 have an arbitrary density $f(X_2)$. Then the joint density of X_1 and X_2 is $f_1(X_1)f(X_2)$. Let us consider the transformation $U = X_2^{1/2}X_1X_2^{1/2}$, $X_2 = V$ so that $X_2 = V$, $X_1 = V^{-1/2}UV^{-1/2}$ and the Jacobian is given by $dX_1 \wedge dX_2 = |V|^{-\frac{p+1}{2}} dU \wedge dV$. If the joint density is denoted by $f(U,V)$ then

$$f(U,V)dU \wedge dV = \frac{\Gamma_p\left(\rho+\alpha+\frac{p+1}{2}\right)}{\Gamma_p(\alpha)\Gamma\left(\rho+\frac{p+1}{2}\right)}\left|V^{-1/2}UV^{-1/2}\right|^{\rho}$$

$$\times\left|I - V^{-1/2}UV^{-1/2}\right|^{\alpha-\frac{p+1}{2}} f(V)|V|^{-\frac{p+1}{2}} dU \wedge dV.$$

Therefore the marginal density of U, denoted by $g(U)$, is available by integrating out V from $f(U,V)$. That

$$g(U) = \int_V f_1\left(V^{-1/2}UV^{-1/2}\right) f(V)|V|^{-1/2} dV$$

$$= \frac{\Gamma_p\left(\rho+\alpha+\frac{p+1}{2}\right)}{\Gamma_p\left(\rho+\frac{p+1}{2}\right)}\int_{V>U} \frac{1}{\Gamma_p(\alpha)}|V|^{-\frac{p+1}{2}}$$

$$\times |U|^{\rho}|V|^{-\rho}|V|^{-\alpha+\frac{p+1}{2}}|V-U|^{\alpha-\frac{p+1}{2}} f(V)dV$$

$$= \frac{\Gamma_p\left(\alpha+p+\frac{p+1}{2}\right)}{\Gamma_p\left(\rho+\frac{p+1}{2}\right)} K_U^{\rho,\alpha} f(U)$$

Hence we have the following theorem :

Theorem 1. *When* X_1 *and* X_2 *are independently distributed p × p positive definite real matrix random variables and when* $X_2 = V$ *and* $U = X_2^{1/2}X_1X_2^{1/2}$ *or* $X_1 = V^{-1/2}UV^{-1/2}$ *and when* X_1 *has a real matrix-*

variate type-1 beta distribution with the parameters $\left(\rho+\frac{p+1}{2},\alpha\right)$ *and if g(U) denotes the density of U then*

$$\frac{\Gamma_p\left(\rho+\frac{p+1}{2}\right)}{\Gamma_p\left(\alpha+\rho+\frac{p+1}{2}\right)}g(U)=K_U^{\rho,\alpha}f(U) \quad ...(2.2)$$

is Kober fractional integral operator of the second kind for the real matrix-variate case.

As a special case of (2.2), or independently, we can derive a result for the right sided Weyl operator for the real matrix-variate case. Let the right sided Weyl fractional integral operator be denoted by $\left(xW_{\infty}^{-\alpha}f\right)(X)$ with infinity here signifying that $T-X$ is positive definite.

Theorem 2. *Let* X_1,X_2,U,V *be as defined in Theorem 1. Let* X_1 *have a type-1 beta density with the parameters* $\left(\frac{p+1}{2},\alpha\right)$. *Let the arbitrary density of* X_2 *be denoted by* $f_2(X_2)=|X_2|^{\alpha}f(X_2)$, *where* $f(X_2)$ *is arbitrary. Let the density of U be again denoted by g (U). Then*

$$\left(XW_{\infty}^{-\alpha}f\right)(X)=\frac{1}{\Gamma(\alpha)}\int_{V>U}|V-U|^{\alpha-\frac{p+1}{2}}f(V)\,dV$$

$$=\frac{\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\alpha+\frac{p+1}{2}\right)}g(U),\quad \mathbf{R}(a)>\frac{p-1}{2} \quad ...(2.3)$$

2.1. A pathway generalization of Kober operator of the second kind in the matrix case

A pathway generalization can be considered. Let X_1 have a special case of a pathway density as follows :

$$f_1(X_1)=C_1|X_1|^{\delta}|I-a(1-q)X|^{\frac{\beta}{1-q}} \quad ...(2.4)$$

for $I-a(1-q)X>O,\ q<1,\beta>0,\ a>0$ where C_1 can be seen to be the following :

$$C_1=\frac{[a(1-q)]^{p\delta+\frac{p(p+1)}{2}}\Gamma_p\left(\delta+\frac{\beta}{1-q}+(p+1)\right)}{\Gamma_p\left(\delta+\frac{p+1}{2}\right)\Gamma_p\left(\frac{\beta}{1-q}+\frac{p+1}{2}\right)} \quad ...(2.5)$$

Let X_2 have an arbitrary density $f(X_2)$ and let X_1 and X_2 be statistically independently distributed. Let

$U = X_2^{1/2} X_1 X_2^{1/2}$, $X_2 = V$ or $X_1 = V^{-1/2} U V^{-1/2}$ and the Jacobian is $|V|^{-\frac{p+1}{2}}$. Let $g(U)$ be the density of U. Then, going through the earlier steps we have the following :

$$g(U) = C_1 |U|^{\delta} \int_{V > \alpha(1-q)U} |V|^{-\delta - \left(\frac{\beta}{1-q} + \frac{p+1}{2}\right)} |V - a(1-q)U|^{\frac{\beta}{1-q}} f(V)\, dV \qquad \ldots(2.6)$$

Then
$$\Gamma_p\left(\delta + \tfrac{p+1}{2}\right) g(U) = \frac{[a(1-q)]^{p\delta + \frac{p(p+1)}{2}} \Gamma_p\left(\delta + \frac{\beta}{1-q} + \frac{(p+1)}{2}\right)}{\Gamma_p\left(\frac{\beta}{1-q} + \frac{p+1}{2}\right)} |U|^{\delta}$$

$$\times \int_{V > a(1-q)U} |V - a(1-q)U|^{\frac{\beta}{1-q}} |V|^{-\delta - \left(\frac{\beta}{1-q} + \frac{p+1}{2}\right)} f(V)\, dV$$

$$= K_{U,a,q}^{\delta, \frac{\beta}{1-q} + \frac{p+1}{2}} f(U) \qquad \ldots(2.7)$$

where $K_{U,a,q}^{\delta, \frac{\beta}{1-q} + \frac{p+1}{2}} f(U)$ can be called the generalized pathway Kober operator of the second kind in the real matrix-variate case. When the pathway parameter q varies from $-\infty$ to 1 it provides a pathway or a class of operators and all these operators in this pathway class will eventually go to the exponential form. For $a = 1$, $q = 0$, $\frac{\beta}{1-q} = \alpha - \frac{p+1}{2}$ and $\delta = \rho$ we have

$$K_{U,a,q}^{\delta, \frac{\beta}{1-q} + \frac{p+1}{2}} f(U) = K_U^{\rho,\alpha} f(U) \qquad \ldots(2.8)$$

the Kober operator of the second kind as a constant multiple of the density of the product of two matrix-variate independently distributed random variables. Note that when $q \to 1_-$ we can evaluate the limit of $g(U)$ by using the following lemmas :

Lemma 1.
$$\lim_{q \to 1_-} C_1 = \frac{(\alpha\beta)^{p\delta + \frac{p(p+1)}{2}}}{\Gamma_p\left(\delta + \frac{p+1}{2}\right)} \qquad \ldots(i)$$

Proof. Open up each $\Gamma_p(\cdot)$ in C_1 of (2.5) in terms of ordinary gamma functions. Then use the following asymptotic approximation for gamma functions. For $|z| \to \infty$ and δ a bounded quantity

$$\Gamma(z+\delta) \approx \sqrt{2\pi}\, z^{z+\delta-\frac{1}{2}}\, e^{-z} \qquad \dots(ii)$$

This is the first term in the asymptotic series. This term is also known as Stirling's approximation. When $q \to 1_-$ we have $\frac{1}{1-q} \to \infty$ and hence take $|z|$ as $\frac{\beta}{1-q}$ and expand all gammas by using Stirling's approximation to see that C_1 reduces to (*i*) above.

Lemma 2.

$$\lim_{q \to 1_-} |I - a(1-q)X|^{\frac{\beta}{1-q}} = e^{-\alpha\beta\, \mathrm{tr}(X)} \qquad \dots(iii)$$

Proof. Writing the determinant in terms of eigenvalues we have

$$|I - a(1-q)X|^{\frac{\beta}{1-q}} = \prod_{j=1}^{p} \left(1 - a(1-q)\lambda_j\right)^{\frac{\beta}{1-q}} \qquad \dots(iv)$$

where $\lambda_1, \dots, \lambda_p$ are the eigenvalues of X. Now

$$\lim_{q \to 1_-} \left(1 - a(1-q)\lambda_j\right)^{\frac{\beta}{1-q}} = e^{-\alpha\beta\lambda_j} \qquad \dots(v)$$

Hence

$$\lim_{q \to 1_-} \left(1 - a(1-q)X\right)^{\frac{\beta}{1-q}} = \prod_{j=1}^{p} e^{-\alpha\beta\lambda_j} = e^{-\alpha\beta\, \mathrm{tr}(X)}$$

which eatablishes (iii). Now by using lemmas 1 and 2 we have

$$\lim_{q \to 1_-} g(U) = \frac{(\alpha\beta)^{p\delta + \frac{p(1+p)}{2}}}{\Gamma_p\left(\delta + \frac{p+1}{2}\right)} |U|^{\delta}$$

$$\times \int_{V>O} |V|^{-\delta - \frac{p+1}{2}}\, e^{-\alpha\beta tr\left(V^{-1/2} U V^{-1/2}\right)} dV \qquad \dots(2.9)$$

This is the limiting form of the pathway Kober operator of the second kind in this class of pathway operators of the second kind.

In the pathway generalization, one can also replace the parameter a with a constant positive definite matrix A. In this case the model will be written as

$$f_1(X_1) = C_1(A)|X_1|^{\delta}\left|I-(1-q)A^{1/2}X_1A^{1/2}\right|^{\frac{\beta}{1-q}} \qquad ...(2.10)$$

for $q<1,\ A>O,\ X_1>O,\ I-(1-q)A^{1/2}X_1A^{1/2}>O$. The pathway parameter is still q. In this case

$$C_1(A) = \frac{(1-q)^{p\delta+\frac{p(p+1)}{2}}|A|^{\delta+\frac{p+1}{2}}\Gamma_p\left(\delta+\frac{\beta}{1-q}+(p+1)\right)}{\Gamma_p\left(\delta+\frac{p+1}{2}\right)\Gamma_p\left(\frac{\beta}{1-q}+\frac{p+1}{2}\right)} \qquad ...(2.11)$$

Then $g(U)$ of (2.6) goes to the following form, denoted by $g_A(U)$

$$g_A(U) = C_1(A)|A|^{\frac{\beta}{1-q}}|U|^{\delta}$$

$$\times\int_{V^*>O}\left|V^{1/2}A^{-1}V^{1/2}-(1-q)U\right|^{\frac{\beta}{1-q}}|V|^{-\delta-\left(\frac{\beta}{1-q}+\frac{p+1}{2}\right)}f(V)\,dV \qquad ...(2.12)$$

where $\qquad V^* = V^{1/2}A^{-1}V^{1/2}-(1-q)U$

Then one can define a pathway generalized Kober operator of the second kind as

$$K_{U,A,q}^{\delta,\frac{\beta}{1-q}+\frac{p+1}{2}}f(U) = \Gamma_p\left(\delta+\frac{p+1}{2}\right)g_A(U)$$

$$= \frac{(1-q)^{p\delta+\frac{p(p+1)}{2}}|A|^{\delta+\frac{\beta}{1-q}+\frac{p+1}{2}}\Gamma_p\left(\delta+\frac{\beta}{1-q}+(p+1)\right)}{\Gamma_a\left(\frac{\beta}{1-q}+\frac{p+1}{2}\right)}|U|^{\delta}$$

$$\times\int_{V^*>O}\left|V^{1/2}A^{-1}V^{1/2}-(1-q)U\right|^{\frac{\beta}{1-q}}|V|^{-\delta-\left(\frac{\beta}{1-q}+\frac{p+1}{2}\right)}f(V)\,dV \qquad ...(2.13)$$

In this case, as $q\to 1_-$ we have

$$\lim_{q\to 1_-} g_A(U) = \frac{|A|^{\delta+\frac{p+1}{2}}\beta^{p\delta+\frac{p(p+1)}{2}}}{\Gamma_p\left(\delta+\frac{p+1}{2}\right)}|U|^{\delta}$$

$$\times \int_{V>A} |V|^{-\delta-\frac{p+1}{2}} e^{-\beta tr\left(A^{1/2}V^{-1/2}UV^{-1/2}A^{1/2}\right)} f(V)dV \quad ...(2.14)$$

3. M-TRANSFORMS OF KOBER OPERATOR OF THE SECOND KIND

The generalized matrix transform of M-transform is defined and illustrated in Mathai (1997). The M-transform of Kober operator of the second kind is the following :

Theorem 3. *For the Kober operator of the second kind defined in (2.1) the M-transform with parameter is given by*

$$M\left\{K_X^{\rho,\alpha} f(X);s\right\} = \int_{X>O} |X|^{s-\frac{p+1}{2}}\left[\int_{T>X} \frac{|X|^{\rho}}{\Gamma_p(\alpha)}|T-X|^{\alpha-\frac{p+1}{2}}|T|^{-\rho-\alpha} f(T)\,dT\right]dX$$

$$= \frac{\Gamma_p(\rho+s)}{\Gamma(\alpha+\rho+s)} f^*(s),\ \ \boldsymbol{R}(\rho+s) > \frac{p-1}{2},\ \ \boldsymbol{R}(a) > \frac{p-1}{2} \quad ...(3.1)$$

where $f^*(s)$ is the M-transform of $f(X)$.

Proof. Interchanging the integral we have

$$M\left\{K_X^{\rho,\alpha} f(X);s\right\} = \int_{T>O} |T|^{-\rho-\alpha} f(T)\left[\frac{1}{\Gamma_p(\alpha)}\int_{O<X<T} |X|^{\rho+s-\frac{p+1}{2}}|T-X|^{\alpha-\frac{p+1}{2}}\,dX\right]dT$$

Note that $|T-X| = |T|\left|I - T^{-1/2}XT^{-1/2}\right|,\ \ Y = T^{-1/2}XT^{-1/2} \Rightarrow dY = |T|^{-\frac{p+1}{2}}\,dX$

Hence $\int_{X<T} |X|^{\rho-\frac{p+1}{2}}|T-X|^{\alpha-\frac{p+1}{2}}\,dX = |T|^{\alpha+\rho+s-\frac{p+1}{2}}\int_Y |Y|^{\rho+s+\frac{p+1}{2}}|I-Y|^{\alpha-\frac{p+1}{2}}\,dY$

We can evaluate the Y-integral by using real matrix-variate type-1 beta integral

$$\int_{O<Y<I} |Y|^{\rho+s-\frac{p+1}{2}}|I-Y|^{\alpha-\frac{p+1}{2}}\,dY = \frac{\Gamma_p(\rho+s)\Gamma_p(\alpha)}{\Gamma_p(\alpha+\rho+s)}$$

for $R(\alpha) > \frac{p-1}{2}$, $R(\rho + s) > \frac{p-1}{2}$ Now the T-integral gives

$$\int_{T>O} |T|^{s-\frac{p+1}{2}} f(T)\,dT = f^*(s)$$

where $f^*(s)$ is the M-transform of $f(X)$. Hence (3.1) follows. Note that for $p = 1$ the result agrees with that in the scalar case, which is available in the literature, see the example Mathai and Haubold (2008).

From (3.1) for $\rho = 0$ and $R(a) > \frac{p-1}{2}$ we have the special case of the Kober operator of the second kind.

$$K_X^{0,\alpha} f(X) = \frac{1}{\Gamma(\alpha)} \int_{T>X} |T - X|^{\alpha - \frac{p+1}{2}} |T|^{-\alpha} f(T)\,dT \qquad \text{...(3.2)}$$

But the right side of (3.2) is Weyl fractional integral of order α in the matrix case, $X W_\infty^{-\alpha} f(X)$, except for the factor $|T|^{-\alpha}$. The Weyl integral in the matrix case is

$$X W_\infty^{-\alpha} f(X) = \frac{1}{\Gamma_p(\alpha)} \int_{T>X} |T - X|^{\alpha - \frac{p+1}{2}} f(T)\,dT, \quad R(\alpha) > \frac{p-1}{2} \qquad \text{...(3.3)}$$

Hence we have the following corollary.

Corollary 3.1. *The M-transform of the right sided Weyl operator in the real matrix case is given by*

$$M\left\{X W_\infty^{-\alpha} |X|^{-\alpha} f(X); s\right\} = \frac{\Gamma_p(s)}{\Gamma_p(\alpha + s)} f^*(s) \qquad \text{...(3.4)}$$

for $R(s) > \frac{p-1}{2}$, $R(\alpha) > \frac{p-1}{2}$ where $f^*(s)$ is the M-transform of $f(X)$.

The proof is parallel to that in Theorem 3. Let us see whether a Mellin convolution type formula holds for Kober operator of the second kind in the matrix case. Let

$$g(U) = \int_V |V|^{-\frac{p+1}{2}} f_1\left(V^{-1/2} U V^{-1/2}\right) f_2(V)\,dV \qquad \text{...(3.5)}$$

where $f_1(X_1)$ is a type-1 matrix-variate beta density with parameters $\left(\rho + \frac{p+1}{2}, \alpha\right)$. That is,

$$f_1(X_1) = \frac{\Gamma_p\left(\alpha+\rho+\frac{p+1}{2}\right)}{\Gamma_p(\alpha)\Gamma_p\left(\rho+\frac{p+1}{2}\right)} |X_1|^{\rho} |I - X_1|^{\alpha-\frac{p+1}{2}}, \quad O < X_1 < I \qquad \ldots(3.6)$$

for $R(\alpha) > \frac{p-1}{2}$, $R(\rho) > -1$ and $f_1(X_1) = 0$ elsewhere. Substituting (3.6) in (3.5) we have

$$\frac{\Gamma_p\left(\rho+\frac{p+1}{2}\right)}{\Gamma_p\left(\alpha+\rho+\frac{p+1}{2}\right)} g(U) = \frac{1}{\Gamma_p(\alpha)} \int_V |V|^{-\frac{p+1}{2}} |U|^{\rho} |V|^{-\rho} \times \left|I - V^{-1/2} U V^{-1/2}\right|^{\alpha-\frac{p+1}{2}} f(V)\, dV$$

$$= \frac{|U|^{\rho}}{\Gamma_p(\alpha)} \int_V |V|^{-\rho-\alpha} |V - U|^{\alpha-\frac{p+1}{2}} f(V)\, dV$$

$$= \frac{|U|^{\rho}}{\Gamma_p(\alpha)} \int_{V>U} |V - U|^{\alpha-\frac{p+1}{2}} |V|^{-\rho-\alpha} f(V)\, dV$$

$$= K_U^{\rho,\alpha} f(U) \qquad \ldots(3.7)$$

which is the Kober operator of the second kind. Hence we have the following theorem :

Theorem 4. *Kober operator of the second kind with real matrix argument can also be represented as a Mellin convolution type formula*

$$K_X^{\rho,\alpha} f(X) = \int_V |V|^{-\frac{p+1}{2}} f_1\left(V^{-1/2} X V^{-1/2}\right) f_2(V)\, dV$$

where $f_1(X_1)$ *is a type-1 beta density with parameter* $\left(\rho+\frac{p+1}{2}, \alpha\right)$ *and* $f_2(V)$ *is an arbitrary function or arbitrary density if the Kober operator is to be taken as a constant multiple of a statistical density.*

4. Generalization in Terms of Hypergeometric Series for Kober Operator of the Second Kind in the Real Matrix Case :

For introducing hypergeometric series of matrix argument we will need the definitions, notation and lemmas. Hypergeometric functions of matrix argument are defined in terms of matrix-variate Laplace transforms. M-transforms and zonal polynomials. Explicit series form for all cases is available through the definition in terms of zonal polynomials and hence we will define in terms of zonal polynomials.

$${}_rF_s(Z) = {}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;Z)$$

$$= \sum_{k=0}^{\infty}\sum_{K}\frac{(a)_K\cdots(a_r)_K}{(b_1)_K\cdots(b_s)_K}\frac{C_K(Z)}{k!} \qquad \ldots(4.1)$$

where $K = (k_1,\ldots,k_p), k_1+\ldots+k_p = k$ is a partition of $k = 0, 1, 2,\ldots$

$$(a)_K = \prod_{j=1}^{p}\left(a-\frac{j-1}{2}\right)k_j,\ (b)_{k_j} = b(b+1)\ldots(b+k_j-1),\ (b)_0 = 1,\ b \neq 0 \qquad \ldots(4.2)$$

and $C_K(Z)$ is a zonal polynomial of order k and Z is a $p \times p$ matrix. The series is defined for the real and complex matrices. Zonal polynomials are certain symmetric functions of the eigenvalues of Z. In our discussions, Z will be real and positive definite. For more details about zonal polynomials see Mathai, Provost and Hayakawa (1995). The following basic results are needed in our discussions. A standard notation in this area is

$$\Gamma_p(\alpha, K) = \Gamma_p(\alpha)(\alpha)_K \qquad \ldots(4.3)$$

The following basic results are needed in our discussion.

Lemma 3. $$\int_O^I |X|^{\alpha-\frac{p+1}{2}}|I-X|^{\beta-\frac{p+1}{2}}C_K(TX)\,dX = \frac{\Gamma_p(\alpha,K)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta,K)}C_K(T) \qquad \ldots(4.4)$$

for $\boldsymbol{R}(a) > \frac{p-1}{2},\ \boldsymbol{R}(B) > \frac{p-1}{2}$

Lemma 4. For $\boldsymbol{R}(a) > \frac{p-1}{2},\ A > O,\ Y > O$

$$\int_{O<Y<A} |Y|^{\alpha-\frac{p+1}{2}}C_K(ZY)\,dY = \frac{\Gamma_p(\alpha,K)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\alpha+\frac{p+1}{2},K\right)}|A|^{\alpha}C_K(ZA) \qquad \ldots(4.5)$$

Let us assume that all the parameters $a_1,\ldots,a_r,b_1,\ldots,b_s$ are real and positive and let the argument matrices $p \times p$ and positive definite. For $A > O$, let the density of X_1 be

$$f_1(X_1) = \frac{1}{c_f}\,{}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;AX_1)|X_1|^{\rho}|I-X_1|^{\alpha-\frac{p+1}{2}}$$

$$= \frac{1}{c_f} \sum_{k=0}^{\infty} \sum_{K} \frac{(a_1)_K \cdots (a_r)_K}{(b_1)_K \cdots (b_s)_K} \frac{1}{k!} C_K(AX_1)|X_1|^{\rho} |I - X_1|^{\alpha - \frac{p+1}{2}} \quad \ldots(4.6)$$

where the normalizing constant c_f is available by integrating out term by term with the help of Lemma 3. It will be available in terms of $_{r+1}F_{s+1}$. Let $f(X_2)$ be an arbitrary density. As before, let $X_1 = V^{-1/2}UV^{-1/2}$, then denoting the density of U again by $g(U)$ we have

$$g(U) = \int_V f_1\left(V^{-1/2}UV^{-1/2}\right) f(V) |V|^{-\frac{p+1}{2}} dV$$

$$= \frac{1}{c_f} \frac{\Gamma_p\left(\alpha + \rho + \frac{p+1}{2}\right)}{\Gamma_p\left(\rho + \frac{p+1}{2}\right)\Gamma_p(\alpha)} \left(\sum_{k=0}^{\infty} \sum_{K} \frac{(a_1)_K \cdots (a_r)_K}{(b_1)_K \cdots (b_s)_K} \frac{1}{k!} \right.$$

$$\left. \times \int_V \left|V^{-1/2}UV^{-1/2}\right|^{\rho} \left|I - V^{-1/2}UV^{-1/2}\right|^{\alpha - \frac{p+1}{2}} |V|^{-\frac{p+1}{2}} C_K\left(AV^{-1/2}UV^{-1/2}\right) f(V) dV \right) \quad \ldots(4.7)$$

This is the generalization of a constant times the Kober operator of the second kind in the matrix case. For $_rF_s = {}_2F_1$ one has the matrix-variate generalization of a constant times the Saigo operator of the second kind in the real matrix-variate case.

5. Kober Fractional Integral Operators of the First Kind in the Matrix Case :

Definition 5.1. We will give the following definition and notation for Kober fractional integral operator of the first kind in the real matrix-variate case :

$$I_X^{\rho,\alpha} f(X) = \frac{|X|^{-\rho-\alpha}}{\Gamma_p(\alpha)} \int_{V<X} |X - V|^{\alpha - \frac{p+1}{2}} |V|^{\rho} f(V) dV \quad \ldots(5.1)$$

for $R(\rho) > -1,\; R(\alpha) > \frac{p-1}{2}$

Theorem 5. *For* $R(\alpha) > \frac{p-1}{2},\; R(\rho) > -1$ *the M-transform, with parameter s, of Kober operator of the first kind in the real matrix-variate case, is given by*

$$M\left\{I_X^{\rho,\alpha} f(X); s\right\} = \int_{X>O} |X|^{s - \frac{p+1}{2}} \left[\frac{|X|^{-\rho-\alpha}}{\Gamma_p(\alpha)} \int_{V<X} |X - V|^{\alpha - \frac{p+1}{2}} |V|^{\rho} f(V) dV \right] dX$$

$$= \frac{\Gamma_p\left(\rho + \frac{p+1}{2} - s\right)}{\Gamma_p\left(\alpha + \rho + \frac{p+1}{2} - s\right)} f^*(s), \quad \boldsymbol{R}(s) < \boldsymbol{R}(\rho + 1), \quad \boldsymbol{R}(a) > \frac{p-1}{2} \qquad \ldots(5.2)$$

where $f^*(s)$ *is the M-transform of* $f(X)$.

Proof. Integrating out X first we have the X-integral

$$\int_{X>V} |X|^{s-\rho-\alpha-\frac{p+1}{2}} |X-V|^{\alpha-\frac{p+1}{2}} dX = \int_{Y>O} |Y+V|^{s-\rho-\alpha-\frac{p+1}{2}} |Y|^{\alpha-\frac{p+1}{2}} dY, \quad Y = X - V$$

$$= |V|^{s-\rho-\alpha-\frac{p+1}{2}} \int_{Y>O} \left|I + V^{-1/2} Y V^{-1/2}\right|^{s-\rho-\alpha-\frac{p+1}{2}} |Y|^{\alpha-\frac{p+1}{2}} dY$$

Put $Z = V^{-1/2} Y V^{-1/2} \Rightarrow dZ = |V|^{-\frac{p+1}{2}} dY$. Then the X-intergral is

$$|V|^{s-\rho-\frac{p+1}{2}} \int_{Z>O} |Z|^{\alpha-\frac{p+1}{2}} |I+Z|^{-\left(\frac{p+1}{2}+\alpha+\rho-s\right)} dZ$$

$$= |V|^{s-\rho-\frac{p+1}{2}} \frac{\Gamma_p(\alpha)\Gamma_p\left(\frac{p+1}{2} + \rho - s\right)}{\Gamma_p\left(\frac{p+1}{2} + \alpha + \rho - s\right)}$$

for $\boldsymbol{R}(a) > \frac{p-1}{2}$, $\boldsymbol{R}(\rho - s) > -1$ by evaluating the integral by using a type-2 matrix-variate beta integral in the real case. Now, the V-integral becomes

$$\int_{V>O} |V|^{s-\frac{p+1}{2}} f(V)\, dV = f^*(s)$$

Hence

$$M\left\{I_X^{\rho,\alpha} f(X); s\right\} = \frac{\Gamma_p\left(\frac{p+1}{2} + \rho - s\right)}{\Gamma_p\left(\frac{p+1}{2} + \alpha + \rho - s\right)} f^*(s) \qquad \ldots(5.3)$$

for $\boldsymbol{R}(a) > \frac{p-1}{2}$, $\boldsymbol{R}(\rho - s) > -1$. Note that for $\boldsymbol{\rho} = 0$,

$$I_X^{0,\alpha} f(X) = |X|^{-\alpha} \, {}_0D_X^{-\alpha} f(X) \qquad \ldots(5.4)$$

where ${}_0D_X^{-\alpha}$ is the left sided Riemann-Liouville fractional integral for the matrix-variate case. Note that for the scalar case, for $p = 1$,

$$M\left\{I_X^{\rho,\alpha} f(x); s\right\} = \frac{\Gamma(1+\rho-s)}{\Gamma(1+\alpha+\rho-s)} \qquad \text{...(5.5)}$$

for $R(a) > 0,\ R(\rho - s) > -1$ agreeing with the corresponding Mellin transform in the scalar case.

Corollary 5.1. *The M-transform of* $|X|^{-\alpha}\ {}_0D_X^{-\alpha} f(X)$ *is given by*

$$M\left\{|X|^{-\alpha}\ {}_0D_X^{-\alpha} f(X); s\right\} = \frac{\Gamma_p\left(\frac{p+1}{2} - s\right)}{\Gamma_p\left(\frac{p+1}{2} + \alpha - s\right)} f^*(s) \qquad \text{...(5.6)}$$

for $R(\alpha) > \frac{p-1}{2},\ R(s) < 1$

The proof is parallel to that in Theorem 5.

Let us treat a Kober operator of the first kind as a statistical density. Let X_2 have an arbitrary real matrix-variate density $f(X_2)$ and X_1 have a real matrix-variate type-1 beta density with parameters (ρ, α). That is,

$$f_1(X_1) = \frac{\Gamma_p(\rho+\alpha)}{\Gamma_p(\rho)\Gamma_p(\alpha)} |X_1|^{\rho - \frac{p+1}{2}} |I - X_1|^{\alpha - \frac{p+1}{2}},\quad O < X_1 < I \qquad \text{...(5.7)}$$

for $R(\rho) > \frac{p-1}{2},\ R(\alpha) > \frac{p-1}{2}$ and $f_1(X_1) = 0$ elsewhere. Let X_1 and X_2 be statistically independently distributed. Consider the transformation $X_2 = V,\ X_1 = V^{1/2}U^{-1}V^{1/2}$. The Jacobian is given by

$$dX_1 \wedge dX_2 = |V|^{\frac{p+1}{2}} |U|^{-(p+1)} dU \wedge dV$$

The marginal density of U, denoted by $g_r(U)$ where r designates that it is coming from a ratio, is given by

$$g_r(U) = \frac{\Gamma_p(\rho+\alpha)}{\Gamma_p(\rho)\Gamma_p(\alpha)} \int_V \left|V^{1/2}U^{-1}V^{1/2}\right|^{\rho - \frac{p+1}{2}}$$

$$\times\left|I - V^{1/2}U^{-1}V^{1/2}\right|^{\alpha-\frac{p+1}{2}} f(V)|V|^{\frac{p+1}{2}}|U|^{-(p+1)}\,dV$$

$$= \frac{\Gamma_p(\rho+\alpha)}{\Gamma_p(\rho)\Gamma_p(\alpha)}|U|^{-\rho-\alpha}\int_{V<U}|U-V|^{\alpha-\frac{p+1}{2}}|V|^{\rho} f(V)\,dV$$

Therefore $$\frac{\Gamma_p(\rho)}{\Gamma_p(\rho+\alpha)} g_r(U) = \frac{|U|^{-\rho-\alpha}}{\Gamma_p(\alpha)}\int_{V<U}|U-V|^{\alpha-\frac{p+1}{2}}|V|^{\rho} f(V)\,dV$$

$$= I_X^{\rho,\alpha} f(X) \quad \ldots(5.8)$$

This is Kober operator of the first kind in the real matrix-variate case and it can be considered as a constant multiple of a real matrix-variate statistical density.

One can also consider a pathway extension for the real matrix-variate Kober operator of the first kind.

5.2. Pathway Extension of Kober operator of the First Kind in the Matrix Case

Consider the following pathway modified form of the density of X_1. That is,

$$f_1(X_1) = C_2|X_1|^{\delta-\frac{p+1}{2}}|I-a(1-q)X_1|^{\frac{\beta}{1-q}},\quad I-a(1-q)X_1 > O \quad \ldots(5.9)$$

for $q < 1,\ a > 0,\ \beta > 0$ where

$$C_2 = \frac{[a(1-q)]^{p\delta}\,\Gamma_p\left(\delta+\frac{\beta}{1-q}+\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{\beta}{1-q}+\frac{p+1}{2}\right)\Gamma_p(\delta)} \quad \ldots(5.10)$$

Consider the same type of transformation as before : $X_2 = V,\ X_1 = V^{1/2}U^{-1}V^{1/2}$. The marginal density of U, denoted by $g_p(U)$, is given by

$$g_p(U) = C_2\int_V \left|V^{1/2}U^{-1}V^{1/2}\right|^{\delta-\frac{p+1}{2}}\left|I-a(1-q)V^{1/2}U^{-1}V^{1/2}\right|^{\frac{\beta}{1-q}}$$

$$\times f(V)|V|^{\frac{p+1}{2}}|U|^{-(p+1)}\,dV \quad \ldots(5.11)$$

$$= C_2 |U|^{-\delta-\left(\frac{\beta}{1-q}+\frac{p+1}{2}\right)} \int_{U>a(1-q)V} |U-a(1-q)V|^{\frac{\beta}{1-q}} |V|^{\delta} f(V)\,dV$$

Then $$\Gamma_p(\delta)\, g_p(U) = \frac{[a(1-q)]^{p\delta}\,\Gamma_p\left(\delta+\frac{\beta}{1-q}+\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{\beta}{1-q}+\frac{p+1}{2}\right)} |U|^{-\delta-\left(\frac{\beta}{1-q}+\frac{p+1}{2}\right)}$$

$$\times \int_{U>a(1-q)V} |U-a(1-q)V|^{\frac{\beta}{1-q}} |V|^{\delta} f(V)\,dV \quad \text{...(5.12)}$$

The right side of (5.12) is the pathway extension of Kober operator of the first kind. The right side divided by $\Gamma_p(\delta)$ is also a statistical density of a type of ratio of independently distributed matrix-variate random variables.

Note that for $a = 1$, $q = 0, \frac{\beta}{1-q}+\frac{p+1}{2} = \alpha$, (5.12) reduces to the special case (5.7) for $\delta = \rho$. Thus, (5.12) describes a vast family of operators which can all be considered as generalizations of the Kober operator of the first kind in the real matrix-variate case. The limiting form when $q \to 1_{-}$ is available from the structure in (5.11). Note that

$$\lim_{q\to 1_{-}} \left|I-a(1-q)V^{1/2}U^{-1}V^{1/2}\right|^{\frac{\beta}{1-q}} = e^{-\alpha\beta\,\mathrm{tr}\left(V^{1/2}U^{-1}V^{1/2}\right)} \quad \text{...(5.13)}$$

Hence $$\lim_{q\to 1_{-}} g_p(U) = \left(\lim_{q\to 1_{-}} C_2\right) \int_{V>O} |U|^{-\delta-\frac{p+1}{2}} |V|^{\delta}$$

$$\times e^{-\alpha\beta\,\mathrm{tr}\left(V^{1/2}U^{-1}V^{1/2}\right)} f(V)\,dV \quad \text{...(5.14)}$$

where $$\lim_{q\to 1_{-}} C_2 = \frac{(\alpha\beta)^{p\delta}}{\Gamma_p(\delta)}$$

That is, $$\lim_{q\to 1_{-}} g_p(U) = \frac{(\alpha\beta)^{p\delta}}{\Gamma_p(\delta)} |U|^{-\delta-\frac{p+1}{2}}$$

$$\times \int_{V>O} |V|^{\delta} e^{-\alpha\beta\, \mathrm{tr}\left(V^{1/2}U^{-1}V^{1/2}\right)} f(V)\, dV \qquad \ldots(5.15)$$

In this case also one can replace the parameter a in $f_1(X_1)$ by a constant positive definite matrix $A > O$. Then $f_1(X_1)$ denoted by $f_{1A}(X_1)$ can be written as

$$f_{1A}(X_1) = C_2(A)|X_1|^{\delta - \frac{p+1}{2}} \left|I - (1-q)A^{1/2}X_1A^{1/2}\right|^{\frac{\beta}{1-q}}$$

where
$$C_2(A) = \frac{(1-q)^{p\delta}|A|^{\delta}\Gamma_p\left(\delta + \frac{\beta}{1-q} + \frac{p+1}{2}\right)}{\Gamma_p(\delta)\Gamma_p\left(\frac{\beta}{1-q} + \frac{p+1}{2}\right)}$$

Then the density of $U = V^{1/2}U^{-1}V^{1/2}$, denoted by $g_A(U)$, is given by

$$g_A(U) = C_2(A)|U|^{-\delta - \left(\frac{\beta}{1-q} + \frac{p+1}{2}\right)}$$

$$\times \int_{U>(1-q)V^{1/2}AV^{1/2}} |V|^{\delta} \left|U - (1-q)V^{1/2}AV^{1/2}\right|^{\frac{\beta}{1-q}} f(V)\, dV \quad \ldots(5.16)$$

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ON THERMONUCLEAR REACTION RATE INTEGRALS THROUGH PATHWAY MODEL

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ABSTRACT

In a recent paper an alternative simple and straight forward analytic proof has been given by Saxena, Mathai and Haubold [25] for the astrophysical thermonuclear functions which are derived on the basic of Boltzmann- Gibbs statistical mechanics. By employing the pathway model given by Mathai [19] ,we present their extensions and analytic proofs. The pathway model introduced by Mathai [19[includes, among others, the Tsallis statistics, regular beta-2 distribution, F-distribution and Levy models etc. The generalized thermonuclear reaction rate integrals are evaluated in a compact form in terms of G and H-functions by the application of the Mellin-Barnes integral representation of the exponential function . Analytical continuation formula and series expansion formula for these integrals are also derived, which enhances the utility of the derived results in applied problems. The results are obtained in a form suitable for numerical computation. This study will open new frontiers for workers in statistics, special functions, fractional calculus, physics and engineering sciences for conducting interdisciplinary research

Key words: Thermonuclear reaction functions ,nuclear reaction rate,physics of atars H-function , pathway parameter, pathway model,matrix-variate distributions, superstatistics, Tsallis statistics.G-function, H-funcvtion.

Mathematics Subject Classification 2010: 26A33, 44A10, 33C60, 35J10

1. INTRODUCTION

In view of their importance and usefulness in the physics of stars and the Sun ,several generalizations of thermonuclear reaction functions were investigated during the last three decades. The general formalism that describes these reactions exists since a long time and there is a wide consensus about understanding of our relevant physics (Clayton, [3];Rolfs and Rodney, [24]). Mathematical and statistical techniques are used in deriving the closed form representations of thermonuclear reaction rates, see (Critchfield [4],, Haubold and John, [7];Haubold and Kumar [8],Haubold and Mathai, [9,10,11],; Haubold, Mathai and Anderson , ;Anderson [14], Haubold and Mathai, [9], Haubold and Mathai,[12.13] , Saxena, Mathai and Haubold [25]. Motivated by these investigations, we present the generalization of the following integrals governing the nuclear reaction rates ,which are expressed in terms of the four astrophysical thermonuclear functions (Anderson et al, [1]) given below:

$$I_1(z,\nu) ::= \int_0^\infty y^\nu \exp[-y - z/y^{1/2}]dy, \qquad \ldots(1.1)$$

$$I_2(z,d,\nu) := \int_0^d y^\nu \exp[-y - z/y^{-1/2}]dy, \qquad \ldots(1.2)$$

$$I_3(z,t;\nu) := \int_0^\infty y^\nu \exp[-y - z/(y+t)^{1/2}]dy, \qquad \ldots(1.3)$$

$$I_4(z,\delta,b,\nu) := \int_0^\infty y^\nu \exp[-y - by^\delta - z/y^{1/2}]dy, \qquad \ldots(1.4)$$

The pathway model given by Mathai [19] is described below

In order to create a pathway from one functional form to another a pathway parameter has been introduced and a pathway model is created by Mathai [19]. By this model one can proceed from a generalized type-1 beta model to a generalized gamma model ,when the variable is restricted to be positive. More families are available when the variable is allowed to vary on the real line. We note that Mathai deals mainly with matrix-variate distributions and the scalar case is a particular case there. For the real scalar case , the pathway model is described below::

$$f(x) = cx^{\gamma-1}[1 - a(1-\alpha)x^\delta]^{1/(1-\alpha)}, \quad \delta > 0, \; 1 - a(1-\alpha)x^\delta > 0; \; \gamma > 0, \qquad \ldots(1.5)$$

where c is the normalizing factor and α is the pathway parameter. For $\alpha < 1$,the model is a generalized type-1 model in the real case. Whereas, $\alpha = \gamma = \delta = 1$ gives rise to Tsallis statistics for $\alpha < 1$ (Tsallis,[26] and [27]). Further we observe that (1.5) is a model with right tail cut off. When $\alpha > 1$ we may write $1 - \alpha = -(\alpha - 1), \alpha > 1$, so that f(x) assumes the form,

$$f(x) = c\,x^\gamma[1 + a(\alpha - 1)x^\delta]^{-1/(\alpha-1)} \qquad ,\alpha > 1, \qquad \ldots(1.6)$$

which is a generalized beta type-2 model for real x. When $\alpha \to 1$, it reduces to

$$f(x) = cx^{\gamma-1}e^{-ax^\delta}, x > 0 \qquad \ldots(1.7)$$

This includes generalized gamma, gamma, exponential , chisquare , Weibull, Maxwell-Boltzmann, Rayleigh, and related models (see, Honerkamp, [15], Mathai ,[19]).

It is also shown by Mathai and Haubold [21] that it also includes the Tsallis statistics (Tsallis,[26]) and superstatistics introduced by Beck and Cohen [2].The normalizing constant c for the three cases can be obtained by setting $u = a(1-\alpha)x^{\delta}$ *for* $\alpha < 1; u = a(\alpha-1)x^{\delta}$ *for* $\alpha > 1; \alpha = ax^{\delta}$ *for* $\alpha \to 1$ and then integrating with the help of a type-1 beta integral, type-2 beta integral and gamma integral, respectively. The value of c obtained is given below:

$$c = \frac{\delta[a(1-\alpha)]^{\gamma/\delta}\,\Gamma(\gamma/\delta+1/(1-\alpha)+1)}{\Gamma(\gamma/\delta)\,\Gamma(1/(1-\alpha)+1)} \quad for \quad \alpha<1 \qquad \ldots(1.8)$$

$$= \frac{\delta[a(\alpha-1)]^{\gamma/\delta}\,\Gamma(1/(\alpha-1))}{\Gamma(\gamma/\delta)\,\Gamma(1/(\alpha-1)-\gamma/\delta)} \quad for \quad \alpha>1 \qquad \ldots(1.9)$$

$$= \frac{\delta a^{\gamma/\delta}}{\Gamma(\gamma/\delta)} \quad for \quad \alpha \to 1. \qquad \ldots(1.10)$$

In a recent paper, Saxena et al [25] presented an alternative simple proof of the four astrophysical thermonuclear functions by the application of the Mellin-Barnes integral for the exponential function ,which were established earlier by Haubold and Mathai([10,11], by employing the statistical technique. The same method is applicable in establishing the generalized form of the astrophysical thermonuclear functions studied in earlier paper . It has been shown here that by the application of the pathway model introduced by Mathai [19] ,which is a generalization of Tsallis statistics, ordinary beta-2 distribution, F-distribution etc. and the Mellin-Barnes integral representation for the exponential function , the generalized forms of astrophysical thermonuclear functions are readily developed in terms of H-function. Analytical continuation formula and a series expansion formula for the integrals are also established. The results (2.1) and (2.2) given in the next section are recently given by Haubold and Kumar [8] by following the statistical technique. Our method is general and straightforward.

2. GENERALIZED DEFINITIONS OF ASTROPHYSICAL THERMONUCLEAR FUNCTIONS

It is proposed to establish the following generalized integral formulas for the derivation of the closed-form representation for the astrophysical thermonuclear functions. By introducing the pathway model introduced by Mathai [19] , the following results will be established

It will be shown here that

$$I^{(1)}_{a,\alpha.\gamma,\rho} \overset{def}{=} \int_0^{1/a(1-\alpha)} x^{\gamma-1} e^{-bx^{-\rho}} [1-a(1-\alpha)x]^{\frac{1}{1-\alpha}} dx$$

$$= \frac{\Gamma(1+\frac{1}{1-\alpha})}{\rho[a(1-\alpha)]^{\gamma/\rho}} H^{2,0}_{1,2}\left[a(1-\alpha)b^{1/\rho} \middle| \begin{matrix} (1+\frac{1}{1-\alpha}+\gamma,1) \\ (0,1/\rho),(\gamma,1) \end{matrix} \right], \qquad \ldots(2.1)$$

$(\rho \neq 0)$; $\mathrm{Re}(b) > 0, \mathrm{Re}(\gamma) > 0, \rho > 0, and \quad \alpha < 1$

and

$$I^{(2)}_{a,\alpha.\gamma,\rho} \overset{def}{=} \int_0^{\infty} x^{\gamma-1} e^{-bx^{-\rho}} [1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}} dx$$

$$= \frac{1}{\rho[a(\alpha-1)]^{\gamma}\Gamma\left(\frac{1}{\alpha-1}\right)} H^{2,1}_{1,2}\left[a\{a(\alpha-1)\}b^{1/\rho} \middle| \begin{matrix} (1-\frac{1}{\alpha-1}+\gamma,1), \\ (\gamma,1),(0,\frac{1}{\rho}) \end{matrix} \right] \quad (\alpha < 1) \qquad \ldots(2.2)$$

where

$$\rho \neq 0; \mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, \mathrm{Re}(\gamma) > 0, \rho > 0, \alpha > 1, \mathrm{Re}(\gamma+s) > 0.$$

Proof of (2,1).Let

$$I_1 = \int_0^d x^{\gamma-1} e^{-ax-bx^{-\rho}} dx, \qquad \ldots(2.3)$$

where $d \le \frac{1}{a(1-\alpha)}$.

Replace e^{-ax} by $[1-a(1-\alpha)x]^{\frac{1}{1-\alpha}}$ with the observation that as

$$\alpha \to 1, \ [1-a(1-\alpha)x]^{\frac{1}{1-\alpha}} \to e^{-ax}.$$

Let us denote the integral in this case by $I^{(1)}_{a,\alpha.\gamma,\rho}$. Then

$$I^{(1)}_{a,\alpha.\gamma,\rho} = \int_0^{1/a(1-\alpha)} x^{\gamma-1}[1-a(1-\alpha)x]^{1/(1-\alpha)} \exp(-bx^{-\rho})dx, \qquad \ldots(2.4)$$

where

$$\mathrm{Re}(\gamma) > 0, \rho > o; \alpha < 1, \mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0;\ 1 - a(1-\alpha)x > 0.$$

Using the fundamental Mellin-Barnes integral for the exponential function

$$\exp(\text{-z}) = \int_L \Gamma(-s) z^s ds, \quad |z| < \infty \qquad \ldots (2.5)$$

We find that

$$I_{\alpha,1}^{(d)} = \int_0^{1/a(1-\alpha)} x^{\gamma-1}[1 - a(1-\alpha)x]^{1/(1-\alpha)} \frac{1}{2\pi i} \int_L \Gamma(-s)(bx^{-\rho})^s \, ds dx \qquad \ldots (2.6)$$

Interchanging the order of integration and making the substitution $a(1-\alpha)x = u$, it gives

$$I_{a,\alpha,\gamma,\rho}^{(1)} = \frac{1}{[a(1-\alpha)]^{\gamma}} \frac{1}{2\pi i} \int_L \Gamma(-s)[a^{\rho}(1-\alpha)^{\rho} b]^s \int_0^1 u^{\gamma-\rho s-1}(1-u)^{1/(1-\alpha)} du ds$$

$$= \frac{\Gamma(1+\frac{1}{1-\alpha})}{[a(1-\alpha)]^{\gamma}} \frac{1}{2\pi i} \int_L \frac{\Gamma(-s)\Gamma(\gamma-\rho s)}{\Gamma(1+\frac{1}{1-\alpha}+\gamma-\rho s)} [a^{\rho}(1-\alpha)^{\rho} b]^s \, ds. \qquad \ldots (2.7)$$

On interpreting it with the help of the definition of the H-function given in (2.10), we finally obtain

$$I_{a,\alpha.\gamma,\rho}^{(1)} = \frac{\Gamma(1+\frac{1}{1-\alpha})}{[a(1-\alpha)]^{\gamma}} H_{1,2}^{2,0}\left[a^{\rho}(1-\alpha^{\rho} b \left| \begin{matrix} (1+\frac{1}{1-\alpha}+\gamma,\rho) \\ (0.1).(\gamma,\rho) \end{matrix} \right. \right], \alpha < 1. \qquad \ldots (2.8)$$

which on applying the following property of the H-function (Mathai and Saxena,1978)

$$H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_p,\delta A_p) \\ (b_q,\delta B_q) \end{matrix} \right. \right] = \frac{1}{\delta} H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix} \right. \right], \delta > 0, \qquad \ldots (2.9)$$

yields the desired result. The H-function occurring in the above results is defined by means of a Mellin- Barnes type integral in the following manner (Mathai and Saxena,and Haubold , [23]):

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix} \right. \right]$$

$$= H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_1,A_1),\ldots,(a_p,A_p) \\ (b_1,B_1),\ldots,(b_q,B_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\Omega} \Theta(\xi) z^{-\xi} d\xi, \qquad \ldots (2.10)$$

where $i = (-1)^{1/2}$,

$$\Theta(\xi) = \frac{\left[\prod_{j=1}^{m}\Gamma(b_j + B_j\xi)\right]\left[\prod_{i=1}^{n}\Gamma(1-a_j-A_j\xi)\right]}{\left[\prod_{j=m+1}^{q}\Gamma(1-b_j-B_j\xi)\right]\left[\prod_{i=n+1}^{p}\Gamma(a_j+A_j\xi)\right]} \quad \ldots (2.11)$$

and an empty product is always interpreted as unity ; $m, n, p, q \in N_0$ with

$0 \le n \le p$, $1 \le m \le q$, $A_i, B_j \in R_+, a_i, b_j \in R$ or C (i=1,...,p ; j=1,...,q) such hat

$$A_i(b_j + k) \neq B_j(a_i - \ell - 1), (k, \ell \in N_0;\ i=1,\ldots,n\ ; j=1,\ldots,m\ ), \quad \ldots (2.12)$$

where we employ the usual notations: N_0 = (0,1,2...); R = $(-\infty,\infty)$,

$R_+ = (0,\infty)$. and C being the complex number field. The contour Ω is either $L_{-\infty}, L_{+\infty}$ or $L_{i\gamma\infty}$. For the explicit definitions of these contours, see Kilbas and Saigo,[16]). A detailed and comprehensive account of the H-function is available from the monograph written by Mathai , Saxena and Haubold [23](,and Kilbas and Saigo [16].

It may be noted that as $\alpha \to 1$, then by virtue of the limit formula for the gamma function ,we obtain the following result given by Haubold and Mathai [13]:

$$I^{(1)}_{a,1.\gamma,\rho} = \frac{1}{\rho a^{\gamma}} H^{2,0}_{0,2}\left[ab^{1/\rho}\Big|^{--}_{(0.1/\rho).(\gamma,1)}\right], \quad \ldots (2.13)$$

Proof of (2,2).Let

$$I_2 = \int_0^{\infty} x^{\gamma-1} e^{-ax-bx^{-\rho}} dx,$$

where Re(a)>0, Re(b)> 0 and $\mathrm{Re}(\rho) > 0$. ... (2.14)

Replacing $\exp(-ax)$ by

$$[1 + a(\alpha - 1)x]^{-\frac{1}{\alpha-1}} \quad \ldots (2.15)$$

and noting that as $\alpha \to 1$, (2.15) becomes $\exp(-ax)$, the integral I_2 transforms into the following form ,denoted by $I^{(2)}_{a,\alpha.\gamma,\rho}$;

$$I^{(2)}_{a,\alpha,\gamma,\rho} = \int_0^\infty x^{\gamma-1} e^{-bx^{-\rho}} [1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}} dx, \qquad \ldots (2.16)$$

Applying the integral formula (2.5) and interchanging the order of integration , we find that

$$I^{(2)}_{a,\alpha,\gamma,\rho} = \frac{1}{2\pi i} \int_C \Gamma(-s) b^s \int_0^\infty x^{\gamma-\rho s-1} [1+a(\alpha-1)x]^{-\frac{1}{\alpha-1}} dx ds \qquad \ldots (2.17)$$

To evaluate the x-integral , we set $a(\alpha-1)x = t$, then we obtain

$$I^{(d)}_{a,\alpha,\gamma,\rho,2} = \frac{1}{[a(\alpha-1)]^\gamma 2\pi i} \int_C \Gamma(-s) b^s \int_0^\infty t^{\gamma-\rho s-1} [1+t]^{-\frac{1}{\alpha-1}} dt ds \qquad \ldots (2.18)$$

The t-integral is a beta type-2 integral, consequently, we finally obtain

$$I^{(2)}_{a,\alpha,\gamma,\rho} = \frac{1}{\rho[a(\alpha-1)]^\gamma} \frac{1}{\Gamma\left(\frac{1}{\alpha-1}\right)} \int_C \Gamma(-s/\rho) \Gamma\left(\frac{1}{\alpha-1} - \gamma + s\right) \Gamma(\gamma - s) [ab^{1/\rho}(\alpha-1)]^s ds, \qquad \ldots (2.19)$$

where $\text{Re}(\gamma+\rho s) > 0$, $\text{Re}(\frac{1}{\alpha-1} - \gamma - \rho s) > 0, \text{Re}(s) > 0$. which , by virtue of the result (2.10) , can be expressed in terms of the H-function in the form

$$I^{(2)}_{a,\alpha,\gamma,\rho} = \frac{1}{\rho[a(\alpha-1)]^\gamma} \frac{1}{\Gamma\left(\frac{1}{\alpha-1}\right)} H^{2,1}_{1,2}\left[\{a(\alpha-1)\} b^{1/\rho} \middle| \begin{matrix} (1-\frac{1}{\alpha-1}+\gamma,1) \\ (\gamma,1),(0,1/\rho) \end{matrix} \right] \quad (\alpha > 1) \qquad \ldots (2.20)$$

This completes the proof of the results (2.1) and (2.2).It is interesting to observe that as $\alpha \to 1$, (2.20) reduces to a result given by Haubold and Mathai (1998 b)

$$I^{(2)}_{a,1,\gamma,\rho} = \frac{1}{\rho a^\gamma} H^{2,0}_{0,2}\left[ab^{1/\rho} \middle| \begin{matrix} - \\ (\gamma,1),(0,1/\rho) \end{matrix} \right], \qquad \ldots (2.21)$$

where $\text{Re}(a) > 0, \text{Re}(b) > 0, \text{Re}(\gamma) > 0$ and $\text{Re}(\rho) > 0$.

3. SPECIAL CASES

If we set $\frac{1}{\rho} = m$, where m is a positive integer, then by the application of the multiplication formula for the gamma functions, the result (2.1) and (2.2) simplify in terms of the G-function (Mathai and Saxena, [22];Mathai [19] in the form

$$I^{(1)}_{a,\alpha,\frac{1}{m},b,\gamma} = \frac{\sqrt{m}(2\pi)^{(1-m)/2}\Gamma(1+\frac{1}{1-\alpha})}{[a(1-\alpha)]^{\gamma}} G^{m+1,0}_{1,m+1}\left[\frac{a(1-\alpha)b^m}{m^m}\middle| \begin{matrix} 1+\gamma+\frac{1}{1-\alpha} \\ \Delta(0,m),\gamma \end{matrix}\right],(\alpha<1) \quad ...(3.1)$$

where $\mathrm{Re}(a)>0, \mathrm{Re}(b)>0, \mathrm{Re}(\gamma)>0, \alpha<1$ and $\Delta(a,m)$ represents the sequence of m parameters

$$\frac{a}{m}, \frac{a+1}{m}, ..., \frac{a+m-1}{m} (m \in N_0)$$

and

$$I^{(2)}_{a,\alpha,\frac{1}{m},b,\gamma} = \frac{\sqrt{m}(2\pi)^{(1-m)/2}}{[a(1-\alpha)]^{\gamma}\Gamma\left(\frac{1}{\alpha-1}\right)} G^{m+1,1}_{1,m+1}\left[\frac{a(1-\alpha)b^m}{m^m}\middle| \begin{matrix} 1+\gamma-\frac{1}{1-\alpha} \\ \Delta(o,m),\gamma \end{matrix}\right],(\alpha>1) \quad ...(3.2)$$

where $\mathrm{Re}(a)>o, \mathrm{Re}(b)>0, \mathrm{Re}(\gamma)>0, \alpha>1$,respectively.

For the non-resonant thermonuclear reactions with high energy cut off $a=1, \rho=\frac{1}{2}, \gamma=1+\nu$,(3.1) gives

$$I^{(1)}_{1,\alpha,\frac{1}{2},b,1+\nu} = \frac{\Gamma(1+\frac{1}{1-\alpha})}{\sqrt{\pi}(1-\alpha)^{\nu+1}} G^{3,0}_{1,3}\left[\frac{(1-\alpha)b^2}{4}\middle| \begin{matrix} \nu+\frac{1}{1-\alpha}+2 \\ 0,\frac{1}{2},\nu+1 \end{matrix}\right], \quad ...(3.3)$$

where $\mathrm{Re}(b)>0, \mathrm{Re}(\nu)>0, \alpha<1$.

In a similar manner , in case of the probability integral for a non-resonant thermonuclear reaction in the Maxwell- Boltzmannian form, from (3.2) for

$$a=1, \rho=\frac{1}{2}, \gamma=1+\nu,$$

one obtains

$$I^{(2)}_{1,\alpha,\frac{1}{2},b,1+\nu} = \frac{1}{\sqrt{\pi}(1-\alpha)^{\nu+1}\Gamma\left(\frac{1}{\alpha-1}\right)} G^{3,1}_{1,3}\left[\frac{(\alpha-1)b^2}{4}\middle|\begin{matrix}\nu-\frac{1}{\alpha-1}+2\\ 0,\frac{1}{2},\nu+1\end{matrix}\right], \quad \ldots(3.4)$$

where $\mathrm{Re}(b)>0, \mathrm{Re}(\nu)>0, \alpha>1$.

4. GENERALIZED FORM OF THE ASTROPHYSICAL THERMONUCLEAR FUNCTIONS $I_3(z,t,\nu,\mu,\alpha)$

It will be shown here that

$$I_3(z,t,\nu,\mu,\alpha) \overset{def}{=} \int_0^\infty y^{\nu-1}[1+(\alpha-1)y]^{-\frac{1}{\alpha-1}} \exp[-z(y+t)^{-\mu}]dy$$

$$= \frac{1}{\left(\frac{1}{\alpha-1}\right)} \sum_{r=0}^{\infty} \frac{[-\frac{1}{t(\alpha-1)}]^r}{r!} H^{2,2}_{3,2}\left[\frac{t^\mu}{z(\alpha-1)}\middle|\begin{matrix}(1,1),(1-r,1),(0,\mu)\\ (r,\mu),(\frac{1}{\alpha-1}-r,1)\end{matrix}\right] \quad \ldots(4.1)$$

$$= \frac{1}{\left(\frac{1}{\alpha-1}\right)} \sum_{r=0}^{\infty} \frac{[-\frac{1}{t(\alpha-1)}]^r}{r!} H^{2,2}_{2,3}\left[\frac{z(\alpha-1)}{t^\mu}\middle|\begin{matrix}(1-r,\mu),(1-\frac{1}{\alpha-1}+r,1)\\ (0,1),(r,1),(1,\mu)\end{matrix}\right] \quad \ldots(4.2)$$

where $\mathrm{Re}(\nu)>0, \mathrm{Re}(z)>0, \alpha>1$ *and* $\mu>0$.

To prove (4.1), we see that in view of the formula (2.5) , the value of the integral is equal to

$$\frac{1}{3\pi i}\int_L \Gamma(s)z^{-s} \int_0^\infty y^{s-1}[1+(\alpha-1)y]^{-\frac{1}{\alpha-1}}(y+t)^{s\mu}\,dyds \quad \ldots(4.3)$$

Upon using the formula

$$(1-x)^{-\alpha} = \sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!}x^r, |x|<1, \quad \ldots(4.4)$$

the above expression (4.3) becomes

$$\frac{1}{2\pi i}\int_L \Gamma(s)z^{-s}\int_0^\infty y^{s-1}[1+(\alpha-1)y]^{-\frac{1}{\alpha-1}} t^{s\mu}\sum_{r=0}^{\infty}\frac{(-y/t)^r(-s\mu)_r}{r!}dyds, \quad |y/t|<1, \quad \ldots(4.5)$$

which on changing the order of integration and summation yields

$$\frac{1}{2\pi i}\int_L \Gamma(s) z^{-s} t^{s\mu} \sum_{r=0}^{\infty} \frac{(-1/t)^r (-s\mu)_r}{r!} \int_0^{\infty} y^{r+s-1} [1+(\alpha-1)y]^{-\frac{1}{\alpha-1}} dy ds, \quad |\, y/t \,| < 1 \qquad \ldots (4.6)$$

On evaluating the y-integral by type-2 beta function formula, namely

$$\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m,n), \text{Re}(m) > 0, \quad \text{Re}(n) > 0, \qquad \ldots (4.7)$$

where Re(m)>0,Re(n)>0,the result (4.6) gives

$$\frac{1}{\Gamma\left(\frac{1}{\alpha-1}\right)} \sum_{r=0}^{\infty} \frac{(-y/t[\alpha-1])^r}{r!} \int_L \frac{\Gamma(s)\Gamma(r+s)\Gamma(r-s\mu)\Gamma(\frac{1}{\alpha-1}-r-s)}{\Gamma(-s\mu)} \left(\frac{t^{\mu}}{z(\alpha-1)}\right)^s ds, \qquad \ldots (4.8)$$

which on interpreting with the help of (2.10) gives the desired result (4.1). By using the transformation formula for the H-function (Mathai and Saxena,1978), the result (4.2) readily follows.

5. GENERALIZED FORM OF THE ASTROPHYSICAL THERMONUCLEAR FUNCTIONS $I_4(z,\delta,b;\nu,\alpha)$

Next, we prove the formula

$$I_4(z,\delta,b;\nu,\alpha) \overset{def}{=} \int_0^{\infty} y^{\nu-1} [1+(\alpha-1)]^{-\frac{1}{\alpha-1}} \exp[-\{by^{\delta} + zy^{-1/2}\}] dy \qquad \ldots (5.1)$$

$$= [(\alpha-1)^{\nu} \Gamma\left(\frac{1}{\alpha-1}\right)]^{-1} \sum_{r=0}^{\infty} \frac{1}{r!} \left[-\frac{b}{z(\alpha-1)^{\delta+1/2}}\right]^r H_{1.2}^{2,1}\left[z(\alpha-1)^{1/2} \left| \begin{matrix} (1+\nu-\frac{1}{\alpha-1}+(\delta+\frac{1}{2})r, \frac{1}{2}) \\ (\nu+(\delta+\frac{1}{2})r, \frac{1}{2}), (r.1) \end{matrix} \right.\right], \qquad \ldots (5.2)$$

where $\text{Re}(\nu) > 0, \text{Re}(b) > 0, \text{Re}(z) > 0 \, and \quad \delta > 0.$

Proof. By virtue of the result (2.5),the integral formula (5.1) can be written as

$$I_4(z,t,\nu,\mu,\alpha) = \int_0^{\infty} y^{s-1} [1+(\alpha-1)y]^{-\frac{1}{\alpha-1}} \frac{1}{2\pi i} \int_L \Gamma(s) y^{s/2} (by^{\delta+1/2} + z)^{-s} ds dy$$

On employing the formula (4.4) and reversing the order of integration and summation, the above expression becomes

$$I_4(z,t,\nu,\mu,\alpha)=\sum_{r=0}^{\infty}\frac{(-b/.z)^r}{r!}\frac{1}{2\pi i}\int_L z^{-s}\Gamma(r+s)\int_0^{\infty} y^{\nu+(\delta+\frac{1}{2})r+\frac{s}{2}-1}[1+(\alpha-1)y]^{-\frac{1}{\alpha-1}}\,\text{dyds} \quad \ldots(5.3)$$

$$=\frac{1}{(\alpha-1)^{\nu}}\sum_{r=0}^{\infty}\frac{\left[-b/\{z(\alpha-1)^{\delta+1/2}\}\right]^r}{r!}$$

$$\times\frac{1}{2\pi i}\int_L \{z(\alpha-1)^{1/2}\}^{-s}\Gamma(r+s)\int_0^{\infty} u^{\nu+(\delta+\frac{1}{2})r+\frac{s}{2}-1}[1+u]^{-\frac{1}{\alpha-1}}duds,$$

which on applying type-2 beta function formula ,the above expression yields

$$I_4(z,t,\nu,\mu,\alpha)=\left[(\alpha-1)^{\nu}\Gamma\left(\frac{1}{\alpha-1}\right)\right]^{-1}\sum_{r=0}^{\infty}\frac{\left[-b/\{z(\alpha-1)^{\delta+1/2}\}\right]^r}{r!}$$

$$\times\frac{1}{2\pi i}\int_L \{z(\alpha-1)^{1/2}\}^{-s}\Gamma(r+s)\Gamma\left[\nu+(\delta+\frac{1}{2})r+\frac{s}{2}\right]\Gamma\left[\frac{1}{\alpha-1}-\nu-(\delta+\frac{1}{2})r-\frac{s}{2}\right]ds, \quad \ldots(5.4)$$

which on being interpreted with the help of the definition of the H-function (2.10) establishes the result (5.1).It may be noted that the result (5.4) can be expressed in terms of the G-function , see (Mathai and Saxena,[22]) in the form

$I_4(z,t,\nu,\mu,\alpha)$

$$=\left[(\alpha-1)^{\nu}\Gamma\left(\frac{1}{\alpha-1}\right)\right]^{-1}\sum_{r=0}^{\infty}\frac{1}{r!}\left[-2b/\{z(\alpha-1)^{\delta+1/2}\}\right]^{r}G_{1,3}^{3,1}\left[\frac{z^2(\alpha-1)}{4}\middle|\begin{matrix}1+\nu-\frac{1}{\alpha-1}+(\delta+\frac{1}{2})r\\ \frac{r}{2},\frac{r+1}{2},\nu+(\delta+\frac{1}{2})r\end{matrix}\right] \quad \ldots(5.5)$$

6. SERIES EXPANSION AND ANALYTICAL CONTINUATION FORMULA FOR THE FUNCTION $I_{a,\alpha,\gamma,\rho}^{(1)}$.

We have
$$I_{a,\alpha,\gamma,\rho}^{(1)}=\frac{1}{\rho[a(\alpha-1)]^{\gamma}\Gamma\left(\frac{1}{\alpha-1}\right)}H_{1,2}^{2,1}\left[a(\alpha-1)b^{1/\rho}\middle|\begin{matrix}(1-\frac{1}{\alpha-1}+\gamma,1)\\(\gamma,1),(0,1/\rho)\end{matrix}\right] \quad \ldots(6.1)$$

$$= \frac{1}{\rho[a(\alpha-1)]^{\gamma}\Gamma\left(\frac{1}{\alpha-1}\right)} \frac{1}{2\pi i}\int_L \Gamma(\gamma-s)\Gamma(-s/\rho)\Gamma(\frac{1}{\alpha-1}-\gamma+s)[a(\alpha-1)b^{1/\rho}]^s\, ds. \qquad \ldots(6.2)$$

Assuming that the poles of the integrand of (6.2) are simple, then for calculating the sum of the residues at the poles of the integrand of (6.2) at the points given by $s=\gamma+\nu(\nu\in N_0)$ and $s=\rho\nu(\nu\in N_0)$, it is found that

$$I^{(1)}_{a,\alpha,\gamma,\rho} = \frac{1}{\rho\Gamma\left(\frac{1}{\alpha-1}\right)}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(\frac{-\nu-\gamma}{\rho})\Gamma(\frac{1}{\alpha-1}+\nu)\left[a(1-\alpha)b^{1/\rho}\right]^{\nu}$$

$$+\left[\{a(\alpha-1)\}^{\gamma}\Gamma\left(\frac{1}{\alpha-1}\right)\right]^{-1}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(\gamma-\rho\nu)\Gamma(\frac{1}{\alpha-1}-\gamma+\rho\nu)\left[a(\alpha-1)b^{1/\rho}\right]^{\rho\nu}, \qquad \ldots(6.3)$$

which holds for $\alpha>1, |a(\alpha-1)b^{1/\rho}|<1$. On the other hand, if we calculate the residues at the poles $s=\gamma-\nu-\frac{1}{\alpha-1}$ $(\nu\in N_0)$ of the integrand of (6.2) ,we obtain the following analytical continuation formula

$$I^{(1)}_{a,\alpha,\gamma,\rho} = \frac{1}{\rho[a(\alpha-1)]^{\gamma}\Gamma\left(\frac{1}{\alpha-1}\right)} H^{2,1}_{1,2}\left[a(\alpha-1)b^{1/\rho}\left|\begin{matrix}(1-\frac{1}{\alpha-1}+\gamma,1)\\(\gamma,1),(0,1/\rho)\end{matrix}\right.\right]$$

$$= \frac{b^{\frac{1}{\rho}(\gamma-\frac{1}{\alpha-1})}}{\rho[a(\alpha-1)]^{\frac{1}{\alpha-1}}\Gamma\left(\frac{1}{\alpha-1}\right)}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(\frac{1}{\alpha-1}+\nu)\Gamma[\frac{1}{\rho(\alpha-1)}-\frac{\gamma}{\rho}+\frac{\nu}{\rho})\left[a(1-\alpha)b^{1/\rho}\right]^{-\nu} \qquad \ldots(6.4)$$

$$= \frac{b^{\frac{1}{\rho}(\gamma-\frac{1}{\alpha-1})}}{\rho[a(\alpha-1)]^{\frac{1}{\alpha-1}}\Gamma\left(\frac{1}{\alpha-1}\right)}\,{}_2\Psi_0\left[\left.\begin{matrix}(\frac{1}{\alpha-1},1),(\frac{1}{\rho(\alpha-1)}-\frac{\gamma}{\rho},\frac{1}{\rho})\\ ---\end{matrix}\right|\frac{-1}{a(\alpha-1)b^{1/\rho}}\right], \qquad \ldots(6.5)$$

where $\alpha>1, |a(\alpha-1)b^{1/\rho}|>1$. and ${}_2\Psi_0(.)$ is the Fox-Wright generalized hypergeometric function, defined by (Erdélyi et al, [6],p.183; Wright ([27], [28]),Mathai , Saxena and Haubold, [23] Also see [17]. [18])],

$$ {}_p\Psi_q\left[\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\bigg| z\right] = \frac{1}{2\pi i}\int_L \frac{\{\Gamma(s)\}\{\prod_{j=1}^{p}\Gamma(a_j - A_j s)\}}{\{\prod_{j=1}^{q}\Gamma(b_j - B_j s)\}}(-z)^{-s}\,ds, \qquad \ldots (6.6) $$

where the path of integration L separates all the poles at $s = -\ell(\ell \in N_0)$ to the left and all the poles at $s = (a_r + n_r)/A_r\ (r = 1,\ldots,p; n_r \in N\ )$ to the right. If $L = (\xi - i\infty, \xi + i\infty)(\xi \in R)$, then the representation holds if either of the following conditions are satisfied (Kilbas et al, 2006):

$$ \theta < 1;\ |\arg(\text{-z})| < \frac{(1-\theta)\pi}{2}, z \neq 0;\ (\theta = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j); $$

or,
$$ \theta = 1, (\theta + 1)\xi + \frac{1}{2} < \text{Re}(\phi), \arg) - z) = 0, z \neq 0. $$

7. EXPANSION OF THE GENERALIZED ASTROPHYSICAL THERMONUCLEAR FUNCTION $I^{(2)}_{a,\alpha.\gamma,\rho}$

We have

$$ I^{(d)}_{a,\alpha.\gamma,\rho,2} = \frac{\Gamma\left(1+\frac{1}{1-\alpha}\right)}{\rho[a(\alpha-1)]^{\gamma}} H^{2,0}_{1,2}\left[a(\alpha-1)b^{1/\rho}\left|\begin{matrix}(1+\frac{1}{1-\alpha}+\gamma,1)\\(\gamma,1),(0,1/\rho)\end{matrix}\right.\right] (\alpha < 1) \qquad \ldots (7.1) $$

$$ = \frac{\Gamma\left(1+\frac{1}{1-\alpha}\right)}{\rho[a(\alpha-1)]^{\gamma}}\frac{1}{2\pi i}\int_L \frac{\Gamma(\gamma - s)\Gamma(-s/\rho)}{\Gamma(1+\gamma+\frac{1}{1-\alpha}-s)}[a(1-\alpha)b^{1/\rho}]^s\,ds, \qquad \ldots (7.2) $$

where it is assumed that the poles of the integrand of (7.2) are simple. On evaluating the residues at the poles of the gamma function $\Gamma(\gamma - s)$ at the points $s = \gamma + \nu\ (\nu \in N_0)$ and the gamma function $\Gamma(-s/\rho)$ at the points $s = \nu\rho(\nu \in N_0)$, we find that

$$ I^{(2)}_{a,\alpha.\gamma,\rho} = \frac{\Gamma\left(1+\frac{1}{1-\alpha}\right)b^{\gamma/\rho}}{\rho}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\{\Gamma(\frac{-\nu-\gamma}{\rho})/\Gamma(1+\frac{1}{\alpha-1}-\nu)\}\left[a(1-\alpha)b^{1/\rho}\right]^{\nu} \qquad \ldots (7.3) $$

$$+\ \frac{\Gamma\left(1+\frac{1}{1-\alpha}\right)}{[a(1-\alpha)]^{\gamma}}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\{\Gamma(\gamma-\rho\nu)/\Gamma(1+\frac{1}{\alpha-1}+\gamma-\rho\nu)\}\left[a(1-\alpha)b^{1/\rho}\right]^{\rho\nu}, \quad \ldots(7.4)$$

where $\alpha<1$ and $|a(1-\alpha)b^{1/\rho}|<1$.

As a concluding remarks, it is observed that the results in this paper are in much better compact forms than the ones obtained in an earlier paper (Saxena et al ,[25]) and are suitable for numerical computation. The reason lies in the fact that the introduction of the parameter α in the pathway model by Mathai [19]) has simplified the results . A further study of the expansions of the integrals given in the paper in logarithmic cases may give some useful results and may form the subject -matter of a future communication.

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SOME GLIMPSES OF FLUID DYNAMICS *

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The earlier investigations in the field of fluid dynamics, the science of motion of liquids and gases together known as fluids, were based on the concept of ***perfect (ideal) fluid*** or inviscid fluid. The mathematical theory of perfect fluids based on ***Euler's equation*** is extremely developed and gives satisfactory explanations for some flow phenomena such as motion of surface waves, the lift and induced drag of an aerofoil or the formation of liquid jets and the like.

However, the theory of perfect fluids fails to explain other phenomena such as skin- friction, form drag of a body, no slippage on the surface of a solid body, flow separation, vortex formation and the like. In order to understand these phenomenon, we have to investigate the flow of ***real fluids*** which are, in general, viscous and compressible. The various types of flow are distinguished according to the fluid properties which characterize the physical situation. The motion of real fluids which may be steady laminar, unsteady laminar, and turbulent can be classified, generally, into two categories known as Newtonian and Non-Newtonian fluid motion.

(I) Newtonian Fluids :

In Newtonian fluids there is a linear relation between the magnitude of applied shear stress and the resulting rate of deformation. If the relation between stress components and the rate of strain components is invariant to orientation of the coordinate axes then the fluid is said to be ***isotropic***. Newtonian fluids have been found to describe adequately the mechanical behavior of many real fluids under a wide range of situations.

From the theoretical point of view, the fundamental equations for the dynamics of real fluids, which are Newtonian and isotropic, are the Navier-Stokes (N-S) equations with no dissipation of energy in a spherically symmetrical expansion or compression, so that there is only one coefficient of viscosity. Due to the non-linear character of these equations only in few special cases exact solutions for steady laminar flow through pipes and

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channels known as ***Hagen Poiseuille flow*** and ***Couette flow*** have been obtained including the problems of ***fully developed*** flow region, ***transpiration cooling*** and the flow in the ***entrance region*** of ducts. Flow in convergent ***and divergent channels (Jeffery-Hamel flow)***, stagnation point flows ***(Himenz and Homann flow)*** and flow due to a rotating disc ***(Kármán flow)*** have also been studied. The well known unsteady flow problems, known as ***Stokes' first and second problems*** have become the basis of numerous generalizations.

Theory of very slow motion :

In order to study the flow past bodies of finite size approximate solutions of the N-S equations have been obtained for the extreme cases, viz.,

(i) when the Reynolds number is very small and

(ii) when Reynolds number is large but less than the critical value at which the flow ceases to be laminar and turns turbulent.

If the Reynolds number is very small the viscous forces will be considerably greater than the inertia forces and as a first approximation the inertia terms may be neglected altogether from the N-S equations to yield Stokes' equations for the theory of slow motion. Solution of these' simplified equations for flow past a sphere and a circular cylinder has been studied which give rise to the concept of ***Stokeslet*** and ***Stokes paradox*** respectively.

An improvement of the Stokes' solution was later given by Oseen, who took the inertia terms partly into account and improved the picture of the flow field. The other important industrial application of the theory of slow motion is the hydrodynamic theory of lubrication.

Theory of boundary layers :

It was Prandtl (1904) who introduced the concept of boundary layer so that the N-S equations were simplified to a mathematically tractable form. Thus succeeded in giving a physically penetrating explanation of the importance of viscosity in the case of thin fluids like water and air, which in earlier investigations were regarded as non-viscous (ideal), in the assessment of frictional drag, flow separation and vortex formation.

Boundary layer flow past a flat plate ***(Blasius flow)*** and near a cuspidal and blunt edge have been extensively studied with the help of ***Prandtl boundary layer equations*** using both analytical and numerical methods. Approximate methods, such as ***Kármán-Pohlhausen*** method based on ***Kármán momentum integral equation*** and other energy integral equations have been used to give quicker, although less accurate but practically acceptable, results.

Jet Flow :

The boundary layer assumptions are not only applicable to regions near a solid wall, but can also be applied when two layers of fluid with different velocities meet. One such example is the jet mixing when a fluid is discharged through a slit or small orifice and mixes with the surrounding fluid which is at rest. Three types of jet flow with the without swirl viz., (i) ***Plane free jet***, (ii) ***Circular free jet*** and (iii) ***plane wall jet*** have been studied extensively. The other example is the region of ***wakes*** behind solid bodies.

Theory of Compressible flow :

In classical hydrodynamics the flow of an ideal incompressible fluid has been extensively investigated. However, the density of some fluids, particularly gases, changes considerably with the speed as well as with the temperature of the flow. For high speed flow determined by the Mach number, or larger temperature gradient flow or both, the effect of compressibility must be considered in the study of the fluid flow, known as ***gasdynamics***. It is a combination of two sciences, fluid dynamics and thermodynamics.

In addition to Mach number, the controlling factors in this case are the Reynolds number, Prandtl number and the ratio of specific heats. The inviscid theory of gas dynamics is important in the calculation of nozzle characteristics, shock waves, lift and wave drag of a body, while the viscous theory is applicable to the calculation of skin-friction and heat transfer characteristics of a body moving through a gas at high speeds.

In a rarefied gas flow the gas adjacent to a solid surface no longer takes the velocity and temperature of the surface as we have in noslip conditions. In such a flow the gas at the surface has a finite tangential velocity, i.e. it ***slips*** along the surface and a temperature ***jump*** between the surface and the adjacent gas takes place, which depends on the molecular mean free path and the Prandtl number of the gas.

Flow of multicomponent mixtures :

The flow of a single, homogeneous fluid, is enlarged upon by the consideration of the flow of binary mixture, chemically reacting or non-reacting, or multicomponent mixtures. In addition to the momentum and the heat transport, mass transport must also be included in such fluids flow. The property controlling the mass transfer is the diffusion parameter called the ***Schmidt number***.

Flow through the porous media :

The study of the flow through porous media has one of the most useful applications in industries, viz., in the field of chemical engineering for filteration and purification process, in petroleum technology to study the movement of natural gases, oil and water through the oil reservoirs, and seepage of water in river beads to study the underground water resources.

Two important properties of porous medium are ***porosity*** and ***permeability***. The porosity in the ratio of the volume of the void space to the bulk volume of a porous medium and the permeability is the measure of the ease with which a fluid flows through the medium. The Darcy's law govern the flow of homogeneous fluid through the porous medium, provided the permeable material is isotropic and homogeneous. A large number of studies both analytical and experimental have been made in this field.

Turbulent flow :

Turbulent flow is random in nature. It is an irregular condition of flow in which various quantities show a random variation with time and space coordinates. If the same experiment is performed several times under apparently identical conditions, the measured property of the fluid, say pressure, will not be the same in different experiments, but will fluctuate randomly.

According to ***Taylor*** and ***Von Kármán turbulence*** can be generated by fluid flow past solid surfaces or by the flow of layers of fluid at different velocities, known as ***Wall turbulence*** and ***free turbulence*** respectively.

In the mathematical description of the turbulent flow it is convenient to assume that the motion consists of a ***mean flow*** and a superimposed fluctuation or ***eddying flow*** about the mean value. The mean flow equations known as ***Reynolds equations***, when compared with the N-S equations of laminer flow, contain additional stresses called ***Reynolds stresses*** or ***Virtual stresses*** of the turbulent flow.

The importance of stability in connection with tubulence arises because a motion which is definitely unstable for small disturbances connot remain steady for speeds higher than that at which instability sets in. On the other hand, a motion which is definitely stable for small disturbances may become turbulent when finite disturbances are imposed on it. One of the controlling parameter in the stability problems, studied extensively by ***Chandrasekhar*** and others, is the Rayleigh number.

Different theories have been provided to study the impact of eddying flow on the mean flow, such as (i) statistical theories, (ii) mixing length theories, (iii) momentum transfer theory, and (iv) vorticity transfer theory. The field of turbulant flow requires many further investigations.

Magnetofluid dynamics :

The subject of Magneto Fluid Dynamics (MFD) is an extension of fluid dynamics when an electrically conducting fluid moves in the presence of a magnetic field. Several alternative names which are widely used are; ***magnetohydrodynamics*** (MHD), ***magnetogasdynamics*** (MGD), and ***hydromagnetics.***

A gas at ordinary and moderately high temperatures is a nonconductor, but at very high temperature thermal excitation takes place leading to dissociation and ionization. Ionized gas is often called a ***Plasma*** which is an electrically conducting gas by virtue of its charged particles due to free electrons. Highly ionized gases are present in the Sun and other stars. Such problems are handled in ***plasmadynamics.***

Examples of engineering problems involving the flow of electrically conducting fluids are the MHD power generator, MFD submarines, plasma jet, confinement of plasma in nuclear fusion (pinch effect) and reentry problems of missiles and satellites.

(II) Non-Newtonian fluids :

Many real fluids exhibit behaviors which are not accounted for by the theory of Newtonian fluids. Fluids which do not follow the linear law are called Non-Newtonian. Examples of such substances are polymeric solution, paints, tooth paste, thick long chained hydrocarbons, molasses etc.

A ***dilatant***, or shear thickening, fluid increases resistance with increasing applied stress. Alternately, a ***pseudoplastic***. or shear thinning, fluid decrease resistance with increasing stress. If the thinning effect in very strong, the fluid is termed ***plastic***. The limiting case of a plastic substance is one which requires a finite yield stress before it begins to flow. An idealization of such a flow is known as ***Bingham-plastic***. An example of an yielding fluid is ***toothpaste***. which will not flow out of the tube until a finite stress is applied by squeezing.

Some fluids requires a gradually increasing shear stress to maintain a constant strain rate and are called ***rheopectic***. The opposite case of a fluid which thins out with time and requires decreasing stress is termed ***thixotropic*** such as printer's ink. The study of Non-Newtonian fluids is treated in the books on ***rheology*** and a large number of problems, parallel to the dynamics of Newtonian fluids, have been studied in the literature.

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SOME GLIMPSES OF FRACTIONAL CALCULUS

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INTRODUCTION

When $f(z) = z^4$, we know that its derivative is $4z^3$, second order derivative is $12z^2$ and so on. Infact, the concept of n^{th} derivative of a function is familiar if n is a positive integer. But now we can find its derivatives of orders 1/3, $\sqrt{2}$, 2+i. The branch of mathematics that studies and deals with the analysis of derivatives and integrals of functions to an arbitrary order (fractional, irrational, complex) is known as Fractional Calculus (F.C.) Now Fractional Calculus stands on a firm footing by valuable contributions of a large number of eminent research workers such as Leibniz, L'Hospital, Euler, Lagrange, Laplace, Lacroix, Fourier, Abel, Liouville, Peacock Gregory, DeMorgan, Riemann, Center, Grünwald, Letnikov, Heaviside, Pincherle, Hardy, Weyl, Post, Berg, Zygmund, Davis, Erdélyi, Kober, Widder, Riesz, Srivastava, Buschman, Higgins, Samko, Osler, Love, Sneddon, Kesarwani, Prabhakar and several others. I have gone through some of these works and find them interesting and useful.

The familiar calculus of integral order is very comfortable with trigonometric functions, exponential function and the polynomials. Certain higher transcendental functions such as Gamma function, incomplete gamma function, Mittag-Leffler function, Bessel function, hyper geometric functions Legendre and Laguerre functions are admirably suited to the study and development of Fractional Calculus.

THE GAMMA FUNCTION

The gamma function has great importance in analysis and applications. It is a building block of several higher transcendental functions. The simplest definition of gamma function $\Gamma(z)$ is in terms of Euler's second integral is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad \text{Re}(z) > 0$$

Also it is known that

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$\Gamma(1) = 1$, $\Gamma(½) = \sqrt{\pi}$ and gamma of 0 and negative integers is infinite.

On integration by parts, the defining integral of $\Gamma(z)$, we easily arrive at the most important property $\Gamma(z+1) = z\ \Gamma(z)$ of gamma function. Though the defining integral of gamma function cannot be obtained in terms of elementary functions, but because of its importance it has been tabulated and has been studied by eminent mathematicians Euler, Gauss and Weierstrass.

It can be easily established that k^{th} order derivative of a constant (say) c is given by

$$D^k(c) = \frac{c\,z^{-k}}{\Gamma(1-k)}$$

where k is an arbitrary number and we are dealing with Riemann Liouville version of F.C.

If k is any positive number (say 4) in the above formula we get

$$D^4(c) = \frac{c\,z^{-4}}{\Gamma(-3)}$$

which is zero because $\Gamma(-3)$ is infinity.

From the main formula it is clear that only integral order derivations of a constant are all equal to zero but all other order derivatives of a constant will be functions of the variable.

Thus if $k = ½$ in the main formula, we get semi-derivative of constant c as

$$D^{1/2}(c) = \frac{c\,z^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} = \frac{c}{\sqrt{\pi z}}$$

The above formula is more interesting for its historical prospective than for its mathematical content.

Abel used this formula for solving TAUTO CHRONE problem: the problem of determining the shape of the curve such that the time of descent of a frictionless point mass sliding down the curve under the action of gravity is independent of its initial placement on the curve. More generally, the time required for descent is specified as a function of initial height. For the full discussion and the solution of this problem we refer to the text book on the subject by Miller and Ross pp.255-260. Abel was probably the first mathematician to give an application of F.C. This problem should not be confused with BRACHIS TO CHRONE problem in the calculus of variations which is concerned with finding the curve of quickest descent.

Now we shall talk about NH Abel a little more.

Abel was born in 1802 and passed away in 1829 at in an early age of less than 28 years. Abel is considered as one of the greatest mathematicians of all time. The entire development of Algebraic geometry in the 19th century largely depends on the findings of Abel. He has done outstanding work in other areas of mathematics as well. Abel was Norwegian and in honor of Abel Government of Norway has constituted Abel prize (an international prize in Mathematics to be awarded annually) from the year 2003 onwards.

Abel prize is awarded to enhance the public visibility of mathematics, to encourage young people to study mathematics and to point out the growing role and importance of mathematics in modern society. The amount of Abel prize is equal to Nobel Prize recognizing mathematics as being at par with physics, chemistry, economics, etc. Abel prize can be seen as Nobel of Mathematics.

Fractional Calculus is not a hollow extension of conventional theory nor a sterile exercise in pure Mathematics. Besides, serving as a tool in unifying and extending several theorems, formulae, concepts and techniques occurring in calculus of integral order in a versatile and elegant manner, F.C. is capable of solving a large variety of boundary value problems occurring in sciences and engineering: heat conduction in solids, fluid flow, diffusion theory, electrical transmission lines, electrical networks, electromagnetic theory, rheology, viscoelasticity, probability theory and several others.

The first book on F.C. by K.B. Oldham and J. Spanier appeared in 1974 [15]. The book is published by Academic Press and was sent to me for my review by Mathematical Reviews (USA). The book contains a brief historical survey and development of F.C. during 1695-1974, the theoretical foundations of the subject, semi derivatives and integrals of several useful functions and applications of F.C. in solving diverse important problems. During the last 37 years, considerable mathematical activity has emerged out in this useful field by way of international conferences [14], symposiums, workshops, books [12; 13] and a large number of research papers.

The access of F.C. to scientists and engineers is now available through the interesting research work done by a number of persons. Thus, N. Engheta [1; 2; 3] has given the use of fractional calculus in the electromagnetic theory. Lorenzo and Hartley [8; 10] have introduced the concept of initialized fractional calculus that requires the initial condition that is time varying due to the past distributed storage of information and have solved fractional differential equations occurring in various fields. Mainardi et al. [11] have studied the reduced Green function in connection with the solution of the space time fractional order differential equation. Again, M. Garg [4] has studied fractional generalizations of Volterra type integro differential equation and the temperature field problem in oil strata. Podlubny in his book [16] has given an account of the applications of fractional order differential equations in several fields.

In Rajasthan, a number of researchers notably R.K. Saxena, S.L. Kalla, K.C. Gupta [5], C.L. Koul, P.K. Banerjee, R.K. Raina, S.P. goyal, M. Garg, Rashami Jain, Kantesh Gupta, V.B.L. Chaurasia, V.G. Gupta, R.K. Khumbhat, Mridula Purohit, Vandana Agarwal and several others have contributed a lot to the further advancement of F.C. through their valuable research papers.

SPECIAL FUNCTIONS

Lorenzo (Glenn Research Centre NASA Cleveland OHIO, USA) and Hartley (Department of Electrical Engineering, University of Akron, Akron OHIO) have introduced two special functions of fundamental importance [9] which provide solutions and understanding of several fractional order differential equations and the related initial and boundary value problems. Both of these functions and several other useful functions notably Mittag-Leffler function follow as simple special cases of the H-function which is a Mellin-Barnes type contour integral having products and quotients of gamma functions in the integrand [17]. Four books have come out relating this functions. First by Mathai and Saxena (1978), Second by Srivastava, Gupta and Goyal (1982), Third by Kilbas and Saigo [7] and the Fourth by Mathai, Saxena and Haubold (2010). The H-function has embedded in it a large number of simple functions obtainable from it on giving particular values to its parameters [6]. These functions form solutions of fractional order differential or integral equations of practical importance. Lorenzo has pointed out to me in private communication by e-mail that the H-function is interesting. It is now suitable time for young researchers to discover, study, and develop these special cases of the H-function which are solutions of fractional order differential or integral equations relevant to physical problems in engineering.

SEMI DERIVATIVES AND SEMI INTEGRALS

The diffusion equation governs a wide variety of physical problems: Electrical transmission lines, heat conduction in solids, electro chemical problems to mention only a few. The conventional solutions of the diffusion equation range from closed form solution for very simple model problems to computer methods for approximating the concentration of the diffusing substance on a network of points. The F.C. method develops a technique that leads to the replacement of the diffusion equation together with an initial and asymptotic boundary condition, to a semi differential equation that involves proportionality of first order spatial derivative to a half order temporal derivative. The proper understanding and the use of Laplace transform theory in finding semi differentiation and its inverse (semi integration) plays a fundamental role in obtaining the final solution of the problem. The semi derivatives and semi integrals of several useful functions are available in Chapter 7 of the book by Oldham and Spanier [15].

CONCLUDING REMARKS

Finally, it is suggested that F.C. should be included in the curricula of Mathematics in Universities and Technical Institutes. further, it will be in the fitness of things if a workshop is arranged in this field for interested teachers to make them familiar with this useful branch of Mathematics. It is expected that these teachers will teach F.C. in their institutions, make the students aware to this field. These young minds will be excited and ignited. They will apply the F.C. in solving certain boundary value problems occurring in diverse field of sciences and engineering.

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ON IMPLICIT SUMMATION FORMULAS OF A SET OF POLYNOMIALS RELATED TO GENERALIZED HERMITE POLYNOMIALS

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ABSTRACT

In this paper, we introduce a new class of the polynomial set which includes ultraspherical and Hermite polynomials as special cases and derive some implicit summation formulae by applying the generating functions. These results extend some known summations of generalized Hermite polynomials.

2000 Mathematics Subject Classification: 33C45, 33C99

Keywords: Hermite polynomials, hypergeometric functions , summation formulae.

1. INTRODUCATION

The 2-variable Kampe de Feriet generalization of the Hermite polynomials [4] reads

$$H_n(x,y) = n! \sum_{k=o}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \qquad .... (1.1)$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=o}^{\infty} H_n(x,y) \frac{t^n}{n!} \qquad .... (1.2)$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ when $y = -1$ and x is replaced by $2x$.

The above result can be expressed in hypergeometric form as

$$e^{xt+yt^2} = \sum_{n=o}^{\infty} {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2}, 4y/x^2\right)\frac{x^n t^n}{n!} \qquad \text{.... (1.3)}$$

so that

$$H_n(x,y) = {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2}, 4y/x^2\right)x^n \qquad \text{.... (1.4)}$$

The following well-known results of Hermite polynomials $H_n(x)$

$$e^{t^2} \cos(2xt) = \sum_{n=0}^{\infty} H_{2n}(x)\frac{(-1)^n t^{2n}}{(2n)!} \qquad \text{.... (1.5)}$$

$$e^{t^2} \sin(2xt) = \sum_{n=0}^{\infty} H_{2n+1}(x)\frac{(-1)^n t^{2n+1}}{(2n+1)!} \qquad \text{.... (1.6)}$$

are generalized by Khan et al [13,p.410(1.20) and (1.21)]) by employing the generating function [15 ,p.196]

$$e^{2x(t+u)-(t+u)^2} = \sum_{k,l=o}^{\infty} H_{k+l}(x)\frac{t^k u^l}{k!l!} \qquad \text{.... (1. 7)}$$

These results are special cases of the following results

$$e^{-yt^2} \cos xt = \sum_{n=0}^{\infty} H_{2n}(x,y)\frac{(-1)^n t^{2n}}{(2n)!} \qquad \text{.... (1. 8)}$$

$$e^{-yt^2} \sin xt = \sum_{n=0}^{\infty} H_{2n+1}(x,y)\frac{(-1)^n t^{2n+1}}{(2n+1)!} \qquad \text{.... (1. 9)}$$

Next we recall the definition of N-variable generalized Hermite polynomials $H_n(\{x\}_1^N)$ defined by Dattoli et al [5,p.602]

$$\exp \sum_{s=1}^{N} x_s t^s = \sum_{n=0}^{\infty} H_n(\{x\}_1^N)\frac{t^n}{n!} \qquad \text{.....(1.10)}$$

where $\{x\}_1^N = x_1, x_2, \ldots\ldots, x_N$

Generalized Hermite polynomials $H_n(\{x\}_1^N)$ for $N = 3$ also belong to those of Bell type [1] as shown in [9,p.403(26)]. The Gould-Hopper polynomials $g_n^m(x,y)$ (see [5]) is a special case of (1.10).The notation $H_n^m(x,y)$ for $g_n^m(x,y)$ was given by Dattoli et al [5].These are specified by

$$e^{zt+yt^s} = \sum_{n=o}^{\infty} H_n^s(z,y)\frac{t^n}{n!} \qquad \text{...(1.11)}$$

Also, another special case of (1.10), the Kampe de Feriet polynomial $H_n^{(3)}(x,y)$ (see [9]) is linked to (1.11) by the generating function

$$e^{zt+yt^3} = \sum_{n=o}^{\infty} H_n^{(3)}(z,y)\frac{t^n}{n!} \qquad \text{...(1.12)}$$

Let us illustrate the case when $N = 3$ in (1.10) which will be used in our next section. In this case we write a generating function

$$\exp \sum_{s=1}^{3} x_s t^s = \sum_{n=0}^{\infty} H_n(3\,x^2, -3x, 1)\frac{t^n}{n!} = \sum_{n=o}^{\infty} H_n^3(x)\frac{t^n}{n!} \qquad \text{...(1.13)}$$

Gould and Hopper [12] shown that

$$e^{3\,x^2t-3xt^2} = e^{-t^3}\sum_{n=o}^{\infty} H_n^3(x)\frac{t^n}{n!} = \sum_{n=o}^{\infty}\frac{t^n}{n!}\sum_{r=o}^{[\frac{n}{3}]}\frac{n!(-1)^r}{r!(n-3r)!}H_{n-3r}^3(x) = \sum_{n=o}^{\infty}\frac{3\,(xt)^n}{n!}(x-t)^n$$

which ultimately yield the formula to express an H_n^3 sum in terms of an ordinary H_n

$$\sum_{r=o}^{[\frac{n}{3}]}\frac{n!(-1)^r}{r!(n-3r)!}H_{n-3r}^3(x) = (\sqrt{3x})^n\; H_n(\tfrac{1}{2}x\sqrt{3x}) \qquad \text{...(1.14)}$$

The object of his paper is to develop certain generating functions which arise from certain forms of generalized Hermite polynomials which were defined and discussed in various papers of Dattoli et al [4] to [11],Brafman [2],Bell [1],Cohen [3] and Gould and Hopper [12].The resulting formulas allow a considerable unification of various results which appear in the literature. Most of the generating functions to be obtained here involve arbitrary parameters and variables, so that generating functions are specialized for certain choices of the parameters and variables.The starting point for the generalizations of the Hermite polynomials which we wish to consider is essentially depending on the set of polynomials in hypergeometric form $f_n(x)$ given in section 2. This approach is not entirely new, however even where a generalization of this nature has been explored in Khan et al [13] and Pathan [14],by using different analytical means on their respective generasting functions ,many of the results we find here do not seem to appear.

2. A SET OF POLYNOMIALS $f_n(x)$.

For each integer $n \geq 0$, consider the polynomial set

$$f_n(x) = p + 2^F{}_q\left(-\tfrac{n}{2}, \tfrac{1-n}{2}, (\alpha)_p, (\beta)_q;\; x\right) \qquad \text{...(2.1)}$$

which includes ultraspherical and Hermite polynomials as special cases by writing generating functions of the type

$$F(x.t) = \sum_{n=o}^{\infty}(a)_n f_n(x)\frac{t^n}{n!} \qquad ...(2.2)$$

For the definitions of ultraspherical , Hermite polynomials and hypergeometric polynomials $p+2^F{}_q$,we refer [14].

Interesting special cases may be obtained from these general sets $f_n(x)$ by giving suitable values to parameters and variables. In equation (2.2) omit the parameters α_1 , α_2 ,...., α_p, take the denominator parameter as $1+\alpha$, replace x by $(x^2-1)/x^2$ and then t by tx. The result is on ultraspherical polynomials. Then omitting the parameter before summing in (2.2) ,Hermite polynomials may be obtained. The result thus obtained (when x is replaced by y) is

$$e^t \ p^F{}_q\,((\alpha)_p;\ (\beta)_q;\ yt^2/4) = \sum_{n=o}^{\infty} p+2^F{}_q\,(-\tfrac{n}{2},\tfrac{1-n}{2},(\alpha)_p;\ (\beta)_q;\ y)\frac{t^n}{n!} \qquad ...(2.3)$$

Replacing t by an xt and then y by y/x^2 yields

$$e^{xt} \ p^F{}_q\,((\alpha)_p;\ (\beta)_q;\ yt^2/4) = \sum_{n=o}^{\infty} p+2^F{}_q\,(-\tfrac{n}{2},\tfrac{1-n}{2},(\alpha)_p;\ (\beta)_q;\ y/x^2)\ \frac{x^n t^n}{n!} \qquad ...(2.4)$$

Now replacing t by it, using $e^{it} = \cos t + i\sin t$ and the result

$$\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n+1), \qquad ...(2.5)$$

in (2.4) and equating real and imaginary parts, we get

$$\cos xt\ p^F{}_q\,((\alpha)_p;\ (\beta)_q;\ -yt^2/4) = \sum_{n=o}^{\infty} p+2^F{}_q\,(-n,\tfrac{1}{2}-n,(\alpha)_p;\ (\beta)_q;\ y/x^2)\ \frac{(-1)^n x^{2n} t^{2n}}{(2n)!} \qquad ...(2.6)$$

$$\sin xt\ p^F{}_q\,((\alpha)_p;\ (\beta)_q;\ -yt^2/4) = \sum_{n=o}^{\infty} p+2^F{}_q\,(\tfrac{1}{2}-n,1-n,(\alpha)_p;\ (\beta)_q;\ y/x^2)\ \frac{(-1)^n x^{2n+1} t^{2n+1}}{(2n+1)!} \qquad ...(2.7)$$

Since (1.3) is a special case of (2.4), we can say that (2.6) and (2.7) are generalizations of (1.5), (1.6), (1.8) and (1.9) and Khan et al [13,p.410(1.20) and (1.21)].For another generalization of these results, see [14].

The non standard type of generating function

$$(1+4t^2)^{-3/2}\,(1+2xt+4t^2)\exp\left(\frac{4x^2t^2}{1+4t^2}\right)=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{[\frac{n}{2}]!} \qquad \ldots(2.8\ )$$

has been known for some time. This generating function and reference to Doetsch are given in Szego [16 ,p.371] and is given at many other references in the literature. A generalization of this result has been given by Brafman [2 ,p.949] in the form

$$(1+4t^2)^{-c}\,{}_1F_1\left(c;\frac{1}{2};\ \frac{4x^2t^2}{1+4t^2}\right)+\frac{32cx^2t^2}{3(1+4t^2)^{c+\frac{1}{2}}}\,{}_1F_1\left(c+1;\frac{5}{2};\ \frac{4x^2t^2}{1+4t^2}\right)+\frac{2xt(1+4t^2-8ct^2)}{(1+4t^2)^{c+1}}\,{}_1F_1\left(c;\frac{3}{2};\ \frac{4x^2t^2}{1+4t^2}\right)$$

$$=\sum_{n=0}^{\infty}H_n(x)\frac{(c)_{[\frac{n}{2}]}t^n}{[\frac{n}{2}]!(1/2)_{[\frac{n}{2}]}} \qquad \ldots(2.9\ )$$

which contains the arbitrary parameter c and reduces to (2.8) for $c = 1/2$.

Now replacing t by it, using $e^{it}=\cos t+i\,sint$ and (2.5) in (2.9) and equating real and imaginary parts, we get

$$(1-4t^2)^{-c}\,{}_1F_1\left(c;\frac{1}{2};\ \frac{-4x^2t^2}{1-4t^2}\right)-\frac{32cx^2t^2}{3(1-4t^2)^{c+\frac{1}{2}}}\,{}_1F_1\left(c+1;\frac{5}{2};\ \frac{-4x^2t^2}{1-4t^2}\right)=\sum_{n=0}^{\infty}H_{2n}(x)\frac{(-1)^n(c)_{[n]}t^{2n}}{n!(1/2)_n} \qquad \ldots(2.10\ )$$

and

$$\frac{2xt(1-4t^2+8ct^2)}{(1-4t^2)^{c+1}}\,{}_1F_1\left(c;\frac{3}{2};\ \frac{-4x^2t^2}{1-4t^2}\right)=\sum_{n=0}^{\infty}H_{2n+1}(x)\frac{(c)_{[n+\frac{1}{2}]}t^n}{[n+\frac{1}{2}]!(1/2)_{[n+\frac{1}{2}]}} \qquad \ldots(2.11\ )$$

Setting $c = 1/2$ and using a connection between the exponential function and confluent hypergeometric functions in the form (see Brafman [2 ,p.949])

$$\exp z=\frac{z}{a+1}\,{}_1F_1\,(a+1;a+2;\ z)+\,{}_1F_1\,(a;a+1;\ z)\,,$$

(2.10) and (2.11) yield

$$(1-4t^2)^{-\frac{1}{2}}\exp\left(\frac{-4x^2t^2}{1-4t^2}\right)=\sum_{n=0}^{\infty}H_{2n}(x)\frac{(-1)^nt^{2n}}{n!} \qquad \ldots(2.12\ )$$

and

$$2x(1-4t^2)^{-3/2}\ \exp\left(\frac{-4x^2t^2}{1-4t^2}\right) = \sum_{n=0}^{\infty} H_{2n+1}(x)\frac{(-1)^n t^{2n}}{[n+\frac{1}{2}]!} \qquad \ldots(2.13)$$

respectively.

Next, we consider a generating function given by (1.13)

$$\sum_{n=0}^{\infty} H_n(3x^2,-3x,1)\frac{t^n}{n!} = \sum_{n=o}^{\infty} H_n^3(x)\frac{t^n}{n!} = \exp(3x^2t-3xt^2+t^3) \qquad \ldots(2.14)$$

On replacing t by $-t$ and adding we get

$$\exp(-3x^2t)[\exp(3x^2t+t^3)+\exp(-3x^2t-t^3)] = 2\sum_{n=o}^{\infty} H_{2n}^3(x)\frac{t^{2n}}{(2n)!}$$

Now replacing t by it, using $e^{it}=\cos t + i\, sint$ and (2.5) in (2.14) and equating real and imaginary parts, we get

$$\exp(3xt^2)\cos(3x^2t) = \sum_{n=o}^{\infty} H_{2n}^3(x)\frac{(-1)^n t^{2n}}{(2n)!}$$

$$\exp(3xt^2)\sin(3x^2t) = \sum_{n=o}^{\infty} H_{2n+1}^3(x)\frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

On the other hand, if we take $s = 2$ in (1.11) and replace x by $2(x + y)$ and y by $-(xy + 1)$ and follow the same method, then we get

$$\exp((xy+1)t^2)\cos(2(x+y)t) = \sum_{n=o}^{\infty}(x+y)^n H_{2n}\left(\frac{x+y}{(x+y)^{1/2}}\right)\frac{(-1)^n t^{2n}}{(2n)!}$$

$$\exp((xy+1)t^2)\cos(2(x+y)t) = \sum_{n=o}^{\infty}(x+y)^{n+1/2} H_{2n+1}\left(\frac{x+y}{(x+y)^{1/2}}\right)\frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

3. IMPLICIT SUMMATION FORMULAE INVOLVING SET OF POLYNOMIALS $f_n(x)$.

Here we prove the following results involving generalized Hermite polynomials These results are generalizations of (1.7) and (1.8)and various well-known results of Hermite polynomials (see also Khan et. al [13,p.410(1.20) and (1.21)])

Theorem 3.1 *For positive integers k,l ≥0, the following implicit summation formula involving generalized Hermite polynomials holds true:*

$$(\frac{z}{x})^k p+2F_q\left(-\frac{k+l}{2},\frac{1-k-l}{2},(\alpha)_p;(\beta)_q;\frac{y}{z^2}\right)$$
$$=\sum_{n,m=o}^{k,l}\binom{k}{n}\binom{l}{m}\frac{(z-x)^{n+m}}{x^n}p+2F_q\left(\frac{-k-l+n+m}{2},\frac{1-k-l+n+m}{2},(\alpha)_p;(\beta)_q;\frac{y}{x^2}\right)$$

...(3.1)

Proof. We replace t by $t+u$ (2.4) and rewrite the generating function as

$$p^F{}_q((\alpha)_p;(\beta)_q;\ y(t+u)^2/4)=e^{-x(t+u)}\sum_{k,l=o}^{\infty}(x)^k p+2F_q\left(-\frac{k+l}{2},\frac{1-k-l}{2},(\alpha)_p;(\beta)_q;\ y/x^2\right)\frac{t^k u^l}{k!l!}$$

...(3.2)

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$\sum_{k,l=o}^{\infty}(z)^k p+2F_q\left(-\frac{k+l}{2},\frac{1-k-l}{2},(\alpha)_p;(\beta)_q;\ y/z^2\right)\frac{t^k u^l}{k!l!}$$
$$=e^{(z-x)(t+u)}\sum_{k,l=o}^{\infty}(x)^k p+2F_q\left(-\frac{k+l}{2},\frac{1-k-l}{2},(\alpha)_p;(\beta)_q;\ y/x^2\right)\frac{t^k u^l}{k!l!}\qquad ...(3.3)$$

On expanding exponential function,(3.3) gives

$$\sum_{k,l=o}^{\infty}(z)^k p+2F_q\left(-\frac{k+l}{2},\frac{1-k-l}{2},(\alpha)_p;(\beta)_q;\ y/z^2\right)\frac{t^k u^l}{k!l!}$$
$$=\sum_{N=0}^{\infty}\frac{[(z-x)(t+u)]^N}{N!}\sum_{k,l=o}^{\infty}(x)^k p+2F_q\left(,-\frac{k+l}{2},\frac{1-k-l}{2}(\alpha)_p;(\beta)_q;\ y/x^2\right)\frac{t^k u^l}{k!l!}$$

...(3.3)

which on using formula [15]

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}=\sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}\qquad ...(3.4)$$

in the right hand side becomes

$$\sum_{k,l=o}^{\infty} (z)^k p + 2^F{}_q \, (-\frac{k+l}{2}, \frac{1-k-l}{2}, (\alpha)_p;\ (\beta)_q;\ \ y/z^2)\ \frac{t^k u^l}{k!\, l!}$$

$$=\sum_{n,m=0}^{\infty} \frac{(z-x)^{n+m}}{n!m!} (t)^n (u)^m \sum_{k,l=o}^{\infty} (x)^k p + 2^F{}_q \, (-\frac{k+l}{2}, \frac{1-k-l}{2}, (\alpha)_p;\ (\beta)_q;\ \ y/x^2)\ \frac{t^k u^l}{k!l!} \quad \ldots(3.5)$$

Now replacing k by $k-n$, l by $l-p$ and using the lemma

$$\sum_{n,k=0}^{\infty} f(n,k) \ = \ \sum_{n=0}^{\infty} \sum_{k=0}^{[n/s]} f(n-sk) \quad \ldots(3.6)$$

with $s = 2$,in the right hand side of (3.5),we get

$$\sum_{k,l=o}^{\infty} (z)^k p + 2^F{}_q \, (-\frac{k+l}{2}, \frac{1-k-l}{2}, (\alpha)_p;\ (\beta)_q;\ \ y/z^2)\ \frac{t^k u^l}{k!\, l!}$$

$$= \ \sum_{k,l=0}^{\infty} \sum_{n,m=o}^{k,l} \frac{(z-x)^{n+m}}{n!m!} \frac{(t)^k}{(k-n)!} \frac{(u)^l}{(l-m)!} (x)^{k-n} p + 2^F{}_q \, (-\frac{k+l-n-m}{2}, \frac{1-k-l+n+m}{2}, (\alpha)_p;\ (\beta)_q;\ \ y/x^2)$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the required result.

Remark 1. *On setting l=0 in (3.1),we immediately deduce the following consequence of Theorem 3.1.*

$$(\tfrac{z}{x})^k \, p + 2^F{}_q \, \left(-\tfrac{k}{2}, \tfrac{1-k}{2}, (\alpha)_p;\ (\beta)_q;\ \tfrac{y}{z^2}\right) = \sum_{n=o}^{k,} \binom{k}{n} \frac{(z-x)^n}{x^n} \ p + 2^F{}_q \, (\tfrac{-k+n}{2}, \tfrac{1-k-n}{2}, (\alpha)_p;\ (\beta)_q; \tfrac{y}{x^2}) \quad \ldots(3.7)$$

which further reduces to [13,p.411(2.7)]

$$H_k(y) = \sum_{n=0}^{k} \binom{k}{n} \, 2^n (y-x)^n H_{k-n}(x) \quad \ldots(3.8)$$

Remark 2. *On replacing y by x+y in (3.7),we get*

$$(\tfrac{z}{x})^k \, p + 2^F{}_q \, \left(-\tfrac{k}{2}, \tfrac{1-k}{2}, (\alpha)_p;\ (\beta)_q;\ \tfrac{y+x}{z^2}\right) = \sum_{n=o}^{k,} \binom{k}{n} \frac{(z-x)^n}{x^n} \ p + 2^F{}_q \, (\tfrac{-k+n}{2}, \tfrac{1-k-n}{2}, (\alpha)_p;\ (\beta)_q; \tfrac{x+y}{x^2}) \quad \ldots(3.9)$$

Remark 3. *When* $\alpha = \beta$ *and p=q so that* $(\alpha)_p = (\beta)_q;\ p,q = 1,2, \ldots$ *, it is possible to give some of the key results for Hermite polynomials* $H_n(x,y)$*.First we note the following result using the hypergeometric form given by (1.3)*

$$(x/y)^l H_k(y,z) = \sum_{n,m=o}^{k,l} \binom{k}{n}\binom{l}{m} \frac{(y-x)^n}{y^m} (x-y)^m H_{k-n}(x,z) \qquad ...(3.10)$$

which is a special case of [13,p.416(4.6)] when ω=0.

Theorem 3.2 *The following implicit summation formula involving generalized Hermite polynomials holds true:*

$$\sum_{n=o}^{\infty} \frac{z^n \exp(-zln)}{n! x^n}\, p+2^F{}_q\left(-\frac{n}{2}, \frac{1-n}{2}, (\alpha)_p;\ (\beta)_q;\ 4y/[x(ln+1)]^2\right) = \frac{\exp z}{1-lz}\, p^F{}_q\left((\alpha)_p;\ (\beta)_q;\ yz^2/x^2\right) \qquad ...(3.11)$$

Proof. We use series expansion of the hypergeometric series $p+2^F{}_q$ in the left hand side of (3.11) to get

$$\sum_{n=o}^{\infty} \frac{z^n \exp(-zln)}{n! x^n} \sum_{m=o}^{[\frac{n}{2}]} \frac{y^m [x(ln+1)]^{n-2m}}{n!(n-2m)!}$$

Now replacing *n* by $n+2j$ and using (3.6) together with a result of Cohen [3 ,p.704(2.9)]

$$\sum_{n=o}^{\infty} \frac{(ln+lsk+1)\, z^n \exp(-zln)}{n!} = \exp z(lsk+1)\,/(1\text{-}lz) \qquad ...(3.12)$$

we get the resulting equation (3.11).

Remark 1. *By taking* $\alpha = \beta$ *and p=q so that* $(\alpha)_p = (\beta)_q;\ p,q = 1,2,\dots,$ *in equation (1.15)and using , we immediately deduce the following result.*

$$\sum_{n=o}^{\infty} \frac{z^n \exp(-zln)}{n! x^n} H_n\,[x(ln+1), y] = \frac{\exp(z+yz^2/x^2)}{1-lz} \qquad ...(3.13)$$

Remark 2. O*n replacing x by 2x and taking y=−1,the above result reduces to the following known result of Cohen [3,p.705(2.14)] for classical Hermite polynomials*

$$\sum_{n=o}^{\infty} \frac{z^n \exp(-zln)}{n!(2x)^n} H_n\,(x) = \frac{\exp(z-z^2/4x^2)}{1-lz} \qquad ...(3.14)$$

Remark 3.*If we use the generating function (1.11) for generalized Hermite polynomials* $H_n^s(z,y)$ *in (3.11) in place of* $p+2^F{}_q$ *then the following generating function for* $H_n^s(z,y)$ *follows*

$$\sum_{n=o}^{\infty} \frac{z^n \exp(-zln)}{n! x^n} H_n^s\,[x(ln+1), y] = \frac{\exp(z+yz^s/x^s)}{1-lz} \qquad ...(3.15)$$

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COSMOLOGICAL MODEL FOR BAROTROPIC FLUID DISTRIBUTION WITH DECAYING VACUUM ENERGY (Λ) IN FRW SPACE-TIME

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ABSTRACT

Cosmological model for barotropic fluid distribution with decaying vacuum energy (Λ) in FRW space-time is investigated. To get deterministic solution and physical requirement of the model, we have assumed two cases: (i) $\Lambda \sim \frac{1}{R^2}$ *as considered by Chen and Wu [18] and (ii)* $\Lambda \sim \rho$, ρ *being the matter density. We find that expansion in the model is large initially but decreases due to lapse of time. The model has decelerating and accelerating phases both in case (i). But in case (ii), the model represents accelerating phase. The vacuum energy (Λ) is initially large but decreases as time increases in both cases. These results match with the recent observations. The special cases for dust, stiff fluid and radiation dominated phases are also discussed.*

1. INTRODUCTION

A barotropic fluid is one whose pressure and density are related by an equation of state that does not contain the temperature as dependent variable. Mathematically, the equation of state can be expressed as $p = p(\rho)$ or $\rho = \rho(p)$. Accordingly a linear equation of state is $p = \gamma\rho$, $0 \leq \gamma \leq 1$ is a special type of barotropic fluid, a polytropic fluid with specific heat at constant pressure same as specific heat at constant volume. The mathematical form of equation of state includes dust ($\gamma = 0$), radiation dominated universe ($\gamma = 1/3$) and stiff fluid ($\gamma = 1$) are considered in cosmological situation as particular cases. Thus the barotropic fluid $p = \gamma\rho$ determines the matter content of the universe.

A wide range of observations suggest that universe possesses a non-zero cosmological constant. Zel'dovich [1], Dreitlein [2], Krauss and Turner [3] have studied its significance from time to time. Riess et al. [4] and Perlmutter et al. [5] used Type Ia supernovae to show that universe is accelerating. This discovery provided the first direct evidence that Λ is non-zero with $\Lambda \sim 1.7 \times 10^{-121}$ Planck units. It is commonly believed by the scientific community that via the cosmological constant, a kind of repulsive pressure dubbed as dark energy is the most suitable candidate to explain recent observations that the universe appears to be expanding at an accelerating rate. Dark energy is a special form of energy that permeates all of space and tends to increases the rate of expansion of the universe. Linde [6] has investigated that Λ is function of temperature and is related to the spontaneous symmetry breaking process. A number of cosmological models in which Λ decays with time have been investigated by number of authors viz. Bertolami [7], Sahni and Starobinski [8], Beesham [9], Berman [10], Abdussattar and Vishwakarma [11], Bronnikov et al. [12], Bali and Singh [13], Ram and Verma [14], Abdussattar and Prajapati [15], Bali and Singh [16]. Recently Barrow and Shaw [17] suggested that cosmological term (Λ) corresponds to a very small value of the order 10^{-122} when applied to FRW space-time.

In this paper, we have investigated cosmological model for barotropic fluid distribution with decaying vacuum energy (Λ) in the frame work of FRW space-time. To get the deterministic solution, we have assumed (i) $\Lambda \sim \frac{1}{R^2}$ and (ii) $\Lambda \propto \rho$ where R is scale factor. The model has decelerating and accelerating phases both in case (i) but accelerating phase in case (ii). The vacuum energy (Λ) is initially large but decreases due to lapse of time in both cases. The special cases for dust distribution i.e. for $\gamma = 0$ or $p = 0$, stiff fluid distribution i.e. for $\gamma = 1$ or $\rho = p$, radiation dominated model i.e. for $\gamma = 1/3$ or $\rho = 3p$ are also discussed.

2. METRIC AND FIELD EQUATIONS

We consider FRW space-time as

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right], \qquad \ldots(2.1)$$

where $k = 0, -1, 1$.

Einstein field equation with time dependent decaying vacuum energy (Λ) is given by

$$R_i^j - \frac{1}{2}\overline{R}\, g_i^j = -8\pi T_i^j - \Lambda(t)\, g_i^j, \qquad \ldots(2.2)$$

where R_{ij} is the Ricci tensor, $\overline{R} = g^{ij} R_{ij}$ scalar curvature which measures the curvature of space. The energy-momentum tensor for matter is taken as

$$T_i^j = (\rho + p)\, v_i\, v^i - p\, _i^j\,, \qquad \ldots(2.3)$$

where ρ is the matter density, p is the isotropic pressure.

We assume the flow vector to be comoving so that $v^1 = 0 = v^2 = v^3, v^4 = 1$.

The Einstein field equation (2.2) for the metric (2.1) leads to

$$\frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} = 8\pi\rho + \Lambda(t) \qquad \ldots(2.4)$$

and

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi p + \Lambda(t) \qquad \ldots(2.5)$$

The curvature equation

$$(8\pi T_i^j + \Lambda g_i^j)_{;j} = 0$$

leads to

$$8\pi\left[\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R}\right] + \dot{\Lambda} = 0 \qquad \ldots(2.6)$$

We assume that the universe is filled with barotropic fluid i.e. $p = \gamma\rho$, $0 \le \gamma \le 1$. Now equations (2.4), (2.5) and condition $p = \gamma\rho$ lead to

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}(1 + 3\gamma) = -(1 + 3\gamma)\frac{k}{R^2} + (1 + \gamma)\Lambda \qquad \ldots(2.7)$$

Case (i)

To get deterministic solution, we assume that $\Lambda \sim \frac{1}{R^2}$ as considered by Chen and Wu [18]. Thus equation (2.7) leads to

$$2\ddot{R} + (1+3\gamma)\frac{\dot{R}^2}{R} = \frac{1}{R}\{(1-\gamma) - (1+3\gamma)k\} \qquad \text{...(2.8)}$$

From equation (2.8), we have

$$\frac{dF^2}{dR} + \frac{(1+3\gamma)}{R}F^2 = \frac{1}{R}\{(1+\gamma) - (1+3\gamma)k\}, \qquad \text{...(2.9)}$$

where

$$\dot{R} = F(R).$$

Equation (2.9) leads to

$$F^2 = \left(\frac{dR}{dt}\right)^2 = \left(\frac{1+\gamma}{1+3\gamma} - k)\right) + \frac{\beta}{R^{1+3\gamma}}, \qquad \text{...(2.10)}$$

where β is constant of integration. Thus, we have

$$\left(\frac{dR}{dt}\right)^2 = \frac{\alpha R^{1+3\gamma} + \beta}{R^{1+3\gamma}}, \qquad \text{...(2.11)}$$

where

$$\alpha = \left(\frac{1+\gamma}{1+3\gamma} - k\right) \qquad \text{...(2.12)}$$

The metric (2.1) leads to the form

$$ds^2 = \left(\frac{dt}{dR}\right)^2 dR^2 - R^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right] \qquad \text{...(2.13)}$$

which leads to the form

$$ds^2 = \frac{\tau^{1+3\gamma}}{\alpha\tau^{1+3\gamma} + \beta} d\tau^2 - \tau^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right], \qquad \text{...(2.14)}$$

where

$$R = \tau \text{ and } t = \int \frac{\tau^{1+3\gamma}\, d\tau}{\alpha\tau^{1+3\gamma} + \beta}.$$

Case (ii).

Let $\Lambda \alpha \rho$ which leads to

$$\dot{\Lambda} = \ell\dot{\rho} \qquad \text{...(2.15)}$$

ℓ being constant. Now conservation equation (2.6) leads to

$$8\pi\dot{\rho} + 24\pi(1+\gamma)\rho\frac{\dot{R}}{R} + \ell\dot{\rho} = 0 \qquad \text{...(2.16)}$$

which leads to

$$(8\pi + \ell)\frac{\dot{\rho}}{\rho} = -24\pi(1+\gamma)\frac{\dot{R}}{R} \qquad \text{...(2.17)}$$

which leads to

$$\frac{\dot{\rho}}{\rho} = -\beta\frac{\dot{R}}{R}, \qquad \text{...(2.18)}$$

where

$$\beta = \frac{24\pi(1+\gamma)}{(8\pi+\ell)} \qquad \text{...(2.19)}$$

From equation (2.18), we have

$$\rho = \frac{\alpha}{R^{\beta}}, \qquad \text{...(2.20)}$$

where α is constant of integration. Equation (2.15) leads to

$$\Lambda = \frac{\alpha\ell}{R^{\beta}} \qquad \text{...(2.21)}$$

Equations (2.4), (2.20) and (2.21), we have

$$\frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} = \frac{8\pi\alpha}{R^{\beta}} + \frac{\alpha\ell}{R^{\beta}} \qquad \text{...(2.22)}$$

In particular, if we take $\beta = 1$ then (2.22) leads to

$$\dot{R}^2 = bR - k\ , \qquad \text{...(2.23)}$$

where

$$b = \frac{8\pi\alpha + \alpha\ell}{3} \qquad \text{...(2.24)}$$

Equation (2.23) leads to

$$R = \frac{4kb + (bt+2m)^2}{4b^2} \qquad \text{...(2.25)}$$

Thus the metric (2.1) leads to

$$ds^2 = dt^2 - \frac{[4kb + (bt+2m)^2]^2}{4b^2}\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\ell^2\right] \qquad \text{...(2.26)}$$

In particular if we take $\alpha = 1 = b$, $k = 0$, $m = 0$ then FRW model for flat space-time leads to

$$ds^2 = dt^2 - \frac{t^4}{4}[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2], \qquad \text{...(2.27)}$$

where

$$R = \frac{t^2}{4} \qquad \text{...(2.28)}$$

3. PHYSICAL AND GEOMETRICAL ASPECTS

The matter density (ρ) and isotropic pressure (p) for the model (2.14) are given by

$$8\pi\rho = \frac{2 - k(1+3\gamma)}{\tau^2} + \frac{3\beta}{\tau^{3(1+\gamma)}} \qquad \text{...(3.1)}$$

and

$$8\pi p = \frac{3\beta\gamma}{\tau^{3(1+\gamma)}} + \frac{2\gamma}{(1+3\gamma)\tau^2} \qquad \text{...(3.2)}$$

The expansion (θ), the deceleration parameter (q), the spatial volume (R^3) are given by

$$\theta = \sqrt{\frac{\alpha}{\tau^2} + \frac{\beta}{\tau^{3(1+\gamma)}}} \qquad \text{...(3.3)}$$

$$q = \frac{\beta(1+3\gamma)\tau^2}{\alpha\tau^{3(1+\gamma)}\{\alpha\tau^{3(1+\gamma)} + \beta\tau^2\}} \qquad \text{...(3.4)}$$

$$R^3 = \tau^3 \qquad \text{...(3.5)}$$

The matter density (ρ), the isotropic pressure (p), the expansion (θ), the deceleration parameter (q), the vacuum energy (Λ) for the model (2.27) are given by

$$\rho=\frac{4}{t^2} \quad \text{...(3.6)}$$

$$p=\frac{4\gamma}{t^2} \quad \text{...(3.7)}$$

$$\theta=\frac{2}{t} \quad \text{...(3.8)}$$

$$q=-\frac{1}{2} \quad \text{...(3.9)}$$

$$\Lambda=\frac{4\ell}{t^2} \quad \text{...(3.10)}$$

$$R^3=\frac{t^6}{64} \quad \text{...(3.11)}$$

SPECIAL MODELS :

***(i)* Dust Model**: If $\gamma = 0$ then we can discuss dust model for $k = 0,-1,1$.

***(ii)* Stiff Fluid Model**: If $\gamma = 1$ then we can find stiff fluid ($\rho = p$) scenario for $k = 0, -1,1$.

***(iii)* Radiation Dominated Model**: If $\gamma = 1/3$ then we can discuss radiation dominated phase of the universe i.e. early universe for $k = 0, -1,1$.

4. DISCUSSION

There is a big-bang in the model (2.14) at $\tau = 0$ and the expansion in the model decreases at τ increases. The matter density (ρ) is infinite at $\tau = 0$ and matter density decreases as τ increases and $\rho \to 0$ when $\tau \to \infty$. The decelerating parameter (q) > 0 if $\beta > 0$ and $q < 0$ if $\beta < 0$. Thus the model (2.14) has decelerating and accelerating phases both. The vacuum energy density is initially large but decreases due to lapse of time. The spatial volume increases as τ increases. These results match with the recent observations.

There is also big-bang in the model (2.27) at $t = 0$ and the expansion in the model decreases as t increases. The matter density $\rho \to \infty$ if $t \to 0$ and $\rho \to 0$ when $t \to \infty$. The deceleration parameter $q < 0$. Thus the model represents accelerating phase. The vacuum energy density (Λ) is initially large but decreases due to lapse of time. The spatial volume increases as t increases. For the model (2.27), the results also match with recent observations.

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A STUDY ON ROLE OF STERILIZATION AND ITS IMPACT ON SOCIETY

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ABSTRACT

In this paper, we studied the impact of family planning program on society, run by state governments and central government of India. We focused our study on the permanent method of family planning program i.e. sterilization and identified some factors which might be helpful in bringing down the total fertility rate(T.F.R.) to a desired level. In this process, we applied statistical techniques like central tendency & chi-square test of independence and based on the study few measures have been suggested which may be immensely useful for the success of the program.

Keywords: Total fertility rate, Sterilization, caste, education, birth rate etc.

INTRODUCTION

Over population is a serious global concern and particular in developing countries like India. Since the 1970's, India's economic growth rate has risen significantly, poverty has declined and social indicators have improved a lot. Nevertheless, a quarter (25%) (CIA world fact book -2012) of India's population currently lives below the national poverty line. In the present form, over population is swallowing all efforts of development made by policymakers to uplift the living and health standard of country. Since the benefit of developmental policies, natural resources which are limited, are not reaching to the society in the required volume. To cope with this problem, in 1952, Indian government, first in the world formulated a national family planning program, which has now become the decades old. During this period the National Family Program has expanded enormously both in resource allocation and development of infrastructure. Setting up such a large organizational structure which penetrates deeply in rural areas is itself a praiseworthy achievement. The program has succeeded in increasing family planning awareness and acceptance too, in the country. It is evident from the fact that among all the methods of family planning 96% (Ramesh et. al., 1992) of currently married women recognized and accepted at least one method of family planning. However, if the impact of the program is measured in terms of the decline in birth rate, it is modest and below the expectation. India's current T.F.R. is 2.50% (S.R.S.-2010 & A.H.S. 2010-11)), which is still very high then the replacement level. India ranked at 81 among 223 countries in terms of T.F.R., which is, clearly, not a healthy sign for future development (Country comparison: T.F.R. CIA world fact book, 2012). India's population is currently growing at a rate of 1.37 % (World Bank report-2012) per year and about 70% (Adlakha, 1997) higher than that of China and will continue growing faster than China for many years in the future. In our present paper, we focused our selves only on sterilization, because, this method is permanent, one time expenditure and has the largest share in family planning program (census-2011). N.F.H.S. 2 &3 report indicates that 95% married women know of female sterilization.

In this study, an attempt has been made to identify the factors which may be helpful in bringing down the fertility rate to the desired level and to suggest some new measures.

DATA COLLECTION, ANALYSIS AND DISCUSSION

For the above said study we have utilized the data of 1040 sterilized women from the two C.H.C's belonging to two states, namely U.P (Baraut) and U.K (Bhagwanpur). For analysis we have two parameters in our mind, one is caste and other is education, because in our society caste and education play important role in overall development.

METHODOLOGY

For the analysis of above data, the technique of measures of central tendency has been used to get the median age at sterilization and average no. of children ever born per couple. We used the chi-square test for testing the interdependence of median age at sterilization and average no. of children with caste and education at 5% level of significance.

AVERAGE NO. OF CHILDREN, MEDIAN AGE AT STERILIZATION AND CASTE

About 96% of sterilized women are Hindu and includes about 25 castes. Keeping in view the large number of caste and too few observations for many of them, the castes have been grouped into 3 categories. The groups are as follows:

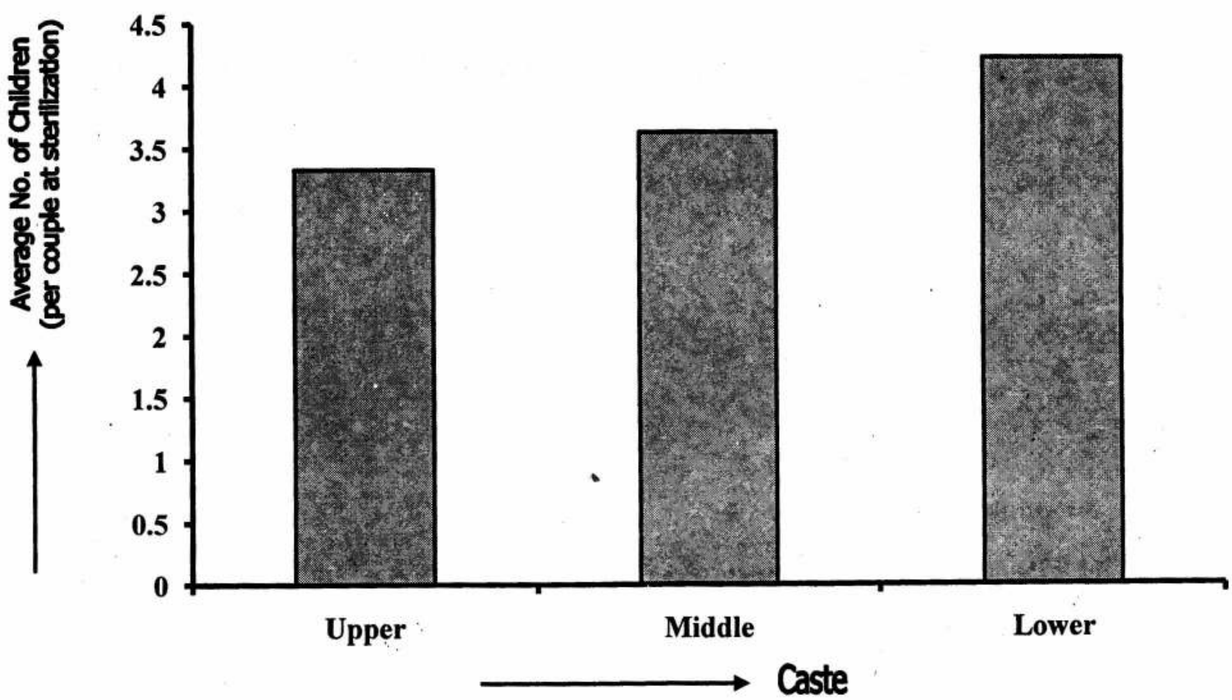
i. **Upper Caste** : Tyagi, Jain, Gupta, Agarwal, Rajput, Brahmin, Rao, Punjabi

ii. **Middle Caste:** Jaat, Gujjar, Saini, Yadav, Naai, Pal, Upadhyay(jogi), Dhobi, Prajapati, Kashyap, Muslim Teli, Badhai, Sunar

iii. **Lower Caste** : Harijan, Valmiki , khatik

TABLE 1: Distribution of Average Number of Children and Median Age at Sterilization According to Caste:

Caste	Median Age at sterilization	Average no. of children per couple at sterilization
Upper Caste	30	3.33
Middle Caste	30	3.63
Lower Caste	30	4.23
Average	30	3.80



It is observed from the Table 1 that median age at sterilization is equal for all the three categories but there is variation in average no. of children. The average no. of children in the caste group I, II and III are 3.33, 3.63 and 4.23 respectively. These figure exhibits upward variation from the upper caste to lower caste which may be due to general feeling of more hand, more earning in the lower/ weaker section of the society and hence unwillingly they keep on increasing the population. The graphic picture also shows the upward movement when we move from Upper Caste to Lower Caste. It clearly indicates that average no. of children ever born per couple depends on caste. Chi-square test also showed that caste and education are independent in terms of age at sterilization and average no. of children per couple, since in former case chi-square (cal.) = 0.7071 and H_0 is accepted at 5% level of significance, similarly in latter chi-square(cal.) = 0.0969. It is not significant at 5% level of significance i.e. age at sterilization is independent of caste as well as education. The data clearly indicated us that education was the major contributor in determining both age at sterilization and average no. of children per couple. This is certainly a very interesting result that caste, now, no longer a decisive factor in determining the demographic variables, rather it is economic status ,which is important now. The living standard of lower class people is also low in comparison to other castes, because of their limited earning for livelihood. They are not well known about the drawbacks of over population and think more members in the family can earn more money. So if our policymakers may be able to eliminate the feeling of more people, more money from the mind of this section and improve their living standard by creating job opportunities to them, then we may have a real check on the population flood originated by this poor section of society. The table clearly indicates the key role played by the different caste in growing population. From the centuries, Indian society system has been dominated by the caste.

AVERAGE NO. OF CHILDREN, MEDIAN AGE AT STERILIZATION AND EDUCATION

There is clear evidence that the fertility percentage is higher of those belonging to an uneducated family, whereas, lower of those who belongs to educated family. Education has been divided into four categories which are as below:

i. **Illiterate:** Neither read, nor put signature
ii. **Primary:** Passed at least fifth standard
iii. **Secondary School:** Passed tenth standard
iv. **Senior secondary and Above:** Passed twelfth standard and above

TABLE 2: Distribution of Average number of Children and Median Age at Sterilization According to education:

Education	Median age at sterilization	Average no. of children per couple at sterilization
Illiterate	31	4.18
Primary	30	3.86
Secondary school	30	3.57
Senior secondary and Above	27	2.60

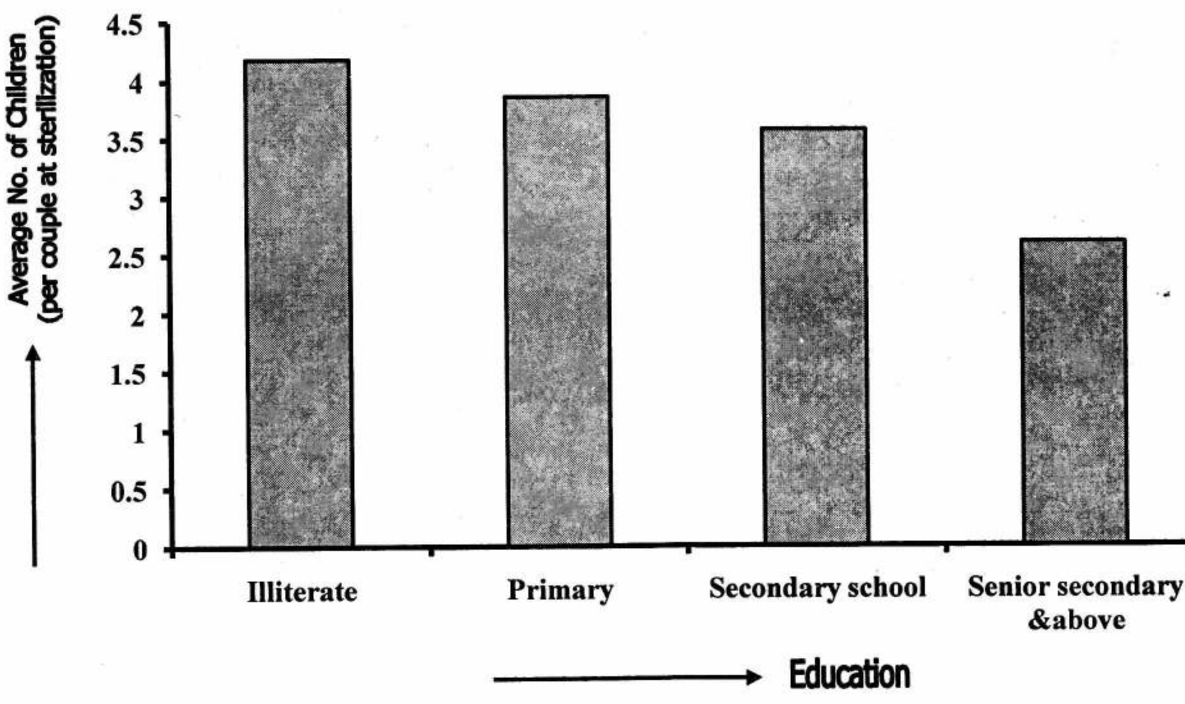


Table 2 depicts that, as the education level of society increases, there is a considerable decrease both in the median age at sterilization and average no. of children per couple. It also indicates that the female education can play a key role in reducing the fertility rate. Actually, with the increasing level of education people get more exposure to the modern society and perception of small size family develops among the people. They tend to think that with their limited earning they can provide better means of living and good education only to their small family. Previous study also showed the inverse relationship between fertility and female literacy (How female literacy affects fertility, 1990). Another important fact of decreasing fertility rate by the educated people is that, they tend to get late marriage as compare to the illiterate people, due to their involvement in study and the child bearing span of women decreases. Singh et. al., (1985) also studied that no. of children depends on age at marriage as well as on caste. So if our policymakers devote their all resources and means only to raise the literacy level of female then this increasing population may be controlled. In this regard, Government of India should take some concrete steps to improve the literacy level of females. Marriages, without having passed senior secondary examination, should be declared as illegal. They can be deprived of contesting elections and holding any democratic positions. The above graph also demonstrates the higher number of children to the illiterate couples and lowers to the educated couples.

CONCLUSION

From the above analysis and discussion we reach at a conclusion that education is the only dominating factor in deciding the age at sterilization, while caste and education both are the dominating factor for average no. of children ever born per couple. So if our policy makers design their future strategies keeping in mind both these factors, and then only we may have a real check on the rapid growth of population. Actually in Indian context caste and education are positively correlated because of high level of unemployment in the lower section of society (Desai and Dubey, 2011). This is the reason that people do not understand the disadvantage of increasing population and either knowingly or unknowingly they get involved in the process of population flood. So our planners should keep their focus on this lower section of society which is the origin of highly increasing population by creating job opportunities and improving education facilities, so that they can understand the benefit of small family.

RECOMMENDATION

After analyzing and discussing the data collected, we make the following recommendation:

i. First of all, we strongly recommend that the age at marriage of women must be increased from the current 18 yrs, in order to reduce the child bearing span.

ii. Female education up to senior secondary school must be made compulsory.

iii. A very handsome amount of cash should be given to those couples who get sterilization after one or two children and give birth to first child on or after 25 years of mother's age.

iv. The current scheme of Janani Suraksha Yojna must be restricted only to two children per couple.

v. All registered private hospitals should also be recognized to provide family planning services.

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UPPER AND LOWER BOUNDS OF WELL KNOWN DIVERGENCES IN TERMS OF RELATIVE *J*-DIVERGENCE MEASURE

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ABSTRACT

During past years Dragomir, Taneja and Pranesh Kumar have contributed a lot of work providing different kinds of bounds on the distance, information and divergence measures. These are very useful and play an important role in many areas like as sensor networks, testing the order in a Markov chain, risk for binary experiments, region segmentation and estimation etc. In this paper we shall establish an upper and lower bounds of information divergence measures in terms of Relative J-divergence measure using a new f-divergence and inequalities.

Keywords: - Chi-square divergence, Jenson-Shannon's divergence, Unified relative Jensen-Shannon and arithmetic-geometric divergence of type s, Triangular discrimination etc.

AMS Classification 62B-10, 94A-17,26D15

1. INTRODUCTION

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, \ldots\ldots p_n) \middle| p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2$$

be the set of all complete finite discrete probability distributions. There are many information and divergence measures exists in the literature of information theory and statistics. Csiszar [2] & [3] introduced a generalized measure of information using f-divergence measure given by

$$I_f(P,Q)=\sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \qquad \ldots (1.1)$$

where $f:\mathbb{R}_+\to\mathbb{R}_+$ is a convex function and $P,Q\in\Gamma_n$. The Csiszar's f-divergence is a general class of divergence measures that includes several divergences used in measuring the distance or affinity between two probability distributions. This class is introduced by using a convex function f, defined on $(0, \infty)$. An important property of this divergence is that many known divergences can be obtained from this measure by appropriately defining the convex function *f*.

There are some examples of divergence measures in the category of Csiszar's f-divergence measure. Bhattacharya divergence [1], Triangular discrimination [5], Relative J-divergence [7], Hellinger discrimination [8], Chi-square divergence [11], Relative Jensen-shannon divergence [12], Relative arithmetic-geometric divergence [13], Unified relative JS and AG divergence of type s [14].

2. NEW *f*-DIVERGENCE MEASURE AND ITS PARTICULAR CASES

In this section we shall consider some properties of a new f-divergence measure [Jain and Saraswat, 9 & 10] and its particular cases which are may be interesting in areas of information theory is given by

$$S_f(P,Q)=\sum_{i=1}^{n} q_i f\left(\frac{p_i+q_i}{2q_i}\right), \qquad \ldots (2.1)$$

where $f:\mathbb{R}_+\to\mathbb{R}_+$ is a convex function and $P,Q\in\Gamma_n$.

It is shown that using new f-divergence measure we shall derive some well known divergence measures such as Chi-square divergence, Relative *J*-divergence, Jenson-Shannon's divergence, Triangular discrimination, Hellinger discrimination, Bhattacharya divergence, Unified relative Jensen-Shannon and arithmetic-geometric divergence of type s etc. in this section. An inequality of *f*-divergence in terms of Relative *J*- divergence measure is established in section 3. Using the inequality of section 3, bounds of various particular measures are found in terms of Relative *J*- divergence measure in section 4.

The following results are on similar lines the result presented by Csisz´ar & K¨orner [4], Dragomir [6] and Jain & Saraswat [9] & [10].

Proposition 2.1 Let $f:[0,\infty)\to\mathbb{R}$ be convex and $P,Q\in\Gamma_n$ then we have the following inequality

$$S_f(P,Q)\geq f(1) \qquad \ldots (2.2)$$

Equality holds in (2.2) iff

$$p_i=q_i \;\; \forall i=1,2,..,n \qquad \ldots (2.3)$$

Corollary 2.1.1 (Non-negativity of new f-divergence measure) Let $f:[0,\infty)\to\mathbb{R}$ be convex and normalized, i.e.

$$f(1)=0 \qquad \ldots (2.4)$$

Then for any $P,Q\in\Gamma_n$ from (2.2) of proposition 2.1 and (2.4), we have the inequality

$$S_f(P,Q)\geq 0 \qquad \ldots (2.5)$$

If f is strictly convex, equality holds in (2.5) iff

$$p_i=q_i \quad \forall\, i\in\left[i,2,................n\right] \qquad \ldots (2.6)$$

and

$$S_f(P,Q)\geq 0 \text{ and } S_f(P,Q)=0 \text{ iff } P=Q \qquad \ldots (2.7)$$

Proposition 2.2 Let $f_1\,\&\,f_2$ are two convex functions and $g=a\,f_1+b\,f_2$ then $S_g(P,Q)=a\,S_{f_1}(P,Q)+b\,S_{f_2}(P,Q)$, where $a\,\&\,b$ are constants and $P,Q\in\Gamma_n$

We now give some examples of well known information divergence measures which are obtained from new *f*-divergence measure.

- **Chi-square divergence measure: -** If $f(t)=(t-1)^2$ then Chi-square divergence measure is given by

$$S_f(P,Q)=\frac{1}{4}\left[\sum_{i=1}^{n}\frac{p_i^2}{q_i}-1\right]=\frac{1}{4}\chi^2(P,Q) \qquad \ldots (2.8)$$

• **Relative Jensen-Shannon divergence measure:**-If $f(t)=-\log t$ then relative Jensen-Shannon divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n}q_i\log\left(\frac{2q_i}{p_i+q_i}\right)=F(Q,P) \qquad \ldots (2.9)$$

• **Relative arithmetic-geometric divergence measure:**-If $f(t)=t\log t$ then relative arithmetic-geometric divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right)=G(Q,P) \qquad \ldots (2.10)$$

• **Triangular discrimination:** - If $f(t)=\frac{(t-1)^2}{t}$, $\forall t>0$ then Triangular discrimination is given by

$$S_f(P,Q)=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{2(p_i+q_i)}=\frac{1}{2}\Delta(P,Q) \qquad \ldots (2.11)$$

• **Relative *J*-divergence measure:** - If $f(t)=(t-1)\log t$ then Relative *J*-divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n}\left(\frac{p_i-q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right)=\frac{1}{2}J_R(P,Q) \qquad \ldots (2.12)$$

• **Hellinger discrimination:** - If $f(t)=(1-\sqrt{t})$ then Hellinger discrimination is given by

$$S_f(P,Q)=\left[1-B\left(\frac{P+Q}{2},Q\right)\right]=h\left(\frac{P+Q}{2},Q\right) \qquad \ldots (2.13)$$

• **Unified relative Jensen-Shannon and arithmetic-geometric divergence of type α :-**

If
$$f(t)=\begin{cases}[\alpha(\alpha-1)]^{-1}\left[t^{\alpha}-1\right], \alpha\neq 0,1 & \\ -\log t & if\,\alpha=0 \\ t\log t & if\,\alpha=1\end{cases} \quad \ldots (2.14)$$

Then Unified relative Jensen-Shannon and Arithmetic-Geometric divergence measure of type α is given by

$$S_f(P,Q)=\Omega_\alpha(Q,P)=\begin{cases}FG_\alpha(Q,P)=[\alpha(\alpha-1)]^{-1}\left[\sum_{i=1}^{n}q_i\left(\frac{p_i+q_i}{2q_i}\right)^{\alpha}-1\right], & \alpha\neq 0,1 \\ F(Q,P)=\sum_{i=1}^{n}q_i\log\left(\frac{2q_i}{p_i+q_i}\right), & \alpha=0 \\ G(Q,P)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right), & \alpha=1\end{cases} \quad \ldots (2.15)$$

3. NEW INEQUALITY

The following theorem concerning an upper and lower bound for the f-divergence measure in terms of the Relative *J*- divergence holds. The results are on similar lines to the result presented by Dragomir [3] and Jain & Saraswat [9] & [10].

Theorem 3.1 Assume that generating mapping $f:(0,\infty)\to\mathbb{R}$ is normalized i.e. $f(1)=0$ and satisfies the assumptions.

(i) *f* is twice differentiable on (*r*, R), where $0\le r\le 1\le R\le\infty$

(ii) there exist constants m, *M* such that

$$m\le\frac{t^2}{(1+t)}f''(t)\le M \quad \ldots (3.1)$$

If *P*, *Q* are discrete probability distributions satisfying the assumptions

$$r<\frac{1}{2}\le r_i=\frac{p_i+q_i}{2q_i}\le R,\ \forall i\in\{1,2,\ldots\ldots\ldots..n\} \quad \ldots (3.2)$$

Then we have the inequality

$$\frac{m}{2}J_R(P,Q) \le S_f(P,Q) \le \frac{M}{2}J_R(P,Q) \qquad \ldots (3.3)$$

Proof: - Define a mapping $F_m:(0,\infty)\to\mathbb{R}, F_m(t)=f(t)-m(t-1)\log t, \forall t>0$. Then $F_m(.)$ is normalized and twice differentiable, Since

$$F_m''(t)=f''(t)-\frac{m(1+t)}{t^2}=\frac{1}{t^2}\left[\frac{t^2}{(1+t)}f''(t)-m\right]\ge 0 \qquad \ldots (3.4)$$

For all $t\in(r,R)$, it follows that $F_m(.)$ is convex on (r,R). Applying non-negativity property of *f*-divergence functional for $F_m(.)$ and the linearity property, we may state that

$$0\le S_{F_m}(P,Q)=S_f(P,Q)-mS_{(t-1)\log t}(P,Q)=S_f(P,Q)-\frac{m}{2}J_R(P,Q)$$

$\Rightarrow$

$$0\le S_f(P,Q)-\frac{m}{2}J_R(P,Q) \qquad \ldots (3.5)$$

from where the first inequality of (3.3) results.

Now we again Define a mapping $F_M:(0,\infty)\to\mathbb{R}, F_M(t)=M(t-1)\log t-f(t)$, which is obviously normalized, twice differentiable and by (3.1), convex on (r,R). Applying non-negativity property of *f*-divergence functional for $F_M(.)$ and the linearity property, we obtain the second part of (3.3) i.e.

$$0\le\frac{M}{2}J_R(P,Q)-S_f(P,Q) \qquad \ldots (3.6)$$

From (3.5) and (3.6) give the result (3.3)

Remark.1 *If we have strict inequality ">" in (3.3) for any* $t \in (r, R)$ *then the mapping* $F_m(.)$ *and* $F_M(.)$ *are strictly convex and equality holds in (3.3) iff* $P = Q$

Remark.2 *It is important note that* f *is twice differentiable on* $(0,\infty)$ *and* $m \le \frac{t^2}{(1+t)} f''(t) \le M < \infty, \forall t \in (0,\infty)$, *then inequality (3.1) holds for any probability distributions* P, Q.

4. SOME PARTICULAR CASES

Using Theorem (3.1) we able to point out the following particular cases which may be interest in Information Theory and Statistics.

Proposition 4.1 Let $P, Q \in \Gamma_n$ be two probability distributions with the property that

$$r < \frac{1}{2} \le r_i = \frac{p_i + q_i}{2q_i} \le R, \ \forall i \in \{1, 2, \ldots\ldots\ldots n\}$$

Then we have the inequality

$$\frac{1}{(1+R)} J_R(P,Q) \le F(Q,P) \le \frac{1}{(1+r)} J_R(P,Q) \qquad \ldots (4.1)$$

Proof:-Consider the mapping $f : (r, R) \to \mathbb{R}$.

$$f(t) = -\log t, f'(t) = -\frac{1}{t}, f''(t) = \frac{1}{t^2} > 0, \ \forall t > 0$$

So function is convex and normalized i.e. $f(1) = 0$

Define $g(t) = \frac{t^2}{(1+t)} f''(t) = \frac{t^2}{(1+t)}\left(\frac{1}{t^2}\right) = \frac{1}{(1+t)}$

Then obviously

$$M = \sup_{t\in[r,R]} g(t) = \frac{1}{(1+r)},\ m = \inf_{t\in[r,R]} g(t) = \frac{1}{(1+R)} \qquad \ldots (4.2)$$

Also from $S_f(P,Q) = F(Q,P)$ (2.9)

From equation (2.9), (3.3) & (4.2)

$$\frac{1}{(1+R)} J_R(P,Q) \le F(Q,P) \le \frac{1}{(1+r)} J_R(P,Q) \qquad \ldots (4.3)$$

Interchange $P \to Q$ and prove of the result (4.1).

Proposition 4.2 Let $P,Q \in \Gamma_n$ be two probability distributions satisfying (3.2)

Then we have the inequality

$$\frac{R}{(1+R)} J_R(P,Q) \le G(P,Q) \le \frac{r}{(1+r)} J_R(P,Q) \qquad \ldots (4.4)$$

Proof:-Consider the mapping $f:(r,R) \to \mathbb{R}$.

$$f(t) = t\log t,\ f'(t) = 1+\log t,\ f''(t) = \frac{1}{t} > 0,\ \forall t > 0$$

So function is convex and normalized i.e. $f(1) = 0$

Define $g(t) = \dfrac{t^2}{(1+t)} f''(t) = \dfrac{t^2}{(1+t)}\left(\dfrac{1}{t}\right) = \dfrac{t}{(1+t)}$

Then obviously

$$M = \sup g(t) = \frac{R}{(1+R)},\ m = \inf g(t) = \frac{r}{(1+r)} \qquad \ldots (4.5)$$

Also

$S_f(P,Q) = G(Q,P)$ from (2.10)

From equation (3.3) & (4.5)

$$\frac{R}{(1+R)} J_R(P,Q) \le G(Q,P) \le \frac{r}{(1+r)} J_R(P,Q) \qquad \text{. . . (4.6)}$$

Interchange $P \to Q$ and prove of the result (4.6).

Proposition 4.3 Let $P,Q \in \Gamma_n$ be two probability distributions satisfying (3.2).

Then we have the inequality

$$\frac{2r^2}{(1+r)} J_R(P,Q) \le \frac{1}{4} \chi^2(P,Q) \le \frac{2R^2}{(1+R)} J_R(P,Q) \qquad \text{. . . (4.7)}$$

Proof:-Consider the mapping $f:(r,R) \to \mathbb{R}$.

$$f(t) = t^2 - 1, f'(t) = 2t, f''(t) = 2 > 0, \ \forall t > 0$$

So function is convex and normalized i.e. $f(1) = 0$

Define $g(t) = \dfrac{t^2}{(1+t)} f''(t) = \dfrac{2t^2}{(1+t)}$

Then obviously

$$M = \sup_{t\in[r,R]} g(t) = \frac{2R^2}{(1+R)}, \ m = \inf_{t\in[r,R]} g(t) = \frac{2r^2}{(1+r)} \qquad \text{. . . (4.8)}$$

Also $S_f(P,Q)=\frac{1}{4}\chi^2(P,Q)$ from (2.8)

From equation (3.3), (4.8) give the result (4.6).

Proposition 4.4 Let $P,Q\in\Gamma_n$ be two probability distributions satisfying (3.2) then we have the following inequality

$$\frac{1}{R(1+R)}J_R(P,Q)\le\frac{\Delta(P,Q)}{4}\le\frac{1}{r(1+r)}J_R(P,Q) \qquad \ldots (4.9)$$

Proof:-Consider the mapping $f:(r,R)\to\mathbb{R}$.

$$f(t)=\frac{(t-1)^2}{t}=\left(t+\frac{1}{t}-2\right),\ f'(t)=\left(1-\frac{1}{t^2}\right),\ f''(t)=\frac{2}{t^3}$$

$f''(t)\ge 0$ and $f(1)=0$, So function f is convex and normalized.

Define $g(t)=\frac{t^2}{(1+t)}f''(t)=\frac{t^2}{(1+t)}\left(\frac{2}{t^3}\right)=\frac{2}{t(1+t)}$

Then obviously

$$M=\sup_{t\in[r,R]}g(t)=\frac{2}{r(1+r)},\ m=\inf_{t\in[r,R]}g(t)=\frac{2}{R(1+R)} \qquad \ldots (4.10)$$

Since $S_f(P,Q)=\frac{1}{2}\Delta(P,Q)$ from (2.11)

From equation (2.11), (3.3) & (4.10) give the result (4.9).

Proposition 4.5 Let $P,Q\in\Gamma_n$ be two probability distributions satisfying (3.2).

Then we have the inequality

$$\frac{r^{\alpha}}{(1+r)} J_R(P,Q) \le FG_{\alpha}(Q,P) \le \frac{R^{\alpha}}{(1+R)} J_R(P,Q) \qquad \text{... (4.11)}$$

Proof:-Consider the mapping $f:(r,R)\to\mathbb{R}$.

$$f(t)=\begin{cases}[\alpha(\alpha-1)]^{-1}\left[t^{\alpha}-1\right], \alpha\neq 0,1 & \\ -\log t & if\, \alpha=0 \\ t\log t & if\, \alpha=1\end{cases}$$

$$f(t)=[\alpha(\alpha-1)]^{-1}\left[t^{\alpha}-1\right], f'(t)=[\alpha-1]^{-1}t^{\alpha-1}, f''(t)=t^{\alpha-2}>0,\ \forall t>0$$

So function is convex and normalized i.e. $f(1)=0$

Define $g(t)=\frac{t^2}{(1+t)}f''(t)=\frac{t^2}{(1+t)}\left(t^{\alpha-2}\right)=\frac{t^{\alpha}}{(1+t)}$

Then obviously

$$M=\sup_{t\in[r,R]} g(t)=\frac{R^{\alpha}}{(1+R)},\ m=\inf_{t\in[r,R]} g(t)=\frac{r^{\alpha}}{(1+r)} \qquad \text{... (4.12)}$$

Also $S_f(P,Q)=FG_{\alpha}(Q,P)$ from (2.15)

From equation (2.3), (3.3) & (4.12) proved of result (4.11).

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