Volume 26, No. 2 December 2012 ISSN 0970-9169

GANITA SANDESH

गणित संदेश

A Half Yearly International Research Journal
of

Rajasthan Ganita Parishad

Registered Head Office
Department of Mathematics
Government College, AJMER - 305 001 (*INDIA*)

Website : www.rgp.co.in Issued April, 2014 E-mail : rgp@rgp.co.in

GAṆITA SANDESH

गणित संदेश

NOTES FOR CONTRIBUTORS

1. The editors will be pleased to receive contributions from the members of the Parishad only from all parts of India / abroad in any area of Mathematics (Research / Teaching etc.).
   In case of joint authorship, each author has to be a member of the Parishad.
2. Manuscripts for publication should be sent through E-Mail directly to the editor or rgp@rgp.co.in along with hard copy in Triplicate duly computerised with double spacing preferably in Adobe Pagemaker 6.5, with text in Times New Roman font 11 pts. and Mathematical symbols (Math Type, Equation editor or Corel Equation).
3. Authors should provide abstract and identify 4 to 5 key words for subject classification.
4. Unduly long papers and papers with many diagrams/tables will not be normaly accepted.
   In general, length of the accepted paper should not exceed 10 printed pages.
5. The contributors are required to meet the partial cost of publication at the rate of ₹ 200/- or equivalent US $ per printed page size A4 (even number of pages) payable in advance on receipt of acceptance of their paper.

Membership Fees / Subcription Rates

	Period	In India (₹)	Outside India (US $)
❑ Admission / Enrolment Fee	First Time Only	100	100 (or equivalent)
❑ Life Membership		2,000	2,000 (or equivalent)
❑ Annual Membership fee for Teachers (Colleges / Universities), T.R.F. Registered Research Scholars	Financial year	250	250 (or equivalent)
❑ Educational / Research Institutions	Calendar year	500	500 (or equivalent)

❋ All payments must be made by Bank **D.D.** in favour of ***Rajasthan Ganita Parishad*** payable at **AJMER** or online **State Bank of India Account No. 10200807636, IFSC Code: SBIN0000603** under intimation to the **Treasurer, Rajasthan Ganita Parishad,** *Department of Maths,* Govt. College, AJMER-305 001 (INDIA)

Ganita Sandesh Vol. 26 No. (2) (*December* 2012) pp. 97-104
ISSN : 0970-9169

Effect of Constant Influx of Toxicant in a Single Species Model Following Generalized Logistic Law

Anjali Sisodia
Department of Mathematics, Mahakal Group of Institutes, UJJAIN (MP) INDIA
E-mail : sisodiyaanjali@yahoo.co.in

Bijendra Singh
School of Studies in Mathematics, Vikram University, UJJAIN (MP) INDIA
E-mail : bijendrasingh@yahoo.com

B. K. Joshi
K-9, M.I.G. Rishi Nagar, UJJAIN (MP) INDIA
E-mail : bk.joshi648@gmail.com

Keywords : Toxicants and Stability.

2000 Mathematics Subject Classification : 93D20, 34D20

Abstract

The model of single-species in a polluted closed environment is analysed by Freedman and Shukla [5]. They have utilized a modified logistic law for concentration of population biomass. We have reanalysed this problem by using generalized logistic equation. The new parameter introduced, gives us more flexibility in specifying growth curves. The mathematical model is more general, thus includes the investigation of Freedman and Shukla [5] as a corollary.

1. Introduction

Pollution is the introduction of contaminantes into an environment. Environmental pollution by various industries is one of the most important ecological problems. Uncontrolled contribution of toxicant to the environment has led many species to extinction. Environmental pollution is closely related to the survival of species. The effects of toxicants on various ecosystems have been studied by many investigators by using mathematical models. Dubey [4] studied the effect of toxicants on forestry resources. Shukla and Dubey [11] studied the depletion of resources in a forest habitat due to the increase of both population and pollution. Singh et al [14] reanalyzed this model and observed that the resource biomass density settles down at a lower equilibrium level than its original carrying capacity. Shukla et al [12] presented a model to study effects of primary and secondary toxicants on the biomass of resources. The effect of toxicants in a single species is also investigated by some investivators [1,2,5,6,7,8,9,13]. In this series Freedman and Shukla [5] presented a model for the interaction of the population and the toxicants in the environment by the means of ordinary differential equations.

In this paper we have reformulated the model of Freedam and Shukla [5] by introducing a new parameter for population biomass. The model is mathematically more general and thus includes the results of Freedman and Shukla [5] as a corollary.

2. Mathematical Model

Non linear systems have always played an important role in the study of natural phenomena. The non linear systems can have several kinds of behavior that are not possible in linear system. Assuming that the population biomass follows generalised logistic law for its growth, the expended mathematical model is represented by the following set of differential equation [5].

$$\begin{aligned} \frac{dx}{dt} &= r(U)x - \frac{r_0}{K(T)}x^{n+1} \\ \frac{dT}{dt} &= -\delta_0 T - \alpha_1 x^n T + \pi_1\gamma_1 x^n U + Q_0 \\ \frac{dU}{dt} &= -\delta_1 U + \alpha_1 x^n T - \gamma_1 x^n U \\ & x(0) \ge 0, T(0) \ge 0, U(0) \ge kx(0), n \ge 1, \end{aligned} \qquad \ldots(2.1)$$

where $x(t)$ is the concentration of population biomass $T(t)$ is concentration of toxicant in the environment and $U(t)$ is the concentration of toxicant in the population. We assume that the toxicant in the environment is washed out with rate δ_0, where δ_0 is the depletion rate of toxicant in the environment. Further it is assumed that toxicant from the environment is absorbed by the population in direct proportion to their concentration i.e. $\alpha_1 x T$, where α_1 in the second equation is the depletion rate of toxicant in the environment due to its intake by the population; The toxicant in the population may also be removed from the total environment directly with rate δ_1, where δ_1 is the depletion rate of the toxicant in the population. Toxicant may also be removed from the population in proportion to their concentrations, some of it re-entering the environment and some removed from the environment.

Finally, toxicant may be externally introduced into the environment according to some rate Q_0. In model (2.1) $\delta_0, \delta_1, \alpha_1, \gamma_1, k$ are positive constants and $0 \le \pi_1 \le 1$. And $r(U)$ represents the growth rate "constant" which is affected by U.

Hence we assume

$$r(0) = r_0 > 0, \quad r'(U) < 0 \text{ for } U \ge 0, \quad r(\bar{U}) = 0 \text{ for some } \bar{U} > 0 \qquad \ldots(2.2)$$

The death component in given by $r_0 x^{n+1} / K(T)$, where $K(T)$ represents the carrying capacity which is affected by T. Hence we assume

$$K(0) = K_0 > 0, \quad K'(T) < 0 \text{ for } T \ge 0 \qquad \ldots(2.3)$$

$$K(\bar{T}) = 0 \text{ for some } \bar{T} > 0.$$

The above assumptions imply the existence of $\bar{U}$ and $\bar{T}$ imply that if the toxicant level is sufficiently high, then the population cannot reproduce or grow, and in fact will $\left(K(\bar{T}) = 0\right)$. Q_0 represents the rate of introduction of toxicant into the environment.

3. Stability Analysis

The system (2.1) has non-negative equilibria $E_2\left(0, Q_0/\delta_0, 0\right)$ and $\tilde{E}\left(\tilde{x}, \tilde{T}, \tilde{U}\right)$ where

$$\tilde{x} = \left[\frac{r\left(\tilde{U}\right)K\left(\tilde{T}\right)}{r_0}\right]^{1/n} \qquad \ldots(2.4)$$

$$\alpha_1\tilde{T} = \left[\gamma_1 + \delta_1\left(\frac{r_0}{r\left(\tilde{U}\right)K\left(\tilde{T}\right)}\right)\right]\tilde{U} \qquad \ldots(2.5)$$

$$\tilde{T} = Q_0/\delta_0 - \left[\delta_1 + (1-\pi_1)\gamma_1\left(\frac{r\left(\tilde{U}\right)K\left(\tilde{T}\right)}{r_0}\right)\right]\tilde{U}/\delta_0 \qquad \ldots(2.6)$$

This implies
$$\tilde{T} \to \frac{Q_0}{\delta_0} \text{ as } \tilde{U} \to 0$$

The variational matrices corresponding to E_2 and $\tilde{E}$ are denoted by M_2 and $\tilde{M}$ respectively and they are computed as follows :

$$M_2 = \begin{bmatrix} r_0 & 0 & 0 \\ \dfrac{-\alpha_1 Q_0}{\delta_0} & -\delta_0 & 0 \\ \dfrac{\alpha_1 Q_0}{\delta_0} & 0 & -\delta_1 \end{bmatrix}$$

$$\tilde{M} = \begin{bmatrix} -nr\left(\tilde{U}\right) & \dfrac{K\left(\tilde{T}\right)^{1/n} K'\left(\tilde{T}\right)\left(r\left(\tilde{U}\right)\right)^{1+1/n}}{r_0^{1/n}K\left(\tilde{T}\right)} & r'\left(\tilde{U}\right)\left(\dfrac{r\left(\tilde{U}\right)K\left(\tilde{T}\right)}{r_0}\right)^{1/n} \\ \left(-\alpha_1\tilde{T} + \pi_1\gamma_1\tilde{U}\right)n\tilde{x}^{n-1} & -\delta_0 - \alpha_1\tilde{x}^n & \pi_1\gamma_1\tilde{x}^n \\ \left(\alpha_1\tilde{T} - \gamma_1\tilde{U}\right)n\tilde{x}^{n-1} & \alpha_1\tilde{x}^n & -\gamma_1\tilde{x}^n - \delta_1 \end{bmatrix}$$

We see that E_2 is a saddle point stable in T-U plane and unstable in x-direction.

Now we set the stability criterion for $\tilde{E}$. We state the following lemma;

Lemma 2.1. The region

$$A = \left\{(x, T, U) : 0 \le x \le (K_0)^{1/n},\ 0 \le T + U \le Q_0 / \delta \ \text{ where } \delta = \min\{\delta_0, \delta_1\}\right\}$$

is a region of attraction.

Proof : $$\frac{dx}{dt} = r(U)x - \frac{r_0}{K(T)}x^{n+1}$$

$$\le r_0 x - \frac{r_0}{K_0}x^{n+1}$$

which gives $\lim\limits_{t \to \infty} x(t) \le (K_0)^{1/n}$

and $$\dot{T} + \dot{U} = -\delta(T + U) + (1 - \pi_1)\gamma_1 x U + Q_0$$

$$\lim_{t \to \infty}\left[T(t) + U(t)\right] \le Q_0 / \delta$$

proving the lemma.

Theorem 2.3. In addition to assumption (2.2) and (2.3), let $r(U)$ and $K(U)$ satisfy in A i.e.

$$K_m \le K(T) \le K_0,\ \ 0 \le -K'(T) \le k \qquad \text{...(2.8)}$$

$$0 \le -r'(U) \le \rho$$

For some positive constant k_m, k, ρ then if the following inequality holds

$$\left[n\frac{r_0 K_0 k}{K_m^2} + \frac{\alpha_1 Q_0}{\delta} + \pi_1\gamma_1\tilde{U}\right]^2 < n\frac{r_0}{K(\tilde{T})}\left(\delta_0 + \alpha_1\tilde{x}^n\right) \qquad \text{...(2.9a)}$$

$$\left[n\rho + \frac{\gamma_1 Q_0}{\delta} + \alpha_1\tilde{T}\right]^2 < n\frac{r_0}{K(\tilde{T})}\left(\delta_1 + \gamma_1\tilde{x}^n\right) \qquad \text{...(2.9b)}$$

$$\left[\pi_1\gamma_1 + \alpha_1\right]^2 K_0^{2n} < \left(\delta_0 + \alpha_1\tilde{x}^n\right)\left(\delta_1 + \gamma_1\tilde{x}^n\right) \qquad \text{...(2.9c)}$$

$\tilde{E}$ is globally asymptotically stable with respect to all solutions initialising in the interior of the positive orthant.

Proof : We consider the positive definite function about $\tilde{E}$

$$V(x,T,U) = x^n - \tilde{x}^n - \tilde{x}^n l_n\left(x^n / \tilde{x}^n\right) + 1/2\left(T - \tilde{T}\right)^2 + 1/2\left(U - \tilde{U}\right)^2$$

The derivative of V along the solution of (2.1) and after some manipulations we get

$$\frac{dV}{dt} = -\frac{r_0 n}{K(\tilde{T})}\left(x^n - \tilde{x}^n\right)^2 - \left(\delta_0 + \alpha_1\tilde{x}^n\right)\left(T - \tilde{T}\right)^2\left(\delta_1 + \gamma_1\tilde{x}^n\right)\left(U - \tilde{U}\right)^2$$

$$\left(x^n - \tilde{x}^n\right)\left(T - \tilde{T}\right)\left[-nr_0 x^n \xi(T) - \alpha_1 T + \pi_1\gamma_1\tilde{U}\right]$$

$$+\left(x^n - \tilde{x}^n\right)\left(U - \tilde{U}\right)\left[n\eta(U) + \alpha_1\tilde{T} - \gamma_1 U\right]\left(T - \tilde{T}\right)\left(U - \tilde{U}\right)\left[\pi_1\gamma_1 x^n + \alpha_1 x^n\right],$$

where
$$\xi(T) = \begin{cases} \dfrac{\dfrac{1}{K(T)} - \dfrac{1}{K(\tilde{T})}}{\left(T - \tilde{T}\right)}, & T \neq \tilde{T} \\ -\dfrac{K'(\tilde{T})}{K(\tilde{T})^2}, & T = \tilde{T} \end{cases} \qquad \text{...(2.10a)}$$

$$\eta(U) = \begin{cases} \dfrac{r(U) - r(\tilde{U})}{\left(U - \tilde{U}\right)}, & U \neq \tilde{U} \\ r'(\tilde{U}), & U = \tilde{U} \end{cases} \qquad \text{...(2.10b)}$$

Hence dV/dt can further be written as

$$\frac{dV}{dt} = -\frac{1}{2}a_{11}\left(x^n - \tilde{x}^n\right)^2 + a_{12}\left(x^n - \tilde{x}^n\right)\left(T - \tilde{T}\right) - \frac{1}{2}a_{22}\left(T - \tilde{T}\right)^2$$

$$-\frac{1}{2}a_{11}\left(x^n - \tilde{x}^n\right)^2 + a_{13}(x - \tilde{x})\left(U - \tilde{U}\right) - \frac{1}{2}a_{33}\left(U - \tilde{U}\right)^2$$

$$-\frac{1}{2}a_{22}\left(T - \tilde{T}\right) + a_{23}\left(T - \tilde{T}\right)\left(U - \tilde{U}\right) - \frac{1}{2}a_{33}\left(U - \tilde{U}\right)^2 \qquad \text{...(2.11)}$$

where $a_{11} = \dfrac{r_0 n}{K(\tilde{T})}$, $\quad a_{22} = \delta_0 + \alpha_1\tilde{x}^n$, $\quad a_{33} = \delta_1 + \gamma_1\tilde{x}^n$

$a_{12} = -r_0 n x^n \xi(T) - \alpha_1 T + \pi_1\gamma_1\tilde{U}$, $\quad a_{13} = n\eta(U) + \alpha_1\tilde{T} - \gamma_1 U$, $\quad a_{23} = \left(\pi_1\gamma_1 + \alpha_1\right)x^n$

Then the sufficient conditions for dV/dt in negative definite are

$$a_{12}^2 - a_{11}a_{22} < 0 \qquad \text{...(2.12a)}$$

$$a_{13}^2 - a_{11}a_{33} < 0 \qquad \text{...(2.12b)}$$

$$a_{23}^2 - a_{22}a_{33} < 0 \qquad \text{...(2.12c)}$$

Equation (2.9a) $\rightarrow$ (2.12a), (2.9b) $\rightarrow$ (2.12b),(2.9c) $\rightarrow$ (2.12c)

Hence dV/dt is negative definite and so V is a Liapunov function with respect to $\tilde{E}$ whose domain contains A, proving the theorem.

Corollary : For $n = 1$ the model coincides with that of Freedman and Shukla [5].

References

[1] Aldo Res Cigno : The struggle for life V. One species living in a limited environment; Bulletin of mathematical biology. 39 **(1977)** 479-485.

[2] Buonomo, B.Lacitignola, D: General conditions for global stability in a single species pollution-toxicant model : Non linear analysis : Real world problem. 5 **(2004)** 749-762.

[3] Dubey, B. : Modelling the depletion and conservation of resources. Effect of two interacting populations Ecol. Model. 101 **(1997)** 123-136.

[4] Dubey, B. : Modelling the effect of toxicants on forestry resources; J. Pure. Appl.Math 28(1)**(1996)** 1-12.

[5] Freedam, H.I., Shukla, J.B. : Models for the effect of toxicants in single species and predator single-species models of dispersal in a patchy environment. Nonlinear analysis 36**(1999)** 981-996.

[6] Freedman, H.I., Qui-Liang peng : Uniform persistence and globle asymptotic stability in periodic single-space models of dispersal in patchy environment. Nonlinear analysis. 36**(1999)** 981-996.

[7] G.P.,Samanta., A, Maiti : Dynamical model of a single species system in a polluted environment J. Appl. Math. and computing. 16**(2004)** 231-242

[8] Jiwei He, Ke Wang : The survival analysis for a single species population model in a polluted environment : Applied mathematical modelling. 31**(2007)**2227-2238.

[9] Lenbury, Y, Tumrasvin. N : Singular perturbation analysis of a model for the effect of toxicants in single-species system. Mathematical and computer modelling. 31**(2000)** 125-134

[10] Shukla, J.B. and Dubey, B : Simultaneous effects of two toxicants on biological species. A mathematical model J.Biol.Systems. 4**(1996)** 109-130.

[11] Shukla, J.B., Dubey, B : Modeling the depletion and conservation of forestry resources : Effects of population and pollution. J. Math. Biol. 36**(1998)** 71-94.

[12] Shukla, J.B., Agarwal, A.K., Singha, P. and Dubey, B : Modelling effects of primary and secondary toxicants on resources. Natural Resources Modelling 16**(2003)** 99-120.

[13] Singh, B, Joshi, B.K., Sisodiya, A and Khandelwal, R. : Effect of toxicant on population. Accepted for publication. J. Indian Academy of Mathematics. 33(1)**(2011)**.

[14] Singh, B,. Joshi B.K., Sisodiya A. Mathematical modelling on the effects of population, pollution and forestry resources : Accepted for publication in Vikram Mathematical Journal.

[15] Sisodiya, A., Singh, B., Joshi, B.K. : Effect of two interacting population on resource following generalized logistic growth. Applied Mathematical Sciences. 5(9) **(2011)**407-420.

Ganita Sandesh Vol. 26 No. (2) (*December* 2012) pp. 105-127 ISSN : 0970-9169

Minimum Shannon Entropy for Prescribed Harmonic Mean and Second Order Moment

Anju Rani
Department of Mathematics
R.R. Govt. College, ALWAR (INDIA)

Shalu Garg
Department of Mathematics
MITRC, ALWAR (INDIA)

Abstract

Shannon entropy is concave function of probability distribution. Maximization of Shannon entropy is simple due to its concave nature whereas minimization is complicated. But minimum entropy probability distribution is necessary for complete information of probability distribution. In the present paper, our aim is to obtain minimum Shannon entropy and switching points for minimum entropy for given values of Harmonic mean and Second order moment.

Keywords: Minimum Shannon entropy, switching point, consistent values of moments.

1. Introduction

Shannon [8] introduced a measure of entropy in 1948, this is given by $S = -\sum_{i=1}^{n} p_i \ln p_i$

It is concave function of probability distribution. After this measure many other measures of entropy came in existence. These are Renyi's [7], Havrda – Charvat [3] measure etc. Since Shannon entropy is concave function, a lot of work is done on its maximization and its applications.

Entropy is maximum when probability distribution is as equal as possible . In this case we have minimum information about system. As we increase information consistent with initial information in the form of moments, entropy decreases. This decreases until we obtain minimum entropy probability distribution. In this case we have complete information about system.

Maximum entropy probability distribution is most unbiased, most uniform and most random while minimum entropy probability distribution is most biased, least uniform and least random. Entropy is concave function so minimization of entropy is complicated than maximization.

Kapur [5] initiated the work to obtain minimum Shannon entropy. Anju Rani [2] obtained minimum entropy for Shannon measure and Havrda- Charvat measure when one moment is prescribed. In this paper, we have obtained analytical expressions for minimum Shannon entropy for given Harmonic mean and Second order moment. Further we have calculated minimum Shannon entropy for six faced dice when Harmonic mean and Second order moment are given.

2. Minimum value of Shannon Entropy when harmonic mean and second order moment are prescribed

Let x be a discrete variate which take all values from 1 to n with probabilities $p_1, p_2, \ldots\ldots, p_n$. Harmonic mean and Second order moment of this probability distribution are prescribed as H and $\left(\mu_2'\right)^{1/2}$ respectively. There will be many distributions having these particular values of H and $\left(\mu_2'\right)^{1/2}$ and each of these distributions will have a particular value of entropy. Out of these entropies our aim is to find minimum value of entropy say S_{min}.

Mathematically we have to minimize

$$S = -\sum_{i=1}^{n} p_i \ln p_i \qquad \ldots(1)$$

subject to,

$$\sum_{i=1}^{n} p_i = 1 \;,\;\; \sum_{i=1}^{n} \frac{p_i}{i} = \frac{1}{H} \;,\;\; \sum_{i=1}^{n} p_i i^2 = \mu_2' \qquad \ldots(2)$$

Since there are three linear constraints, the minimum entropy probability distribution will have at most three non zero components. Let these be p_h, p_k, p_l at points h, k and l respectively. Then from equation (2),

$$p_h + p_k + p_l = 1, \frac{p_h}{h} + \frac{p_k}{k} + \frac{p_l}{l} = \frac{1}{H}, \; h^2 p_h + k^2 p_k + l^2 p_l = \mu_2 \qquad \dots(3)$$

solving these equations, we get

$$p_h = \frac{h\left[l(k+l)(k-H)+H\left(\mu_2'-k^2\right)\right]}{H(k-h)(l-h)(h+k+l)}, \qquad p_k = \frac{k\left[H\left(l^2-\mu_2'\right)-h(h+l)(l-H)\right]}{H(k-h)(l-k)(h+k+l)},$$

$$p_l = \frac{l\left[H\left(\mu_2'-h^2\right)-k(h+k)(H-h)\right]}{H(l-h)(l-k)(h+k+l)} \qquad \dots(4)$$

To calculate p_h , p_k & p_l , we take set of values of Harmonic mean & Second order moment. All values of Second order moment are not possible for given value of Harmonic mean. In fact there is a range for feasible values of Second order moment for given Harmonic mean (Anju Rani [1]). These values are given by –

(i) Minimum value of $\left(\mu_2'\right)^{1/2}$. for given H is

When H takes discrete values

$$\left(\mu_2'\right)_{min}^{1/2} = H \qquad \dots(5)$$

And, when H does not take discrete values,

$$\left(\mu_2'\right)_{min} = \frac{[H]^3+L(3[H]^2+3[H]+1)}{[H]+L} \qquad \dots(6)$$

where H = [H] +L , 0 < L < 1, [H] is integral part of H.

(ii) Maximum value of μ_2' for given H is

$$\left(\mu_2'\right)_{max} = (n^2+n+1) - \frac{n(n+1)}{H} \qquad \dots(7)$$

For given values of H.M. and $\left(\mu_2'\right)_{min}^{1/2}$, probability p_h is zero at point (1, A, A+1) or p_l is zero at point $(A, A+1, n)$ & for the given values of H.M. and $\left(\mu_2'\right)_{max}^{1/2}$, probability $p_k = 0$ at point (1, *n* - 1, *n*). Probability

$p_h = 0$ for $\{1 \le h < k < H \le l \le n\}$ or $\{1 \le h < k \le H < l \le n\}$ and $p_l = 0$ for $\{1 \le h \le H < k < l \le n\}$ or $\{1 \le h < H \le k < l \le n\}$. As we go on increasing the values of p_h & p_l, probability $p_k = 0$ tends to zero. For the given values of H.M. and $\left(\mu_2'\right)^{1/2}_{min}$, the values of entropies are same at all existing points & similarly for the given values of H.M. and $\left(\mu_2'\right)^{1/2}_{max}$, the values of entropies are same at all existing points.

Every interval is divided into many subintervals such that at common values of Second order moment, the values of minimum entropy for any two subintervals are same. These values of Second order moment are called switching points. At these values, we switch over entropy from one set of values of (h,k,l) to another set of values of (h,k,l).

Let $H \in (A, A+1]$, $1 \le A < n$, where A is an integer. h can take values $1,2,\ldots,A$; k can take values $h+1, \ldots, n-1$; l can take values $A+1, \ldots\ldots, n$. We calculate probability distributions in each possible interval for different values of Harmonic mean & Second order moment.

For calculating minimum entropy, we consider four types of points. These points are :

(1) $(A+\alpha, A+\beta, n)$

(2) $1, A+\alpha, A+\gamma)$

(3) $(A+\alpha, A+\delta, A+\xi)$

(4) $(1, A+\alpha, n)$

Before considering these points, we calculate minimum and maximum values of $\left(\mu_2'\right)^{1/2}$ say $\left(\mu_2'\right)^{1/2}_{min}$ and $\left(\mu_2'\right)^{1/2}_{max}$ from equations (5), (6) and (7).

For some value of $\mu_2' \gtrsim \left(\mu_2'\right)_{min}$, we calculate entropies at point $(A, A+1, n)$ and $(1, A, A+1)$ (for $A > 1$). By calculating these entropies, we can observe at which point entropy is minimum for given Harmonic mean and Second order moment. Since at above points probability distributions exist and entropy is minimum. These points are considered as $(A+\alpha, A+\beta, n)$ & $(1, A+\alpha, A+\gamma)$ by taking $\alpha = 0$, $\beta = \alpha + 1$, $\gamma = \beta$.

Now, we consider four types of points to calculate minimum entropy for given values of Harmonic mean and Second order moment. These are as follow:

2(a) When minimum entropy occurs at point $(A + \alpha, A + \beta, n)$:

Minimum entropy shifts from point $(A + \alpha, A + \beta, n)$ to one of the points $(A + \alpha, A + \lambda, A + \beta + 1)$, $(1, A + \alpha, A + \gamma)$, $(A + \alpha, A + \mu, n)$ and $(1, A + \alpha, n)$. Here $\beta > \alpha$, $A + \gamma < n$, $\alpha < \lambda < \beta + 1$, $\mu \neq \beta$.

First, we equate entropies at point $(A + \alpha, A + \beta, n)$ with points $(A + \alpha, A + \lambda, A + \beta + 1)$ and $(1, A + \alpha, A + \gamma)$. By equating entropies we get two values of second order moment. Minimum value out of these two values is considered as switching point. If both values of Second order moment do not lie in the feasible region {i.e. probability distribution does not exist} then we equate entropies at point $(A + \alpha, A + \beta, n)$ with point $(A + \alpha, A + \mu, n)$. If any switching point is obtained then minimum entropy shifts from point $(A + \alpha, A + \beta, n)$ to point $(A + \alpha, A + \mu, n)$ otherwise it shifts from point $(A + \alpha, A + \beta, \text{n})$ to point $(1, A + \alpha, n)$. So, first we are equating entropies as follow:

$$S(A + \alpha, A + \beta, n) = S(A + \alpha, A + \lambda, A + \beta + 1)$$

from equations (1) and (4)

$$p_1 \ln p_1 + q_1 \ln q_1 + r_1 \ln r_1 = p_2 \ln p_2 + q_2 \ln q_2 + r_2 \ln r_2 \qquad \text{...(8)}$$

Here $p_1, q_1, r_1, p_2, q_2, r_2$ represent probabilities at corresponding points and expressions for $p_1, q_1, r_1, p_2, q_2, r_2$ can be obtained from equation (4) by substituting $h = A + \alpha$, $k = A + \beta$, $l = n$ and $h = A + \alpha$, $k = A + \lambda$, $l = A + \beta + 1$

Equation (8) can be solved for value $\left(\mu_2'\right)_A$ (say).

Again, we are equating entropies as follow:

$S(A + \alpha, A + \beta, n) = S(1, A + \alpha, A + \gamma)$

$$p_1 \ln p_1 + q_1 \ln q_1 + r_1 \ln r_1 = p_3 \ln p_3 + q_3 \ln q_3 + r_3 \ln r_3 \qquad \text{...(9)}$$

this equation can be solved for value $\left(\mu_2'\right)_B$.

Now, we check whether $\left(\mu_2'\right)_A$ and $\left(\mu_2'\right)_B$ lie in the feasible region or not. When both values lie in the feasible region then minimum value out of these is considered as switching point. When any one value lies in the feasible region then that value is considered as switching point.

If $\left(\mu_2'\right)_A$ lies in the feasible region and is minimum then minimum entropy shifts from point (A+α, A+β, n) to point (A+α, A+λ, A+β+1), then

$$S_{min} = -p_1\ln p_1 - q_1\ln q_1 - r_1\ln r_1\,, \qquad \text{for } \mu_2' \le \left(\mu_2'\right)_A$$

$$S_{min} = -p_2\ln p_2 - q_2\ln q_2 - r_2\ln r_2, \qquad \text{for } \left(\mu_2'\right)_A \le \mu_2'$$

And, if $\left(\mu_2'\right)_B$ lies In the feasible region and is minimum then minimum entropy shifts from point $(A+\alpha, A+\beta, n)$ to point $(1, A+\alpha,\ A+\gamma)$. Then

$$S_{min} = -p_1\ln p_1 - q_1\ln q_1 - r_1\ln r_1\,, \qquad \text{for } \mu_2' \le \left(\mu_2'\right)_B$$

$$S_{min} = -p_3\ln p_3 - q_3\ln q_3 - r_3\ln r_3, \qquad \text{for } \left(\mu_2'\right)_B \le \mu_2'$$

When both values of second order moment do not lie in the feasible region then we equate entropies at point $(A+\alpha, A+\beta, n)$ with points $(A+\alpha, A+\mu, n)$ and $(1, A+\alpha, n)$.

Hence, $\qquad S(A+\alpha, A+\beta, n) = S(A+\alpha, A+\mu, n)$

$$p_1\ln p_1 + q_1\ln q_1 + r_1\ln r_1 = p_4\ln p_4 + q_4\ln q_4 + r_4\ln r_4 \qquad \ldots(10)$$

The above equation can be solved for value $\left(\mu_2'\right)_C$.

Again, we are equating entropies as follow:

$$S(A+\alpha, A+\beta, n) = S(1, A+\alpha, n)$$

$$p_1\ln p_1 + q_1\ln q_1 + r_1\ln r_1 = p_5\ln p_5 + q_5\ln q_5 + r_5\ln r_5 \qquad \ldots(11)$$

The above equation can be solved for value $\left(\mu_2'\right)_D$.

$$\left(\mu_2'\right)_D = \frac{Hn^2 - (A+\alpha)(n+A+\alpha)(n-H)}{H} \qquad \ldots(12)$$

This expression can also be obtained by equating $q_1 = 0$.

When condition $\left(\mu_2'\right)_P < \left(\mu_2'\right)_C < \left(\mu_2'\right)_D$ holds, then minimum entropy shifts from point $(A+\alpha, A+\beta, n)$ to point $(A+\alpha, A+\mu, n)$ and from this point entropy shifts to point $(1, A+\alpha, n)$. If above condition does not hold then minimum entropy shifts from point $(A+\alpha, A+\beta, n)$ to $(1, A+\alpha, n)$.

If $(\mu_2')_C$ lies in the feasible region then minimum entropy shifts from point $(A + \alpha, A + \beta, n)$ to point $(A + \alpha\, A + \mu, n)$, hence

$$S_{min} = -p_1 \ln p_1 - q_1 \ln q_1 - r_1 \ln r_1 , \qquad \text{for } \mu_2' \leq (\mu_2')_C$$

$$S_{min} = -p_4 \ln p_4 - q_4 \ln q_4 - r_4 \ln r_4, \qquad \text{for } (\mu_2')_C \leq \mu_2'$$

Now, minimum entropy shifts from point $(A + \alpha, A + \mu, n)$ to point $(1, A + \alpha, n)$.

$$S_{min} = -p_4 \ln p_4 - q_4 \ln q_4 - r_4 \ln r_4 , \qquad \text{for } \mu_2' \leq (\mu_2')_D$$

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \qquad \text{for } (\mu_2')_D \leq \mu_2'$$

If $(\mu_2')_C$ does not lie in the feasible region then minimum entropy shifts from point $(A + \alpha, A + \beta, n)$ to point $(1, A + \alpha, n)$.

$$S_{min} = -p_1 \ln p_1 - q_1 \ln q_1 - r_1 \ln r_1 , \qquad \text{for } \mu_2' \leq (\mu_2')_D$$

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \qquad \text{for } (\mu_2')_D \leq \mu_2'$$

2 (b) When minimum entropy occurs at point (1, A+α, A+γ):

While calculating we observe that minimum entropy shifts from point $(1, A + \alpha, A + \gamma)$ to one of the points $(A + \alpha, A + \delta, A + \xi)$, $(A + \alpha - 1, A + v, A + \gamma)$, $(1, A + \alpha, A + \Phi)$, $(1, A + \Psi, A + \gamma)$, or $(1, A + \gamma, n)$.

Here $A + \gamma < n, A + \xi < n, A + \Phi < n, \alpha < \delta < \xi, \alpha - 1 < v < \gamma, \alpha \neq \Psi$.

First, we equate entropies at point (1, A+α, A+γ) with points (A+α, A+δ, A+ξ), (A+α-1, A+v, A+γ) and (1, A+α, A+Φ). By equating entropies we obtain three values of Second order moment. We check whether these three values lie in the feasible region or not. If lie, then minimum value is considered as switching point. When only one value lies in the feasible region then that value is considered as switching point. Hence,

$$S(1, A+\alpha, A+\gamma) = S(A+\alpha, A+\delta, A+\xi)$$

$$p_3 \ln p_3 + q_3 \ln q_3 + r_3 \ln r_3 = p_6 \ln p_6 + q_6 \ln q_6 + r_6 \ln r_6 \qquad \text{...(13)}$$

By solving equation (13), we can get value $\left(\mu_2'\right)_E$.

Again, we are equating entropies as:

$$S(1, A+\alpha, A+\gamma) = S(A+\alpha-1, A+\nu, A+\gamma)$$

$$p_3 \ln p_3 + q_3 \ln q_3 + r_3 \ln r_3 = p_7 \ln p_7 + q_7 \ln q_7 + r_7 \ln r_7 \qquad \text{...(14)}$$

By solving equation (14), we get value $\left(\mu_2'\right)_F$. Again, entropies are equated as:

$$S(1, A+\alpha, A+\gamma) = S(1, A+\alpha, A+\Phi)$$

$$p_3 \ln p_3 + q_3 \ln q_3 + r_3 \ln r_3 = p_8 \ln p_8 + q_8 \ln q_8 + r_8 \ln r_8 \qquad \text{...(15)}$$

here calculated value is $\left(\mu_2'\right)_G$.

Now, we check the existence of $\left(\mu_2'\right)_E$, $\left(\mu_2'\right)_F$ and $\left(\mu_2'\right)_G$ in the feasible region. If $\left(\mu_2'\right)_E$ lies in the feasible region and is minimum then minimum entropy shifts from point $(1, A+\alpha, A+\gamma)$ to point $(A+\alpha, A+\delta, A+\xi)$. Then,

$$S_{\min} = -p_3 \ln p_3 - q_3 \ln q_3 - r_3 \ln r_3, \qquad \text{for } \mu_2' \le \left(\mu_2'\right)_E$$

$$S_{\min} = -p_6 \ln p_6 - q_6 \ln q_6 - r_6 \ln r_6, \qquad \text{for } \left(\mu_2'\right)_E \le \mu_2'$$

Again, if $\left(\mu_2'\right)_F$ lies in the feasible region and is minimum then minimum entropy shifts from point $(1, A+\alpha, A+\gamma)$ to point $(A+\alpha-1, A+\nu, A+\gamma)$. Then,

$$S_{\min} = -p_3 \ln p_3 - q_3 \ln q_3 - r_3 \ln r_3, \qquad \text{for } \mu_2' \le \left(\mu_2'\right)_F$$

$$S_{min} = -p_7 \ln p_7 - q_7 \ln q_7 - r_7 \ln r_7, \qquad \text{for } (\mu_2')_F \leq \mu_2'$$

Again, if $(\mu_2')_G$ is minimum then minimum entropy shifts from point $(1, A + \alpha, A + \gamma)$ to point $(1, A + \alpha, A + \Phi)$. Then,

$$S_{min} = -p_3 \ln p_3 - q_3 \ln q_3 - r_3 \ln r_3, \qquad \text{for } \mu_2' \leq (\mu_2')_G$$

$$S_{min} = -p_8 \ln p_8 - q_8 \ln q_8 - r_8 \ln r_8, \qquad \text{for } (\mu_2')_G \leq \mu_2'$$

When values of $(\mu_2')_E$, $(\mu_2')_F$ and $(\mu_2')_G$ do not lie in the feasible region, then we equate entropies at point $(1, A + \alpha, A + \gamma)$ with point $(1, A + \Psi, A + \gamma)$. By equating entropies as above, if we obtain any switching point then minimum entropy shifts from point $(1, A + \alpha, A + \gamma)$ to point $(1, A + \Psi, A + \gamma)$ and if we do not obtain any switching point then minimum entropy shifts from point $(1, A + \alpha, A + \gamma)$ to point $(1, A + \gamma, n)$. Hence,

$$S(1, A + \alpha, A + \gamma) = S(1, A + \Psi, A + \gamma)$$

$$p_3 \ln p_3 + q_3 \ln q_3 + r_3 \ln r_3 = p_9 \ln p_9 + q_9 \ln q_9 + r_9 \ln r_9 \qquad \text{...(16)}$$

Here calculated value is $(\mu_2')_H$.

Again, we equate entropies as:

$$S(1, A + \alpha, A + \gamma) = S(1, A + \gamma, n)$$

$$p_3 \ln p_3 + q_3 \ln q_3 + r_3 \ln r_3 = p_{10} \ln p_{10} + q_{10} \ln q_{10} + r_{10} \ln r_{10} \qquad \text{... (17)}$$

Here calculated value is $(\mu_2')_I$.

$$(\mu_2')_I = \frac{H(A+\gamma)^2 - (A+\gamma+1)(A+\gamma-H)}{H} \qquad \text{... (18)}$$

The expression of $(\mu_2')_I$ can also be obtained by equating $q_3 = 0$.

When condition $(\mu_2')_P < (\mu_2')_H < (\mu_2')_I$ holds, then minimum entropy shifts from point (1, A+α, A+γ) to point (1, A+Ψ, A+γ). Where $(\mu_2')_P$ is previous switching point and from point (1, A+Ψ, A+γ) minimum entropy shifts to point (1, A+γ, n). In this situation,

$S_{min} = -p_3\ln p_3 - q_3\ln q_3 - r_3\ln r_3$, for $\mu_2' \leq (\mu_2')_H$

$S_{min} = -p_9\ln p_9 - q_9\ln q_9 - r_9\ln r_9$, for $(\mu_2')_H \leq \mu_2'$

Now, minimum entropy shifts from point (1, A+ Ψ, A+γ) to point (1, A+γ, n). So,

$S_{min} = -p_9\ln p_9 - q_9\ln q_9 - r_9\ln r_9$, for $(\mu_2')_H \leq \mu_2' \leq (\mu_2')_I$

$S_{min} = -p_{10}\ln p_{10} - q_{10}\ln q_{10} - r_{10}\ln r_{10}$, for $(\mu_2')_I \leq \mu_2'$

If condition $(\mu_2')_P < (\mu_2')_H < (\mu_2')_I$ does not hold then minimum entropy shifts from point $(1, A+\alpha,\ A+\gamma)$ to point $(1, A+\gamma, n)$. So,

$S_{min} = -p_3\ln p_3 - q_3\ln q_3 - r_3\ln r_3$, for $\mu_2' \leq (\mu_2')_I$

$S_{min} = -p_{10}\ln p_{10} - q_{10}\ln q_{10} - r_{10}\ln r_{10}$, for $(\mu_2')_I \leq \mu_2'$

2 (e) When minimum entropy occurs at point $(A+\alpha, A+\delta, A+\xi)$:

We observe from calculation that minimum entropy shifts from point $(A+\alpha, A+\delta, A+\xi)$ to one of points $(A+\alpha, A+\sigma, A+\xi)$, $(A+\alpha, A+\xi, n)$ or $(1, A+\alpha, A+\xi)$. Here $\delta \neq \sigma$.

We equate entropies at point $(A+\alpha, A+\delta, A+\xi)$ with points $(A+\alpha, A+\sigma, A+\xi)$, $(A+\alpha, A+\xi, n)$ or $(1, A+\alpha, A+\xi)$. We obtain two values of second order moment. (Since by equating entropies at point $(A+\alpha, A+\delta, A+\xi)$ with points $(A+\alpha, A+\xi, n)$ & $(1, A+\alpha, A+\xi)$, we obtain same value of second order moment). Minimum of these two values of second order moment is considered as switching point. If only one value lies in the feasible region then this value is considered as switching point. So, we are equating entropies as:

$S(A+\alpha, A+\delta, A+\xi) = S(A+\alpha, A+\sigma, A+\xi)$

$$p_6\ln p_6 + q_6\ln q_6 + r_6\ln r_6 = p_{11}\ln p_{11} + q_{11}\ln q_{11} + r_{11}\ln r_{11} \quad \ldots(19)$$

Here calculated value is $(\mu_2')_J$.

Now, we are equating entropies as:

$S(A+\alpha,\ A+\delta,\ A+\xi) = S(A+\alpha,\ A+\xi,\ n)$

$$p_6 \ln p_6 + q_6 \ln q_6 + r_6 \ln r_6 = p_{12} \ln p_{12} + q_{12} \ln q_{12} + r_{12} \ln r_{12} \qquad \ldots(20)$$

Again,

$S(A+\alpha,\ A+\delta,\ A+\xi) = S(1,\ A+\alpha,\ A+\xi)$

$$p_6 \ln p_6 + q_6 \ln q_6 + r_6 \ln r_6 = p_{13} \ln p_{13} + q_{13} \ln q_{13} + r_{13} \ln r_{13} \qquad \ldots(21)$$

By solving equations (20) and (21) calculated value is $\left(\mu_2'\right)_K$.

Here, $$\left(\mu_2'\right)_K = \frac{H(A+\xi)^2 - (A+\alpha)(2A+\alpha+\xi)(A+\xi-H)}{H} \qquad \ldots(22)$$

The expression $\left(\mu_2'\right)_K$ can also be obtained by equating $q_6 = 0$.

When $\left(\mu_2'\right)_P < \left(\mu_2'\right)_J < \left(\mu_2'\right)_K$, then minimum entropy shifts from point $(A+\alpha, A+\delta, A+\xi)$ to point $(A+\alpha, A+\sigma, A+\xi)$ and from this point entropy shifts to point $(1, A+\alpha, A+\xi)$ or $(A+\alpha, A+\xi, n)$. So,

$S_{min} = -p_6 \ln p_6 - q_6 \ln q_6 - r_6 \ln r_6$, for $\mu_2' \le \left(\mu_2'\right)_J$

$S_{min} = -p_{11} \ln p_{11} - q_{11} \ln q_{11} - r_{11} \ln r_{11}$, for $\left(\mu_2'\right)_J \le \mu_2'$

Now, minimum entropy shifts from point $(A+\alpha, A+\sigma, A+\xi)$ to point $(1, A+\alpha, A+\xi)$ or $(A+\alpha, A+\xi, n)$. So,

$S_{min} = -p_{11} \ln p_{11} - q_{11} \ln q_{11} - r_{11} \ln r_{11}$, for $\left(\mu_2'\right)_J \le \mu_2' \le \left(\mu_2'\right)_K$

$S_{min} = -p_{12} \ln p_{12} - q_{12} \ln q_{12} - r_{12} \ln r_{12}$ or

$S_{min} = -p_{13} \ln p_{13} - q_{13} \ln q_{13} - r_{13} \ln r_{13}$, for $\left(\mu_2'\right)_K \le \mu_2'$

If condition $\left(\mu_2'\right)_P < \left(\mu_2'\right)_J < \left(\mu_2'\right)_K$ does not hold, then minimum entropy shifts from point $(A+\alpha, A+\delta, A+\xi)$ to point $(A+\alpha, A+\xi, n)$ or point $(1, A+\alpha, A+\xi)$.

$S_{min} = -p_6 \ln p_6 - q_6 \ln q_6 - r_6 \ln r_6$, for $\mu_2' \le \left(\mu_2'\right)_K$

$S_{min} = -p_{12} \ln p_{12} - q_{12} \ln q_{12} - r_{12} \ln r_{12}$ or

$S_{min} = -p_{13} \ln p_{13} - q_{13} \ln q_{13} - r_{13} \ln r_{13}$, for $\left(\mu_2'\right)_K \le \mu_2'$

Now, we check that minimum entropy shifts from point $(A + \alpha, A + \delta, A + \xi)$ to point $(A + \alpha, A + \xi, n)$ or $(1, A + \alpha, A + \xi)$. For this, we calculate entropies at point $(A + \alpha, A + \xi, n)$ and point $(1, A + \alpha, A + \xi)$ for value $\mu_2' \gtrsim (\mu_2')_K$. The point at which entropy is smaller for $(\mu_2')_K$, minimum entropy shifts to that point.

2 (d) When minimum entropy occurs at point $(1, A + \alpha, n)$:

Now, there are two cases:

(i) $A < n - 1$ (ii) $A = n - 1, \alpha = 0$

i) **When $A < n - 1$:** To study shifting of minimum entropy, entropies are equated at point $(1, A + \alpha, n)$ and $(A + \alpha - 1, A + \alpha, A + \alpha + 1)$. If we obtain any switching point then minimum entropy shifts from point $(1, A + \alpha, n)$ to point $(A + \alpha - 1, A + \alpha, A + \alpha + 1)$. If we do not obtain any switching point then entropies are equated at point $(1, A + \alpha, n)$ with points $(A + \alpha - 1, A + \epsilon, A + \alpha + 2)$, $(A + \alpha - 1, A + \alpha + 1, n)$, $(1, A + \alpha - 1, A + \alpha + 1)$ or $(A + \alpha - 2, A + \omega, A + \alpha + 1)$. Here $\alpha - 1 < \epsilon < \alpha + 2$, $\alpha - 2 < \omega < \alpha + 1$. By equating entropies as above we obtain four values of second order moment. Minimum value out of these four values is considered as switching point. If only one value lies in the feasible region then that value is considered as switching point. After this if we do not obtain any switching point then entropies are equated as above with points $(A + \alpha - 1, A + \sigma, A + \alpha + 2)$ and $(A + \alpha - 2, A + \omega, A + \alpha + 1)$ as discussed with $(A + \alpha - 1, A + \alpha, A + \alpha + 1)$. Now, we are equating entropies as follow:

$$S(1, A + \alpha, n) = S(A + \alpha - 1, A + \alpha, A + \alpha + 1)$$

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{14} \ln p_{14} + q_{14} \ln q_{14} + r_{14} \ln r_{14} \qquad \ldots(23)$$

Here calculated value is $(\mu_2')_L$. When this value lies in the feasible region, minimum entropy shifts from point $(1, A + \alpha, n)$ to point $(A + \alpha - 1, A + \alpha, A + \alpha + 1)$. So, in this situation

$$S_{\min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5 , \qquad \text{for } \mu_2' \le (\mu_2')_L$$

$$S_{\min} = -p_{14} \ln p_{14} - q_{14} \ln q_{14} - r_{14} \ln r_{14}, \qquad \text{for } (\mu_2')_L \le \mu_2'$$

If $(\mu_2')_L$ does not lie in the feasible region, then we equate entropies as:

$$S(1, A + \alpha, n) = S(A + \alpha - 1, A + \epsilon, A + \alpha + 2)$$

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{15} \ln p_{15} + q_{15} \ln q_{15} + r_{15} \ln r_{15} \qquad \ldots(24)$$

By solving equation (24), we get value $\left(\mu_2'\right)_M$.

Again,

$$S(1, A+\alpha, n) = S(A+\alpha-1, A+\alpha+1, n)$$

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{16} \ln p_{16} + q_{16} \ln q_{16} + r_{16} \ln r_{16} \qquad ...(25)$$

By solving equation (25), we get value $\left(\mu_2'\right)_N$.

$$S(1, A+\alpha, n) = S(1, A+\alpha-1, A+\alpha+1)$$

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{17} \ln p_{17} + q_{17} \ln q_{17} + r_{17} \ln r_{17} \qquad ...(26)$$

By solving equation (26) we get value $\left(\mu_2'\right)_O$.

$$S(1, A+\alpha, n) = S(A+\alpha-2, A+\omega, A+\alpha+1)$$

$$p_5 \ln p_5 + q_5 \ln q_5 + r_5 \ln r_5 = p_{18} \ln p_{18} + q_{18} \ln q_{18} + r_{18} \ln r_{18} \qquad ...(27)$$

By solving equation (27), we get value $\left(\mu_2'\right)_Q$.

Now, we check the existence of values $\left(\mu_2'\right)_M, \left(\mu_2'\right)_N, \left(\mu_2'\right)_O$ and $\left(\mu_2'\right)_Q$ in the feasible region. Minimum value out of these is considered as switching point.

If $\left(\mu_2'\right)_M$ lies in the feasible region and is minimum then minimum entropy shifts from point $(1, A+\alpha, n)$ to point $(A+\alpha-1, A+\epsilon, A+\alpha+2)$, then

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \qquad \text{for } \mu_2' \le \left(\mu_2'\right)_M$$

$$S_{min} = -p_{15} \ln p_{15} - q_{15} \ln q_{15} - r_{15} \ln r_{15}, \qquad \text{for } \left(\mu_2'\right)_M \le \mu_2'$$

If $\left(\mu_2'\right)_N$ lies in the feasible region and is minimum then minimum entropy shifts from point $(1, A+\alpha, n)$ to point $(A+\alpha-1, A+\alpha+1, n)$, then

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \qquad \text{for } \mu_2' \le \left(\mu_2'\right)_N$$

$$S_{min} = -p_{16} \ln p_{16} - q_{16} \ln q_{16} - r_{16} \ln r_{16}, \qquad \text{for } \left(\mu_2'\right)_N \le \mu_2'$$

If $\left(\mu_2'\right)_O$ lies in the feasible region and is minimum then minimum entropy shifts from point $(1, A+\alpha, n)$ to point $(1, A+\alpha-1, A+\alpha+1)$, then

$$S_{min} = -p_5 \ln p_5 - q_5 \ln q_5 - r_5 \ln r_5, \qquad \text{for } \mu_2' \le \left(\mu_2'\right)_O$$

$$S_{min} = -p_{17} \ln p_{17} - q_{17} \ln q_{17} - r_{17} \ln r_{17} \qquad \text{for } \left(\mu_2'\right)_O \le \mu_2'$$

If $(\mu_2')_Q$ lies in the feasible region and is minimum then minimum entropy shifts from point (1, A+α, n) to point (A+α-2, A+ω, A+α+1), then

$$S_{min} = -p_5\ln p_5 - q_5\ln q_5 - r_5\ln r_5, \qquad \text{for } \mu_2' \le (\mu_2')_Q$$

$$S_{min} = -p_{18}\ln p_{18} - q_{18}\ln q_{18} - r_{18}\ln r_{18}, \qquad \text{for } (\mu_2')_Q \le \mu_2'$$

If we do not obtain any switching point, then we consider shifting of minimum entropy for point $(A+\alpha-1, A+\epsilon, A+\alpha+2)$ and point $(A+\alpha-2, A+\omega, A+\alpha+1)$, as we have discussed above for point $(A+\alpha-1, A+\alpha, A+\alpha+1)$.

ii) **When $A = n-1$, $\alpha = 0$:** In this case, minimum entropy shifts from point $(1, A+\alpha, n)$ to point $(A+\alpha-1, A+\alpha, n)$. So,

$$S(1, A+\alpha, n) = S(A+\alpha-1, A+\alpha, n)$$

$$p_5\ln p_5 + q_5\ln q_5 + r_5\ln r_5 = p_{19}\ln p_{19} + q_{19}\ln q_{19} + r_{19}\ln r_{19} \qquad \ldots(28)$$

By solving equation (28), we get value $(\mu_2')_R$.

After this, minimum entropy shifts from point $(A+\alpha-1, A+\alpha, n)$ to point $(1, A+\alpha-1, n)$, then

$$S(A+\alpha-1, A+\alpha, n) = S(1, A+\alpha-1, n)$$

$$p_{19}\ln p_{19} + q_{19}\ln q_{19} + r_{19}\ln r_{19} = p_{20}\ln p_{20} + q_{20}\ln q_{20} + r_{20}\ln r_{20} \qquad \ldots(29)$$

solving equation (29) we get value $(\mu_2')_S$. This value can be obtained by equating $q_{19} = 0$.

$$(\mu_2')_S = \frac{Hn^2-(A+\alpha-1)(n+A+\alpha-1)(n-H)}{H} \qquad \ldots(30)$$

Further shifting of minimum entropy is discussed as point $(1, A+\alpha, n)$ upto $A+\alpha-1 = 2$.

In this way we have observed shifting of minimum entropy. But in the particular range entropies exist only at points $(1, A+\tau, n)$, where $(A+\tau)$ vary from 2 to $n-1$. We can obtain this range from equation (4) by $p_k = 0$ at points $(2, 3, n)$ and $(1, n-2, n-1)$ and from equat I on (7) i.e. $(\mu_S')_{max}$. We get following expressions.

$$\frac{Hn^2-2(n+2)(n-H)}{H} < \mu_2' < (n^2+n+1) - \frac{n(n+1)}{H}, \text{ for } A > 1 \qquad \ldots(31)$$

$$\frac{H(n-1)^2-n(n-1-H)}{H} < \mu_2' < (n^2+n+1) - \frac{n(n+1)}{H}, \text{ for all values of A} \qquad \ldots(32)$$

If these two conditions are satisfied then minimum entropy shifts from point $(1, A+\alpha, n)$ to point $(1, A+\tau, n)$, where $\alpha \neq \tau$.

$$S(1, A+\alpha, n) = S(1, A+\tau, n)$$

$$p_5\ln p_5 + q_5\ln q_5 + r_5\ln r_5 = p_{21}\ln p_{21} + q_{21}\ln q_{21} + r_{21}\ln r_{21} \qquad \ldots(33)$$

By solving equation (33), we get value $(\mu_2')_T$.

$S_{min} = -p_5\ln p_5 - q_5\ln q_5 - r_5\ln r_5$, for $\mu_2' \le (\mu_2')_T$

$S_{min} = -p_{21}\ln p_{21} - q_{21}\ln q_{21} - r_{21}\ln r_{21}$, for $(\mu_2')_T \le \mu_2'$

Hence, we have obtained the expressions for minimum Shannon entropy and switching points for the given values of Harmonic mean & Second order moment.

3. Minimum value of Shannon entropy when harmonic mean and second order moment are prescribed : Six faced dice :

Now, we calculate minimum Shannon entropy for a six faced dice i.e. $n = 6$. In this case we take H∈(1,2], (2,3], …, (5,6] and for a particular interval we give different values to Harmonic mean. For the given value of Harmonic mean, minimum and maximum values of second order moment are taken by Anju Rani [6]. Further we calculate entropies for the different values of Harmonic mean & Second order moment and out of these entropies we obtain minimum entropy. While calculating we observe that minimum entropy shifts from one set of values of (h, k, l) to another set of values of (h, k, l).

CASE-1 we consider the case when H ∈ (1,2]. In this interval we take values H=1.25, 1.5, 1.75, 2.0. But in the present paper we are considering only for H=1.25 and for others values of H entropies are given in table 2. Values of entropies are given in the table 1 for given values of Harmonic mean and Second order moment.

Here $h = 1$, $k = 2,3,4,5$, $l = 3,4,5,6$.

H=1.25

First, we obtain range of Second order moment, for given value of Harmonic mean.

From equations (6) & (7), $(\mu_2')^{1/2}_{min} = 1.4832$ and $(\mu_2')^{1/2}_{max} = 3.0659$

for $(\mu_2')^{1/2}_{min}$, $S_{min} = 0.67291$ and for $(\mu_2')^{1/2}_{max}$, $S_{min} = 0.55111$ [table 1]

$(\mu_2')^{1/2}$ / h,k,l	**1.4832**	**1.6**	**1.8**	**1.8439**	**2.0**	**2.2**	**2.2361**	**2.4**
1,2,3	.67291	.86426	**.76471**	**.61088**				
1.2,4	.67291	.7808	.85658	.86107	.83911	**.66634**	**.57994**	
1,2,5	.67291	.73969	.81221	.82162	.84493	.8422	.837	.78908
1,2,6	**.67291**	**.72329**	.77825	.78777	.81559	.8367	.8386	.83887
1.3,4				.61088	.78567	.67749	.57994	
1,3,5				.61088	.7234	.77548	.77793	.75869
1,3,6				**.61088**	**.68881**	.74183	.74826	.76697
1,4,5							.57994	.74428
1,4,6							**.57994**	.68975
1,5,6								

Contd…..

$(\mu_2')^{1/2}$ / h,k,l	2.6	2.6458	2.8	3.0	3.0659
1,2,3					
1,2,4					
1,2,5	.64077	.56229			
1,2,6	.81679	.8072	.75919	.63592	.55111
1,3,4					
1,3,5	**.63594**	**.56229**			
1,3,6	.76421	.75905	.72563	**.62489**	**.55111**
1,4,5	.66046	.56229			
1,4,6	.73814	.74093	.72712	.63259	.55111
1,5,6		**.56229**	**.71656**	.66409	.55111

Table [1]

Initially S_{min} occurs at point $(A+\alpha, A+\beta, n)$ i.e. (1, 2, 6). From this point S_{min} may shifts to any one out of points $(A+\alpha, A+\lambda, A+\beta+1)$ i.e. (1, 2, 3) or $(A+\alpha, A+\mu, n)$ i.e. (1,3,6), (1,4,6),(1,5,6).

Here $A = 1, \lambda = 1, \quad \mu = 2,3,4$

We observe from table [1] that it shifts from point (1,2,6) to point (1,2,3). To calculate switching point we equate entropies at these two points i.e. $S_{min}(1,2,6) = S_{min}(1,2,3)$

From equation (8)

$$\frac{[\mu_2'+24.8]}{45}\ln\frac{[\mu_2'+24.8]}{45}+\frac{[9.4-\mu_2']}{18}\ln\frac{[9.4-\mu_2']}{18}+\frac{[\mu_2'-2.2]}{30}\ln\frac{[\mu_2'-2.2]}{30}$$

$$=\frac{[\mu_2'+5]}{12}\ln\frac{[\mu_2'+5]}{12}+\frac{[3.4-\mu_2']}{3}\ln\frac{[3.4-\mu_2']}{3}+\frac{[\mu_2'-2.2]}{4}\ln\frac{[\mu_2'-2.2]}{4}$$

By solving this equation, we get $\left(\mu_2'\right)_1^{1/2} = 1.7943$ and $S_{min} = 0.77698$

S_{min} occurs at point (1,2,6) for $\left(\mu_2'\right)^{1/2} \in [1.4832, 1.7943]$

Now, S_{min} shifts from point (1,2,3) to point $(A+\alpha, A+\beta+1, n)$ i.e. (1,3,6).

So, $\quad S_{min}(1,2,3) = S_{min}(1,3,6)$

For $A+\alpha = 1$ & $A+\xi = 3$ from equation (22), we obtain $\left(\mu_2'\right)_2^{1/2} = 1.8439$ and $S_{min} = 0.61088$ [table 1].

S_{min} occurs at point (1,2,3) for $\left(\mu_2'\right)^{1/2} \in [1.7943, 1.8439]$

S_{min} shifts from point (1,3,6) to one of point $(A+\alpha, A+\lambda, A+\beta+2)$ i.e (1,2,4), (1,3,4) or $(A+\alpha, A+\mu, n)$ i.e. (1,2,6), (1,4,6), (1,5,6). But, here it shifts to point (1,2,4). $\lambda = 1,2$, $\alpha < \lambda < \beta+2$.

$$S_{min}(1,3,6) = S_{min}(1,2,4)$$

Again, from equation (8),

$$\frac{[\mu_2'+66.6]}{100}\ln\frac{[\mu_2'+66.6]}{100}+\frac{[9.4-\mu_2']}{20}\ln\frac{[9.4-\mu_2']}{20}+\frac{[\mu_2'-3.4]}{25}\ln\frac{[\mu_2'-3.4]}{25}$$

$$=\frac{[\mu_2'+10.4]}{21}\ln\frac{[\mu_2'+10.4]}{21}+\frac{[5-\mu_2']}{7}\ln\frac{[5-\mu_2']}{7}+\frac{[\mu_2'-2.2]}{10.5}\ln\frac{[\mu_2'-2.2]}{10.5}$$

By solving this equation, we get $\left(\mu_2'\right)_3^{1/2} = 2.1542$ and $S_{min} = 0.73224$

S_{min} occurs at point (1,3,6) for $\left(\mu_2'\right)^{1/2} \in [1.8439, 2.1542]$

S_{min} may shifts from point (1,2,4) to one of the points $(A+\alpha, A+\sigma, A+\beta+2)$ i.e. (1,3,4) or $(A+\alpha, A+\beta+2, n)$ i.e. (1,4,6). Here $\sigma \neq \lambda$, $\alpha < \sigma < \beta+2$. But we observe from table [1], it shifts to point (1,4,6). Switching point can be calculated from equation (22) for $A+\alpha = 1$, $\xi = \beta+2 = 3$, so $\left(\mu_2'\right)_4^{1/2} = 2.2361$ and $S_{min} = 0.57994$

S_{min} occurs at point (1,2,4) for $\left(\mu_2'\right)^{1/2} \in [2.1542, 2.2361]$

Further, S_{min} may shifts from point (1,4,6) to one of points $(A+\alpha, A+\lambda, A+\beta+3)$ i.e. (1,2,5), (1,3,5), (1,4,5). We observe that it shifts to (1,3,5). To obtain switching point for this shifting we equate entropies as:

$$S_{min}(1,4,6) = S_{min}(1,3,5)$$

Again, from equation (8)

$$\frac{[\mu_2'+116]}{165}\ln\frac{[\mu_2'+116]}{165}+\frac{[9.4-\mu_2']}{16.5}\ln\frac{[9.4-\mu_2']}{16.5}+\frac{[.6\mu_2'-3]}{11}\ln\frac{[.6\mu_2'-3]}{11}$$

$$=\frac{[\mu_2'+47]}{72}\ln\frac{[\mu_2'+47]}{72}+\frac{[7-\mu_2']}{12}\ln\frac{[7-\mu_2']}{12}+\frac{[\mu_2'-3.4]}{14.4}\ln\frac{[\mu_2'-3.4]}{14.4}$$

by solving this equation, we get $\left(\mu_2'\right)_5^{1/2} = 2.4943$ and $S_{min} = 0.72006$

S_{min} occurs at point (1,4,6) for $\left(\mu_2'\right)^{1/2} \in [2.2361, 2.4943]$

S_{min} may shifts from point (1,3,5) to one of points $(A+\alpha,\ A+\sigma,\ A+\beta+3)$ i.e. (1,2,5), (1,4,5) or $(A+\alpha,\ A+\beta+3,\ n)$ i.e. (1,5,6). Here $\sigma \neq \lambda,\ \alpha < \sigma < \beta + 3$. But minimum entropy shifts to point (1,5,6). Switching point can be calculated from equation (22) for $A+\alpha = 1,\ \xi = \beta + 3 = 4$, so

$$\left(\mu_2'\right)_6^{1/2} = 2.6458 \text{ and } S_{min.} = 0.56229$$

S_{min} occurs at point (1,3,5) for $\left(\mu_2'\right)^{1/2} \in [2.4943, 2.6458]$

For $H = 1.25$, $n = 6$ from equation (32), we get $7 < \left(\mu_2'\right) < 9.4$

Since $A+\beta+4 = n$, S_{min} may shifts from point (1,5,6) to one of points $(A+\alpha,\ A+\lambda,\ A+\beta+4)$ i.e. (1,2,6), (1,3,6), (1,4,6) and we find from calculation that it shifts to (1,3,6).

So, $\qquad S_{min}(1,5,6) = S_{min}(1,3,6)$

From equation (8),

$$\frac{[\mu_2'+173]}{240}\ln\frac{[\mu_2'+173]}{240} + \frac{[9.4-\mu_2']}{9.6}\ln\frac{[9.4-\mu_2']}{9.6} + \frac{[\mu_2'-7]}{10}\ln\frac{[\mu_2'-7]}{10}$$

$$= \frac{[\mu_2'+66.6]}{100}\ln\frac{[\mu_2'+66.6]}{100} + \frac{[9.4-\mu_2']}{20}\ln\frac{[9.4-\mu_2']}{20} + \frac{[\mu_2'-3.4]}{25}\ln\frac{[\mu_2'-3.4]}{25}$$

by solving this equation, we get $\left(\mu_2'\right)_7^{1/2} = 2.8149$ and $S_{min} = 0.72085$

S_{min} occurs at point (1,5,6) for $\left(\mu_2'\right)^{1/2} \in [2.6458, 2.8149]$

S_{min} occurs at point (1,3,6) for $\left(\mu_2'\right)^{1/2} \in [2.8149, 3.0659]$

Similarly, we can obtain minimum value of Shannon entropy for other intervals like (2,3], (3,4], (4,5], (5,6]. For these intervals values of minimum entropy and switching points are given in table [2].

4. The values of minimum entropy for given harmonic mean & second order moment in the table form :

We have calculated minimum value of Shannon entropy and values of switching points for the given values of Harmonic mean and Second order moment. These values are given in the table [2].

H	$(\mu_2')^{1/2}$	s_{min}
1.25	**1.4832**	**.67291**
	1.6	.72329
	1.7943	**.77698**
	1.8439	**.61088**
	2.0	.68881
	2.1542	**.73224**
	2.2	.66634
	2.2361	**.57994**
	2.4	.68975
	2.4943	**.72006**
	2.6	.63594
	2.6458	**.56229**
	2.8	.71656
	2.8149	**.72085**
	3.0	.62489
	3.0659	**.55111**
1.5	**1.7321**	**.63641**
	1.8	.68349
	2.0	.78199
	2.2	.8618
	2.2005	**.86201**
	2.2361	**.69308**
	2.4	.80719
	2.6	.89299
	2.6602	**.91327**
	2.7689	**.68688**
	2.8	.73218
	3.0	.88177
	3.1131	**.92955**
	3.2	.86172
	3.3166	**.67291**

H	$(\mu_2')^{1/2}$	s_{min}
1.5	3.4	.8424
	3.5256	**.94256**
	3.6	.91324
	3.8	.77706
	3.873	**.67291**
1.75	**1.8898**	**.40994**
	2.0	.49945
	2.2	.62459
	2.4	.73342
	2.4612	**.76429**
	2.4785	**.6519**
	2.6	.75967
	2.8	.87603
	2.9715	**.95115**
	3.0	.91231
	3.0938	**.68301**
	3.2	.81948
	3.4	.96119
	3.4871	**1.0019**
	3.6	.9057
	3.7225	**.69049**
	3.8	.87489
	3.951	**1.0275**
	4.0	1.0097
	4.3589	**.69271**
2.0	**2.0**	**0**
	2.2	.22019
	2.4	.38604
	2.6	.53141
	2.6457	**.56265**

H	$(\mu_2')^{1/2}$	s_{min}
2.0	**2.6458**	**.5623**
	2.646	**.56298**
	2.8	.66185
	3.0	.77793
	3.2	.87872
	3.2172	**.88644**
	3.3166	**.63641**
	3.6	.93584
	3.7297	**1.0091**
	3.8	.962
	4.0	**.66156**
	4.2	1.0226
	4.2392	**1.052**
	4.4	.98336
	4.6	.82224
	4.6904	**.67291**
2.25	**2.3805**	**.63641**
	2.4	.65597
	2.4996	**.7168**
	2.582	**.52978**
	2.6	.55415
	2.7408	**.62326**
	2.7689	**.45063**
	2.8	.49329
	2.8047	**.49838**
	2.8087	**.47931**
	2.9399	**.58617**
	3.0	.53794
	3.0551	**.45064**
	3.2	.58996
	3.4	.73516

H	$(\mu_2')^{1/2}$	s_{min}	H	$(\mu_2')^{1/2}$	s_{min}	H	$(\mu_2')^{1/2}$	s_{min}
2.25	**3.4238**	**.75029**	**2.5**	**4.9698**	**.98601**	**3.0**	**4.1554**	**.79978**
	3.4801	**.57239**		5.0	.78879		4.2	.80353
	3.6	.75044		**5.1186**	**.59297**		4.4	.73857
	3.8	.9234					**4.5826**	**.45663**
	3.9001	**.98406**	**2.75**	**2.8445**	**.47378**		4.6	.53059
	4.0	.92482		2.9	.38812		**4.7362**	**.84121**
	4.2	.62965		**2.9388**	**.18521**		4.8	.854
	4.2032	**.61551**		3.0	.27083		5.0	.8485
	4.4	1.0049		3.2	.46498		5.2	.75819
	4.4297	**1.0321**		3.4	.61671		**5.3198**	**.63436**
	4.6	.98148		**3.5697**	**.72477**		**5.3852**	**.5004**
	4.6765	**.94207**		3.6	.68834			
	4.8	.84092		**3.705**	**.42521**	**3.25**	**3.3397**	**.61726**
	4.9329	**.63641**		3.8	.5903		3.4	.68691
				4.0	.79909		**3.4286**	**.69698**
2.5	**2.6458**	**.67291**		**4.0549**	**.84235**		**3.4752**	**.48943**
	2.6646	**.70152**		**4.1174**	**.73017**		3.6	.52456
	2.8	.61123		**4.1341**	**.67644**		**3.6268**	**.42926**
	2.8636	**.32517**		4.2	.82664		**3.6374**	**.44175**
	3.0	.48022		**4.2933**	**.80458**		**3.7364**	**.52579**
	3.2	.63944		4.4	.69079		3.8	.43809
	3.3002	**.70652**		**4.4823**	**.50669**		**3.8531**	**.27123**
	3.3166	**.63641**		4.6	.82598		4.0	.51742
	3.4	.73502		**4.665**	**.92323**		**4.1716**	**.69735**
	3.4752	**.79872**		4.8	.92065		**4.1787**	**.65278**
	3.6	.53165		5.0	.85811		4.2	.67597
	3.6056	**.5004**		**5.1698**	**.72016**		4.4	.71296
	3.6812	**.63303**		5.2	.67913		4.6	.55246
	3.6878	**.61088**		**5.2657**	**.54703**		**4.6658**	**.39509**
	3.8	.71819					**4.7718**	**.71223**
	4.0	.84582	**3.0**	**3.0**	**0**		4.8	.73173
	4.1147	**.89991**		3.2	.26347		5.0	.79257
	4.2	.83574		3.4	.44664		5.2	.76283
	4.3589	**.56229**		3.6	.59663		5.4	.61352
	4.4	.70828		**3.6548**	**.6332**		**5.4842**	**.45458**
	4.5648	**.98601**		**3.7859**	**.34891**			
	4.6	.98413		3.8	.38516	**3.5**	**3.6056**	**.68301**
	4.8	.93219		4.0	.66152		3.7	.68032

Contd...

H	$\left(\mu_2'\right)^{1/2}$	s_{min}
3.5	**3.7431**	**.64629**
	3.7796	**.41004**
	3.8	.41089
	3.9	.23967
	3.9097	**.19144**
	4.0	.36673
	4.2	.60672
	4.3328	**.72182**
	4.3589	**.59823**
	4.4	.62707
	4.6	.57674
	4.7359	**.34042**
	4.8	.56845
	4.9119	**.78825**
	4.957	**.6519**
	5.0	.68628
	5.2	.72553
	5.4	.6477
	5.5435	**.47488**
	5.5678	**.41004**
3.75	**3.821**	**.5004**
	3.873	**.42383**
	3.8987	**.24506**
	3.9	.24547
	3.9581	**.10637**
	4.0	.20441
	4.2	.48653
	4.4	.67448
	4.4448	**.70816**
	4.4497	**.67291**
	4.4715	**.68907**
	4.5	.58688
	4.5092	**.52977**
	4.6	.544
	4.7958	**.28681**
	4.8	.31304
	5.0	.7853

H	$\left(\mu_2'\right)^{1/2}$	s_{min}
3.75	**5.0311**	**.82426**
	5.0569	**.81853**
	5.099	**.73856**
	5.1381	**.61088**
	5.2	.64779
	5.4	.63763
	5.6	.45877
	5.6213	**.41785**
	5.6391	**.36689**
4.0	**4.0**	**0**
	4.2	.35606
	4.4	.57172
	4.5114	**.66273**
	4.5918	**.62667**
	4.6	.60542
	.4507	**.46368**
	4.7	.45129
	4.8	.35356
	4.8477	**.23395**
	5.0	.66138
	5.0724	**.76996**
	5.2	.75611
	5.2214	**.75159**
	5.2726	**.63793**
	5.2915	**.56229**
	5.4	.59417
	5.6	.4965
	5.6877	**.36503**
	5.7009	**.32517**
4.25	**4.3182**	**.60575**
	4.4	.61785
	4.4192	**.46592**
	4.5	.54854
	4.5558	**.59666**
	4.5568	**.57789**
	4.6	.57402

H	$\left(\mu_2'\right)^{1/2}$	s_{min}
4.25	**4.7108**	**.50954**
	4.7465	**.36205**
	4.8	.34747
	4.893	**.18079**
	4.9	.22267
	5.0	**.52532**
	5.103	**.70791**
	5.2	.71039
	5.3629	**.67848**
	5.4233	**.50841**
	5.7533	**.29122**
	5.7548	**.2845**
4.5	**4.5826**	**.68688**
	4.6	.69043
	4.7019	**.65722**
	4.7258	**.4507**
	4.8	.3958
	4.8164	**.37698**
	4.8419	**.26426**
	4.9	.21248
	4.9329	**.12692**
	5.0	.37959
	5.184	**.71796**
	5.1962	**.63641**
	5.2	.63844
	5.4	.63203
	5.4868	**.60067**
	5.5377	.4507
	5.6	.46023
	5.7974	**.26268**
	5.8	.25419
	5.8023	**.24506**
4.75	**4.8068**	**.51477**
	4.8515	**.4511**
	4.872	**.27609**
	4.9	.24121

Contd...

H	$\left(\mu_2'\right)^{1/2}$	S_{min}	H	$\left(\mu_2'\right)^{1/2}$	S_{min}	H	$\left(\mu_2'\right)^{1/2}$	S_{min}
4.75	4.9109	.22297	5.0	5.6933	.43368	5.5	5.6889	.47402
	4.9258	.15201		5.7271	.32517		5.7	.46861
	4.9683	.07008		5.8813	.17191		5.7731	.40868
	5.0	.21557		5.8822	.16789		5.7918	.30462
	5.2	.64926					5.8	.30143
	5.3308	.77724	5.25	5.305	.59823		5.8597	.24824
	5.3754	.57629		5.4	.69271		5.8775	.18500
	5.4	.58005		5.4926	.73589		5.9467	.09078
	5.5965	.51875		5.5033	.59823			
	5.6	.51082		5.6	.5653	5.75	5.7973	.52365
	5.6382	.38947		5.6458	.53761		5.8	.52034
	5.8	.30081		5.6695	.41004		5.8358	.52365
	5.8421	.21585		5.781	.3437		5.8533	.29548
	5.8445	.20618		5.8	.28893		5.8891	.24869
				5.8064	.25732		5.9	.19234
5.0	5.0	0		5.9144	.13681		5.9014	.17867
	5.2905	.67307		5.9161	.12984		5.932	.1421
	5.2915	.67291					5.9417	.10475
	5.4	.67288	5.5	5.5678	.68892		5.9746	.04996
	5.5051	.64262		5.6	.68973			
	5.5317	.5004		5.6712	.66527	6.0	6.0	0

Table [2]

4. Conclusion

We have calculated minimum Shannon entropy for the given values of Harmonic Mean H and Second order moment $\left(\mu_2'\right)^{1/2}$. So, we observe that

1. For given values of H & $\left(\mu_2'\right)^{1/2}_{min}$, entropies are same for all existing points and similarly for given values of H & $\left(\mu_2'\right)^{1/2}_{max}$, entropies are same for all existing points.
2. When both moments take discrete and equal values, S_{min} is zero.
3. We observe from calculation, number of switching points are small when $\left(\mu_2'\right)^{1/2}$ is near to 1 & n and number of switching point are large when $\left(\mu_2'\right)^{1/2}$ is far away from 1 & n.

4. S_{min} is a piecewise concave function.
5. On increasing value of second order moment for the fixed value of H, S_{min} first increases and then decreases in a subinterval.

References

[1] Anju Rani, "Maximum and Minimum value of a moment of a probability distribution when value of some other moment is prescribed", *Pure and Applied Mathematical Science* 66, **(1997)** 57-67.

[2] Anju Rani, "Minimum Entropy Probability Distribution in presence inequalities constraints" , ***Recent** Advances in Information Theory. Statistics and Computer Application, Department of Mathematics and Statistics, C.C.S., Haryana Agriculture University, Hisar,* **(1997)** 166-170.

[3] Havrda, J.H.& Charvat, F., "Quantification Methods of Classificatory Process: Concept of **Structural** α Entropy", *Kybernatica* 3, **(1967)** 30-35.

[4] Kapur, J.N., "Generalised entropy of order α and type β", *The Maths Seminal* 4, **(1967)** 78-94.

[5] Kapur, J.N., "On minimum entropy probability distribution", *MI&TA, JW&S,* **(1994)** 203-218.

[6] Kapur, J.N., "An unorthodox measure of entropy", *MI&TA, JW&S,* **(1994)** 3-11.

[7] Renyi, A., "On measures of information and entropy",

Proc. 4th Berkely Symposium Maths Stat. Prob. **(1961)** 547-561.

[8] Shannon, C.E.(1948), "A mathematical theory of communication",

Journal of Bell System Tech. 27, 379-423.

Ganita Sandesh Vol. 26 No. (2) (*December* 2012) pp. 129-144 ISSN : 0970-9169

On Certain Generalized Hypergeometric Functions of Three Variables

Shanu Sharma

Department of Mathematics, Jai Narain Vyas University, JODHPUR (INDIA)

E-mail : shanu_19802006@yahoo.com

Nini Verma

Department of Mathematics, Vyas Engg.College, JODHPUR (INDIA)

Abstract

In this paper, we have given the definitions as well as integral representations of the complete set of Lauricella's hypergeometric functions of three variables by introducing three additional parameters. Formulae of differentiation of generalized functions have also been worked out. These formulae with parameters τ_1, τ_2, τ_3 *lead to many related applications of hypergeometric series of one and several variables.*

1. Introduction

Srivastava [5] has given the definitions and the integral representations of the complete set of Lauricella's hypergeometric functions of three variables. Single integral representation of Appell's function F_3 and then derivation of hypergeometric transformations and simple integral representation of certain hypergeometric functions of three variables have been attempted appreciably by Srivastava and Singhal [8]. Al-Shammery and Kalla [1] also presented the generalizations of Appell's functions $F_i^{\tau,\tau'}(w,z)$ $(i=1,2,3)$ in terms of Gaussian hypergeometric series ${}_2R_1(z)$ in order to establish some integral representations, recurrence relations, differentiation formulae for generalized Appell's functions.

The hypergeometric series in one and several variables appear naturally in variety of problems in mathematical, applied mathematics, statistical distributions, investigation of operations, theoretical physics and engineering sciences. The results can be further applied in the derivation of integral representations of certain hypergeometric functions of three variables as well as in the evaluation of the sums of certain triple series of the hypergeometric type.

In Mathai [6] Lauricella four hypergeometric functions F_A, F_B, F_C and F_D are defined. The set was completed by Shanti Saran [7] who gave ten more functions of three variables.

The integrals in the paper have been derived by means of fractional derivatives and integrals. The necessary rules of fractional integration by parts are as follows :

$$\int_a^b u \frac{\partial^\nu v}{\partial (b-x)^\nu} dx = \int_a^b v \frac{\partial^\nu u}{\partial (x-a)^\nu} dx \qquad \ldots (1.1)$$

The fractional derivatives occurring in this rule can be defined by integrals, if real part of ν is negative. Thus,

$$\left.\begin{aligned} \frac{\partial^\nu u}{\partial (x-a)^\nu} &= \frac{1}{\Gamma(-\nu)} \int_a^x (x-y)^{-\nu-1}\, u\,(y)\, dy \\ \frac{\partial^\nu v}{\partial (b-x)^\nu} &= \frac{1}{\Gamma(-\nu)} \int_x^b (y-x)^{-\nu-1}\, v\,(y)\, dy \end{aligned}\right\} R(\nu) < 0 \qquad \ldots (1.2)$$

If u and v can be expressed by the series :

$$\left.\begin{aligned} u &= \sum_{r=0}^{\infty} A_r (x-a)^{\rho+r-1} \\ \text{and} \qquad v &= \sum_{r=0}^{\infty} B_s (b-x)^{\sigma+s-1} \end{aligned}\right\} \qquad \ldots (1.3)$$

then the fractional derivatives are obtained by differentiating these series term by term and using the definition :

$$\frac{\partial^\nu\, w^{\mu-1}}{\partial\, w^\nu} = \frac{\Gamma(\mu)\, w^{\mu-\nu-1}}{\Gamma(\mu-\nu)} \qquad \ldots (1.4)$$

for fractional derivatives which holds for all values of ν except for $\nu = \mu$.

2. Definitions

In this section, we consider only the first four Lauricella functions R_A, R_B, R_C and R_D and the remaining ten functions have been considered in the next section. These functions are defined by introducing three additional parameters namely τ_1, τ_2 and τ_3. Their integral representations have also been derived as per the rules given in section 1. These functions of Lauricella set are defined as

$$R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 m+\tau_2 n+\tau_3 p)\,\Gamma(\beta_1+\tau_1 m)\,\Gamma(\beta_2+\tau_2 n)\,\Gamma(\beta_3+\tau_3 p)}{m\,!\,n\,!\,p\,!\,\Gamma(\gamma_1+\tau_1 m)\,\Gamma(\gamma_2+\tau_2 n)\,\Gamma(\gamma_3+\tau_3 p)\,\Gamma(\alpha_1)}$$

$$\cdot\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}\cdot x^m y^n z^p \qquad \ldots(2.1)$$

for $|x|+|y|+|z|<1,\ \tau_1,\tau_2,\tau_3>0,\ \tau_1,\tau_2,\tau_3\in R^+$

$$R_B^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3;\gamma_1;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 m)\,\Gamma(\alpha_2+\tau_2 n)\,\Gamma(\alpha_3+\tau_3 p)\,\Gamma(\beta_1+\tau_1 m)\,\Gamma(\beta_2+\tau_2 n)\,\Gamma(\beta_3+\tau_3 p)}{m\,!\,n\,!\,p\,!\,\Gamma(\gamma_1+\tau_1 m+\tau_2 n+\tau_3 p)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\alpha_3)}$$

$$\cdot\frac{\Gamma(\gamma_1)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}\cdot x^m y^n z^p \qquad \ldots(2.2)$$

for $|x|<1, |y|<1, |z|<1,\ \tau_1,\tau_2,\tau_3>0,\ \tau_1,\tau_2,\tau_3\in R^+$

$$R_C^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 m+\tau_2 n+\tau_3 p)\,\Gamma(\beta_1+\tau_1 m+\tau_2 n+\tau_3 p)\,\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}{m\,!\,n\,!\,p\,!\,\Gamma(\gamma_1+\tau_1 m)\,\Gamma(\gamma_2+\tau_2 n)\,\Gamma(\gamma_3+\tau_3 p)\,\Gamma(\alpha_1)\,\Gamma(\beta_1)}\cdot x^m y^n z^p \qquad \ldots(2.3)$$

for $\left|\sqrt{x}\right|+\left|\sqrt{y}\right|+\left|\sqrt{z}\right|<1,\ \tau_1,\tau_2,\tau_3>0,\ \tau_1,\tau_2,\tau_3\in R^+$

$$R_D^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1;x,y,z)$$

$$=\sum_{m,n,p=0}^{\infty}\frac{\Gamma(\alpha_1+\tau_1 m+\tau_2 n+\tau_3 p)\,\Gamma(\beta_1+\tau_1 m)\,\Gamma(\beta_2+\tau_2 n)\,\Gamma(\beta_3+\tau_3 p)\Gamma(\gamma_1)}{m!\,n!\,p!\,\Gamma(\gamma_1+\tau_1 m+\tau_2 n+\tau_3 p)\,\Gamma(\alpha_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_3)}\cdot x^m y^n z^p \quad \ldots(2.4)$$

for $\left|x\right|<1,\left|y\right|<1,\left|z\right|<1,\ \tau_1,\tau_2,\tau_3>0,\ \tau_1,\tau_2,\tau_3\in R^+$

The integral representations of the functions R_A, R_B, R_C and R_D are as follows :

$$\frac{\Gamma(\nu_1)\,\Gamma(\gamma_1-\nu_1)\,\Gamma(\nu_2)\,\Gamma(\gamma_2-\nu_2)\,\Gamma(\nu_3)\,\Gamma(\gamma_3-\nu_3)}{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}\cdot R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;s^{\tau_1},t^{\tau_2},u^{\tau_3})$$

$$=\int_0^1\int_0^1\int_0^1 x^{\nu_1-1}(1-x)^{\gamma_1-\nu_1-1}\,y^{\nu_2-1}(1-y)^{\gamma_2-\nu_2-1}\,z^{\nu_3-1}(1-z)^{\gamma_3-\nu_3-1}$$

$$\cdot R_A^{\tau_1,\tau_2,\tau_3}[\alpha_1,\beta_1,\beta_2,\beta_3;\nu_1,\nu_2,\nu_3;(sx)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_3}]\,dx\,dy\,dz, \quad \ldots(2.5)$$

where

$$\left|s\right|+\left|t\right|+\left|u\right|<1,0<R(\nu_1)<R(\gamma_1);0<R(\nu_2)<R(\gamma_2);0<R(\nu_3)<R(\gamma_3)$$

$$\frac{\Gamma(\alpha_1)\,\Gamma(\lambda_1-\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\lambda_2-\alpha_2)\,\Gamma(\alpha_3)\,\Gamma(\lambda_3-\alpha_3)}{\Gamma(\lambda_1)\,\Gamma(\lambda_2)\,\Gamma(\lambda_3)}\cdot R_B^{\tau_1,\tau_2,\tau_3}[\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3;\gamma_1;s^{\tau_1},t^{\tau_2},u^{\tau_3}]$$

$$=\int_0^1\int_0^1\int_0^1 x^{\alpha_1-1}(1-x)^{\lambda_1-\alpha_1-1}\,y^{\alpha_2-1}(1-y)^{\lambda_2-\alpha_2-1}\,z^{\alpha_3-1}(1-z)^{\lambda_3-\alpha_3-1}$$

$$\cdot R_B^{\tau_1,\tau_2,\tau_3}[\lambda_1,\lambda_2,\lambda_3,\beta_1,\beta_2,\beta_3;\gamma_1;(sx)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_3}]\,dx\,dy\,dz, \quad \ldots(2.6)$$

where

$$|s|<1,|t|<1,|u|<1;0<R(\alpha_1)<R(\lambda_1);0<R(\alpha_2)<R(\lambda_2);0<R(\alpha_3)<R(\lambda_3)$$

$$\frac{\Gamma(\nu_1)\Gamma(\gamma_1-\nu_1)\Gamma(\nu_2)\Gamma(\gamma_2-\nu_2)\Gamma(\nu_3)\Gamma(\gamma_3-\nu_3)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}\cdot R_C^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1;\gamma_1,\gamma_2,\gamma_3;s^{\tau_1},t^{\tau_2},u^{\tau_3})$$

$$=\int_0^1\int_0^1\int_0^1 x^{\nu_1-1}(1-x)^{\gamma_1-\nu_1-1}y^{\nu_2-1}(1-y)^{\gamma_2-\nu_2-1}z^{\nu_3-1}(1-z)^{\gamma_3-\nu_3-1}$$

$$\cdot R_C^{\tau_1,\tau_2,\tau_3}[\alpha_1,\beta_1;\nu_1,\nu_2,\nu_3;(sx)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_3}]\,dx\,dy\,dz\,, \quad \ldots(2.7)$$

where

$$\left|\sqrt{x}\right|+\left|\sqrt{y}\right|+\left|\sqrt{z}\right|<1,0<R(\nu_1)<R(\gamma_1);0<R(\nu_2)<R(\gamma_2);0<R(\nu_3)<R(\gamma_3)$$

$$\frac{\Gamma(\beta_1)\Gamma(\mu_1-\beta_1)\Gamma(\beta_2)\Gamma(\mu_2-\beta_2)\Gamma(\beta_3)\Gamma(\mu_3-\beta_3)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)}\cdot R_D^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1;s^{\tau_1},t^{\tau_2},u^{\tau_3})$$

$$=\int_0^1\int_0^1\int_0^1 x^{\beta_1-1}(1-x)^{\mu_1-\beta_1-1}y^{\beta_2-1}(1-y)^{\mu_2-\beta_2-1}z^{\beta_3-1}(1-z)^{\mu_3-\beta_3-1}$$

$$\cdot R_D^{\tau_1,\tau_2,\tau_3}[\alpha_1,\mu_1,\mu_2,\mu_3;\gamma_1;(sx)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_3}]\,dx\,dy\,dz\,, \quad \ldots(2.8)$$

where

$$|s|<1,|t|<1,|u|<1;0<R(\beta_1)<R(\mu_1);0<R(\beta_2)<R(\mu_2);0<R(\beta_3)<R(\mu_3)$$

Now, we give the proof of (2.5) and proofs for other functions are similar.

Let us consider

$$\frac{\partial^{\nu_1-\gamma_1+\nu_2-\gamma_2+\nu_3-\gamma_3}}{\partial s^{\nu_1-\gamma_1}\,\partial t^{\nu_2-\gamma_2}\,\partial u^{\nu_3-\gamma_3}}\left[s^{\nu_1-1}t^{\nu_2-1}u^{\nu_3-1}R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\nu_1,\nu_2,\nu_3;s^{\tau_1},t^{\tau_2},u^{\tau_3})\right]$$

$$\Rightarrow\quad \frac{\partial^{\nu_1-\gamma_1+\nu_2-\gamma_2+\nu_3-\gamma_3}}{\partial s^{\nu_1-\gamma_1}\,\partial t^{\nu_2-\gamma_2}\,\partial u^{\nu_3-\gamma_3}}\sum_{m,n,p=0}^{\infty}\frac{\Gamma(\alpha_1+\tau_1 m+\tau_2 n+\tau_3 p)\,\Gamma(\beta_1+\tau_1 m)}{m\,!\,n\,!\,p\,!\,\Gamma(\nu_1+\tau_1 m)\,\Gamma(\nu_2+\tau_2 n)\,\Gamma(\nu_3+\tau_3 p)}$$

$$\cdot\frac{\Gamma(\beta_2+\tau_2 n)\,\Gamma(\beta_3+\tau_3 p)\,\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)}{\Gamma(\alpha_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_3)}\cdot s^{\nu_1+\tau_1 m-1}\,t^{\nu_2+\tau_2 n-1}\,u^{\nu_3+\tau_3 p-1}$$

$$\Rightarrow\quad \sum_{m,n,p=0}^{\infty}\frac{\Gamma(\alpha_1+\tau_1 m+\tau_2 n+\tau_3 p)\,\Gamma(\beta_1+\tau_1 m)\,\Gamma(\beta_2+\tau_2 n)\,\Gamma(\beta_3+\tau_3 p)}{m\,!\,n\,!\,p\,!\,\Gamma(\nu_1+\tau_1 m)\,\Gamma(\nu_2+\tau_2 n)\,\Gamma(\nu_3+\tau_3 p)}$$

$$\cdot\frac{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)\,\Gamma(\nu_1+\tau_1 m)\,\Gamma(\nu_2+\tau_2 n)\,\Gamma(\nu_3+\tau_3 p)}{\Gamma(\alpha_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_3)\,\Gamma(\gamma_1+\tau_1 m)\,\Gamma(\gamma_2+\tau_2 n)\,\Gamma(\gamma_3+\tau_3 p)}$$

$$\cdot s^{\gamma_1+\tau_1 m-1}\,t^{\gamma_2+\tau_2 n-1}\,u^{\gamma_3+\tau_3 p-1}$$

By using (1.4) and getting

$$\Rightarrow\quad \frac{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\nu_3)}{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;s^{\tau_1},t^{\tau_2},u^{\tau_3})\cdot s^{\gamma_1-1}\,t^{\gamma_2-1}\,u^{\gamma_3-1}$$

Hence

$$s^{\gamma_1-1}\,t^{\gamma_2-1}\,u^{\gamma_3-1}\,R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;s^{\tau_1},t^{\tau_2},u^{\tau_3})$$

$$=\frac{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}{\Gamma(\nu_1)\,\Gamma(\gamma_1-\nu_1)\,\Gamma(\nu_2)\,\Gamma(\gamma_2-\nu_2)\,\Gamma(\nu_3)\,\Gamma(\gamma_3-\nu_3)}\int_0^1\int_0^1\int_0^1 p^{\nu_1-1}\,q^{\nu_2-1}\,r^{\nu_3-1}$$

$$\cdot(s-p)^{\gamma_1-\nu_1-1}(t-q)^{\gamma_2-\nu_2-1}(u-r)^{\gamma_3-\nu_3-1}R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\nu_1,\nu_2,\nu_3;$$

$$\cdot p^{\tau_1},q^{\tau_2},r^{\tau_3})\,dp\,dq\,dr \qquad \ldots(2.9)$$

Putting $p = sx$, $q = ty$, $r = uz$, we get the result by using (1.2)

$$\frac{\Gamma(\nu_1)\Gamma(\gamma_1-\nu_1)\Gamma(\nu_2)\Gamma(\gamma_2-\nu_2)\Gamma(\nu_3)\Gamma(\gamma_3-\nu_3)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)} R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;$$

$$\cdot\gamma_1,\gamma_2,\gamma_3;s^{\tau_1},t^{\tau_2},u^{\tau_3})$$

$$= \int_0^1\int_0^1\int_0^1 x^{\nu_1-1}(1-x)^{\gamma_1-\nu_1-1}y^{\nu_2-1}(1-y)^{\gamma_2-\nu_2-1}z^{\nu_3-1}(1-z)^{\gamma_3-\nu_3-1}$$

$$\cdot R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\nu_1,\nu_2,\nu_3;(sx)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_3})\,dx\,dy\,dz\,, \quad \ldots (2.10)$$

where

$$|s|+|t|+|u|<1, 0<R(\nu_1)<R(\gamma_1); 0<R(\nu_2)<R(\gamma_2); 0<R(\nu_3)<R(\gamma_3)$$

Similar representations hold for other functions.

i) The remaining ten functions of the Lauricella set are defined as :

$$R_E^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 m+\tau_2 n+\tau_3 p)\Gamma(\beta_1+\tau_1 m)\Gamma(\beta_2+\tau_2 n+\tau_3 p)\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}{m!\,n!\,p!\,\Gamma(\gamma_1+\tau_1 m)\Gamma(\gamma_2+\tau_2 n)\Gamma(\gamma_3+\tau_3 p)\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)}$$

$$\cdot x^m y^n z^p \quad \ldots (2.11)$$

$$R_F^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 m+\tau_2 n+\tau_3 p)\Gamma(\beta_1+\tau_1 m+\tau_3 p)\Gamma(\beta_2+\tau_2 n)\Gamma(\gamma_1)\Gamma(\gamma_2)}{m!\,n!\,p!\,\Gamma(\gamma_1+\tau_1 m)\Gamma(\gamma_2+\tau_2 n+\tau_3 p)\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(\beta_2)}$$

$$\cdot x^m y^n z^p \quad \ldots (2.12)$$

$$R_G^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$=\sum_{m,n,p=0}^{\infty}\frac{\Gamma(\alpha_1+\tau_1 m+\tau_2 n+\tau_3 p)\,\Gamma(\beta_1+\tau_1 m)\,\Gamma(\beta_2+\tau_2 n)\,\Gamma(\beta_3+\tau_3 p)\,\Gamma(\gamma_1)\,\Gamma(\gamma_2)}{m!\,n!\,p!\,\Gamma(\gamma_1+\tau_1 m)\,\Gamma(\gamma_2+\tau_2 n+\tau_3 p)\,\Gamma(\alpha_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_3)}$$

$$\cdot x^m y^n z^p \qquad \ldots (2.13)$$

$$R_K^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z)$$

$$=\sum_{m,n,p=0}^{\infty}\frac{\Gamma(\alpha_1+\tau_1 m)\Gamma(\alpha_2+\tau_2 n+\tau_3 p)\,\Gamma(\beta_1+\tau_1 m+\tau_3 p)\,\Gamma(\beta_2+\tau_2 n)\,\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}{m!\,n!\,p!\,\Gamma(\gamma_1+\tau_1 m)\,\Gamma(\gamma_2+\tau_2 n)\,\Gamma(\gamma_3+\tau_3 p)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\beta_1)\,\Gamma(\beta_2)}$$

$$\cdot x^m y^n z^p \qquad \ldots (2.14)$$

$$R_M^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$=\sum_{m,n,p=0}^{\infty}\frac{\Gamma(\alpha_1+\tau_1 m)\Gamma(\alpha_2+\tau_2 n)\Gamma(\alpha_3+\tau_3 p)\,\Gamma(\beta_1+\tau_1 m+\tau_3 p)\,\Gamma(\beta_2+\tau_2 n)\,\Gamma(\gamma_1)\,\Gamma(\gamma_2)}{m!\,n!\,p!\,\Gamma(\gamma_1+\tau_1 m)\,\Gamma(\gamma_2+\tau_2 n+\tau_3 p)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\alpha_3)\,\Gamma(\beta_1)\Gamma(\beta_2)}$$

$$\cdot x^m y^n z^p \qquad \ldots (2.15)$$

$$R_N^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$=\sum_{m,n,p=0}^{\infty}\frac{\Gamma(\alpha_1+\tau_1 m)\Gamma(\alpha_2+\tau_2 n)\Gamma(\alpha_3+\tau_3 p)\,\Gamma(\beta_1+\tau_1 m+\tau_3 p)\,\Gamma(\beta_2+\tau_2 n)\,\Gamma(\gamma_1)\,\Gamma(\gamma_2)}{m!\,n!\,p!\,\Gamma(\gamma_1+\tau_1 m)\,\Gamma(\gamma_2+\tau_2 n+\tau_3 p)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\alpha_3)\,\Gamma(\beta_1)\Gamma(\beta_2)}$$

$$\cdot x^m y^n z^p \qquad \ldots (2.16)$$

$$R_P^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_1,\beta_1,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 m+\tau_3 p)\Gamma(\alpha_2+\tau_2 n)\Gamma(\beta_1+\tau_1 m+\tau_2 n)\Gamma(\beta_2+\tau_3 p)\Gamma(\gamma_1)\Gamma(\gamma_2)}{m!n!p!\Gamma(\gamma_1+\tau_1 m)\Gamma(\gamma_2+\tau_2 n+\tau_3 p)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)} \cdot x^m y^n z^p \quad \ldots (2.17)$$

$R_R^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 m+\tau_3 p)\Gamma(\alpha_2+\tau_2 n)\Gamma(\beta_1+\tau_1 m+\tau_3 p)\Gamma(\beta_2+\tau_2 n)\Gamma(\gamma_1)\Gamma(\gamma_2)}{m!n!p!\Gamma(\gamma_1+\tau_1 m)\Gamma(\gamma_2+\tau_2 n+\tau_3 p)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)} \cdot x^m y^n z^p \quad \ldots (2.18)$$

$R_S^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_1,\gamma_1;x,y,z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 m)\Gamma(\alpha_2+\tau_2 n+\tau_3 p)\Gamma(\beta_1+\tau_1 m)\Gamma(\beta_2+\tau_2 n)\Gamma(\beta_3+\tau_3 p)\Gamma(\gamma_1)}{m!n!p!\Gamma(\gamma_1+\tau_1 m+\tau_2 n+\tau_3 p)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)} \cdot x^m y^n z^p \quad \ldots (2.19)$$

$R_T^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_1,\gamma_1;x,y,z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 m)\Gamma(\alpha_2+\tau_2 n+\tau_3 p)\Gamma(\beta_1+\tau_1 m+\tau_3 p)\Gamma(\beta_2+\tau_2 n)\Gamma(\gamma_1)}{m!n!p!\Gamma(\gamma_1+\tau_1 m+\tau_2 n+\tau_3 p)\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)} \cdot x^m y^n z^p \quad \ldots (2.20)$$

The regions of convergence of the above series have been discussed by Shanti Saran[7].The triple summation in the above series runs for $m, n, p = 0$ to ∞.

Particular Case : When $\tau_1, \tau_2, \tau_3 = 1$ in the definitions of the entire set of Lauricella's functions, above definitions reduces to functions given earlier by Srivastava [5]. By taking n variables and n parameters

$\tau_1, \tau_2, \ldots, \tau_n$, we can define and give their integral representation for Generalized Hypergeometric Functions in a similar manner to that of Garg, Mishra and Kalla [3].

The integral representations of the above functions are :

$$\frac{\Gamma(\nu_1)\,\Gamma(\gamma_1-\nu_1)\,\Gamma(\nu_2)\,\Gamma(\gamma_2-\nu_2)\,\Gamma(\nu_3)\,\Gamma(\gamma_3-\nu_3)}{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}$$

$$\cdot R_E^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;s^{\tau_1},t^{\tau_2},u^{\tau_3})$$

$$=\int_0^1\int_0^1\int_0^1 x^{\nu_1-1}(1-x)^{\gamma_1-\nu_1-1}\,y^{\nu_2-1}(1-y)^{\gamma_2-\nu_2-1}\,z^{\nu_3-1}(1-z)^{\gamma_3-\nu_3-1}$$

$$\cdot R_E^{\tau_1,\tau_2,\tau_3}[\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\nu_1,\nu_2,\nu_3;(sx)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_3}]\,dx\,dy\,dz\,, \quad \ldots (2.21)$$

where $0 < R(\nu_1) < R(\gamma_1)\,; 0 < R(\nu_2) < R(\gamma_2)\,; 0 < R(\nu_3) < R(\gamma_3)$

Similar representations hold for R_G, R_K, R_N and R_S viz.

$$\frac{\Gamma(\beta_1)\,\Gamma(\lambda_1-\beta_1)\,\Gamma(\beta_2)\,\Gamma(\lambda_2-\beta_2)\,\Gamma(\beta_3)\,\Gamma(\lambda_3-\beta_3)}{\Gamma(\lambda_1)\,\Gamma(\lambda_2)\,\Gamma(\lambda_3)}$$

$$\cdot R_G^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;s^{\tau_1},t^{\tau_2},u^{\tau_3})$$

$$=\int_0^1\int_0^1\int_0^1 x^{\beta_1-1}(1-x)^{\lambda_1-\beta_1-1}\,y^{\beta_2-1}(1-y)^{\lambda_2-\beta_2-1}\,z^{\beta_3-1}(1-z)^{\lambda_3-\beta_3-1}$$

$$\cdot R_G^{\tau_1,\tau_2,\tau_3}[\alpha_1,\alpha_1,\alpha_1,\lambda_1,\lambda_2,\lambda_3;\gamma_1,\gamma_2,\gamma_2;(sx)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_3}]\,dx\,dy\,dz\,, \quad \ldots (2.22)$$

where $0 < R(\beta_1) < R(\lambda_1)\,; 0 < R(\beta_2) < R(\lambda_2)\,; 0 < R(\beta_3) < R(\lambda_3)$

$$\frac{\Gamma(\nu_1)\Gamma(\gamma_1-\nu_1)\Gamma(\nu_2)\Gamma(\gamma_2-\nu_2)\Gamma(\nu_3)\Gamma(\gamma_3-\nu_3)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_3)}$$

$$\cdot R_K^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;s^{\tau_1},t^{\tau_2},u^{\tau_3})$$

$$=\int_0^1\int_0^1\int_0^1 x^{\nu_1-1}(1-x)^{\gamma_1-\nu_1-1}y^{\nu_2-1}(1-y)^{\gamma_2-\nu_2-1}z^{\nu_3-1}(1-z)^{\gamma_3-\nu_3-1}$$

$$\cdot R_K^{\tau_1,\tau_2,\tau_3}[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\nu_1,\nu_2,\nu_3;(sx)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_3}]\,dx\,dy\,dz\,, \quad \ldots (2.23)$$

where $0<R(\nu_1)<R(\gamma_1);0<R(\nu_2)<R(\gamma_2);0<R(\nu_3)<R(\gamma_3)$

$$\frac{\Gamma(\alpha_1)\Gamma(\lambda_1-\alpha_1)\Gamma(\alpha_2)\Gamma(\lambda_2-\alpha_2)\Gamma(\alpha_3)\Gamma(\lambda_3-\alpha_3)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)}$$

$$\cdot R_N^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;s^{\tau_1},t^{\tau_2},u^{\tau_3})$$

$$=\int_0^1\int_0^1\int_0^1 x^{\alpha_1-1}(1-x)^{\lambda_1-\alpha_1-1}y^{\alpha_2-1}(1-y)^{\lambda_2-\alpha_2-1}z^{\alpha_3-1}(1-z)^{\lambda_3-\alpha_3-1}$$

$$\cdot R_N^{\tau_1,\tau_2,\tau_3}[\lambda_1,\lambda_2,\lambda_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;(sx)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_3}]\,dx\,dy\,dz\,, \quad \ldots (2.24)$$

where $0<R(\alpha_1)<R(\lambda_1);0<R(\alpha_2)<R(\lambda_2);0<R(\alpha_3)<R(\lambda_3)$

$$\frac{\Gamma(\beta_1)\Gamma(\lambda_1-\beta_1)\Gamma(\beta_2)\Gamma(\lambda_2-\beta_2)\Gamma(\beta_3)\Gamma(\lambda_3-\beta_3)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)}$$

$$\cdot R_S^{\tau_1,\tau_2,\tau_3}(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_1,\gamma_1;s^{\tau_1},t^{\tau_2},u^{\tau_3})$$

$$=\int_0^1\int_0^1\int_0^1 x^{\beta_1-1}(1-x)^{\lambda_1-\beta_1-1}y^{\beta_2-1}(1-y)^{\lambda_2-\beta_2-1}z^{\beta_3-1}(1-z)^{\lambda_3-\beta_3-1}$$

$$\cdot R_S^{\tau_1,\tau_2,\tau_3}[\alpha_1,\alpha_2,\alpha_3,\lambda_1,\lambda_2,\lambda_3;\gamma_1,\gamma_1,\gamma_1;(sx)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_3}]\,dx\,dy\,dz\,, \quad \ldots (2.25)$$

where $0 < R(\beta_1) < R(\lambda_1)\,; 0 < R(\beta_2) < R(\lambda_2)\,; 0 < R(\beta_3) < R(\lambda_3)$

Let us now consider

$$\frac{\partial^{\nu_1-\gamma_1+\beta_2-\gamma_2+\nu_2-\gamma_2}}{\partial s^{\nu_1-\gamma_1}\,\partial t^{\beta_2-\gamma_2}\,\partial u^{\nu_2-\gamma_2}}\left[s^{\nu_1-1}t^{\beta_2-1}u^{\nu_2-1}R_F^{\tau_1,\tau_3,\tau_3}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\gamma_2,\beta_1;\nu_1,\nu_2,\nu_3;s^{\tau_1},t^{\tau_3},u^{\tau_3})\right]$$

$$= \quad s^{\gamma_1-1}t^{\gamma_2-1}u^{\gamma_2-1}\frac{\Gamma(\nu_1)\,\Gamma(\nu_2)\,\Gamma(\beta_2)}{\Gamma(\gamma_2)\,\Gamma(\gamma_1)\,\Gamma(\gamma_2)}R_F^{\tau_1,\tau_3,\tau_3}[\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;s^{\tau_1},(tu)^{\tau_3},u^{\tau_3}]$$

Hence

$$s^{\gamma_1-1}t^{\gamma_2-1}u^{\gamma_2-1}R_F^{\tau_1,\tau_2,\tau_3}[\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;s^{\tau_1},(tu)^{\tau_3},u^{\tau_3}]$$

$$= \frac{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_2)}{\Gamma(\nu_1)\,\Gamma(\gamma_1-\nu_1)\,\Gamma(\nu_2)\,\Gamma(\gamma_2-\nu_2)\,\Gamma(\beta_2)\,\Gamma(\gamma_2-\beta_2)}\int_0^s\int_0^t\int_0^u p^{\nu_1-1}(s-p)^{\gamma_1-\nu_1-1}\,.$$

$$q^{\beta_2-1}(t-q)^{\gamma_2-\beta_2-1}r^{\nu_2-1}(u-r)^{\gamma_2-\nu_2-1}$$

$$.R_F^{\tau_1,\tau_3,\tau_3}[\alpha_1,\alpha_1,\alpha_1,\beta_1,\gamma_2,\beta_1;\nu_1,\nu_2,\nu_2;p^{\tau_1},(qr)^{\tau_3},r^{\tau_3}]\,dp\,dq\,dr \qquad \ldots (2.26)$$

Putting $p = sx$, $q = ty$, $r = uz$, we get

$$\frac{\Gamma(\nu_1)\,\Gamma(\gamma_1-\nu_1)\,\Gamma(\nu_2)\,\Gamma(\gamma_2-\nu_2)\,\Gamma(\beta_2)\,\Gamma(\gamma_2-\beta_2)}{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_2)}$$

$$R_F^{\tau_1,\tau_3,\tau_3}[\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;s^{\tau_1},(tu)^{\tau_3},u^{\tau_3}]$$

$$= \int_0^1\int_0^1\int_0^1 x^{\nu_1-1}(1-x)^{\gamma_1-\nu_1-1}y^{\beta_2-1}(1-y)^{\gamma_2-\beta_2-1}z^{\nu_2-1}(1-z)^{\gamma_2-\nu_2-1}$$

$$\cdot R_F^{\tau_1,\tau_2,\tau_3}[\alpha_1,\alpha_1,\alpha_1,\beta_1,\gamma_2,\beta_1;\nu_1,\nu_2,\nu_3;(sx)^{\tau_1},(tuyz)^{\tau_2},(uz)^{\tau_3}]\,dx\,dy\,dz, \quad \ldots (2.27)$$

where $0 < R(\nu_1) < R(\gamma_1); 0 < R(\nu_2) < R(\gamma_2); 0 < R(\beta_2) < R(\gamma_2)$

Similar representations hold for R_M, R_P, R_R and R_T

$$R_M^{\tau_1,\tau_2,\tau_2}[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;s^{\tau_1},(tu)^{\tau_2},u^{\tau_2}]$$

$$=\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_2)}{\Gamma(\nu_1)\Gamma(\gamma_1-\nu_1)\Gamma(\nu_2)\Gamma(\gamma_2-\nu_2)\Gamma(\beta_2)\Gamma(\gamma_2-\beta_2)}\int_0^1\int_0^1\int_0^1 x^{\nu_1-1}(1-x)^{\gamma_1-\nu_1-1}y^{\beta_2-1}$$

$$\cdot(1-y)^{\gamma_2-\beta_2-1}z^{\nu_2-1}(1-z)^{\gamma_2-\nu_2-1}$$

$$R_M^{\tau_1,\tau_2,\tau_2}[\alpha_1,\alpha_2,\alpha_2,\beta_1,\gamma_2,\beta_1;\nu_1,\nu_2,\nu_3;(sx)^{\tau_1},(tyuz)^{\tau_2},(uz)^{\tau_2}]\,dx\,dy\,dz, \quad \ldots (2.28)$$

where $0 < R(\nu_1) < R(\gamma_1); 0 < R(\nu_2) < R(\gamma_2); 0 < R(\beta_2) < R(\gamma_2)$

$$R_P^{\tau_1,\tau_2,\tau_1}[\alpha_1,\alpha_2,\alpha_1,\beta_1,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_1;(su)^{\tau_1},t^{\tau_2},u^{\tau_1}]$$

$$=\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_2)}{\Gamma(\alpha_2)\Gamma(\gamma_1-\alpha_2)\Gamma(\nu_1)\Gamma(\gamma_2-\nu_1)\Gamma(\nu_2)\Gamma(\gamma_2-\nu_2)}\int_0^1\int_0^1\int_0^1 x^{\alpha_2-1}(1-x)^{\gamma_1-\alpha_2-1}y^{\nu_1-1}$$

$$\cdot(1-y)^{\gamma_2-\nu_1-1}z^{\nu_2-1}(1-z)^{\gamma_2-\nu_2-1}$$

$$R_P^{\tau_1,\tau_2,\tau_1}[\alpha_1,\gamma_2,\alpha_1,\beta_1,\beta_1,\beta_2;\nu_1,\nu_2,\nu_2;(suxz)^{\tau_1},(ty)^{\tau_2},(uz)^{\tau_1}]\,dx\,dy\,dz, \quad \ldots (2.29)$$

where $0 < R(\alpha_2) < R(\gamma_1); 0 < R(\nu_1) < R(\gamma_2); 0 < R(\nu_2) < R(\gamma_2)$

$$R_R^{\tau_1,\tau_3,\tau_3}[\alpha_1,\alpha_2,\alpha_1,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;s^{\tau_1},(tu)^{\tau_3},u^{\tau_3}]$$

$$= \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_2)}{\Gamma(\nu_1)\Gamma(\gamma_1-\nu_1)\Gamma(\nu_2)\Gamma(\gamma_2-\nu_2)\Gamma(\beta_2)\Gamma(\gamma_2-\beta_2)} \int_0^1\int_0^1\int_0^1 x^{\nu_1-1}(1-x)^{\gamma_1-\nu_1-1}y^{\beta_2-1}$$

$$\cdot(1-y)^{\gamma_2-\beta_2-1}z^{\nu_2-1}(1-z)^{\gamma_2-\nu_2-1}$$

$$R_R^{\tau_1,\tau_3,\tau_3}[\alpha_1,\alpha_2,\alpha_1,\beta_1,\gamma_2,\beta_1;\nu_1,\nu_2,\nu_2;(sx)^{\tau_1},(tuyz)^{\tau_3},(uz)^{\tau_3}]\,dx\,dy\,dz, \qquad \ldots(2.30)$$

where $0 < R(\nu_1) < R(\gamma_1); 0 < R(\beta_2) < R(\gamma_2); 0 < R(\nu_2) < R(\gamma_2)$

$$R_T^{\tau_1,\tau_1,\tau_1}[\alpha_1,\alpha_2,\alpha_2,\beta_2,\beta_2,\beta_2;\gamma_1,\gamma_1,\gamma_1;(su)^{\tau_1},(tu)^{\tau_1},u^{\tau_1}]$$

$$= \frac{\Gamma(\xi)\Gamma(\eta)\Gamma(\gamma_1)}{\Gamma(\alpha_1)\Gamma(\xi-\alpha_1)\Gamma(\beta_2)\Gamma(\eta-\beta_2)\Gamma(\nu_1)\Gamma(\gamma_1-\nu_1)} \int_0^1\int_0^1\int_0^1 x^{\alpha_1-1}(1-x)^{\xi-\alpha_1-1}y^{\beta_2-1}$$

$$\cdot(1-y)^{\eta-\beta_2-1}z^{\nu_1-1}(1-z)^{\gamma_1-\nu_1-1}$$

$$R_T^{\tau_1,\tau_1,\tau_1}[\xi,\alpha_2,\alpha_2,\beta_1,\eta,\beta_1;\nu_1,\nu_1,\nu_1;(suxz)^{\tau_1},(tuyz)^{\tau_1},(uz)^{\tau_1}]\,dx\,dy\,dz, \qquad \ldots(2.31)$$

where $0 < R(\alpha_1) < R(\xi); 0 < R(\beta_2) < R(\eta); 0 < R(\nu_1) < R(\gamma_1)$

The integrals deduced in the article 2(ii) can result into many special cases for hypergeometric functions of two variables and generalized hypergeometric functions by Erdelyi [2].

3. Differentiation for the function $R_A^{\tau_1,\tau_2,\tau_3}$

The formula of differentiation for the function $R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;x,y,z)$ is :

$$\frac{\partial^3}{\partial x\partial y\partial z}\left[R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;x,y,z)\right]$$

$$= \frac{\Gamma(\alpha_1+\tau_1+\tau_2+\tau_3)\,\Gamma(\beta_1+\tau_1)\,\Gamma(\beta_2+\tau_2)\,\Gamma(\beta_3+\tau_3)\,\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}{\Gamma(\gamma_1+\tau_1)\,\Gamma(\gamma_2+\tau_2)\,\Gamma(\gamma_3+\tau_3)\,\Gamma(\alpha_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_3)}$$

$$R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1+\tau_1+\tau_2+\tau_3,\beta_1+\tau_1,\beta_2+\tau_2,\beta_3+\tau_3;\gamma_1+\tau_1,\gamma_2+\tau_2,\gamma_3+\tau_3;x,y,z) \quad \dots (3.1)$$

$$\frac{\partial^6}{\partial x^2 \partial y^2 \partial z^2}\left[R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;x,y,z)\right]$$

$$=\frac{\Gamma(\alpha_1+2\tau_1+2\tau_2+2\tau_3)\,\Gamma(\beta_1+2\tau_1)\,\Gamma(\beta_2+2\tau_2)\,\Gamma(\beta_3+2\tau_3)\,\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}{\Gamma(\gamma_1+2\tau_1)\,\Gamma(\gamma_2+2\tau_2)\,\Gamma(\gamma_3+2\tau_3)\,\Gamma(\alpha_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_3)}$$

$$R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1+2\tau_1+2\tau_2+2\tau_3,\beta_1+2\tau_1,\beta_2+2\tau_2,\beta_3+2\tau_3;\gamma_1+2\tau_1,\gamma_2+2\tau_2,\gamma_3+2\tau_3;x,y,z) \quad \dots (3.2)$$

Now we can get the generalized formula as :

$$\frac{\partial^n}{\partial x^n \partial y^n \partial z^n}\left[R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_3;x,y,z)\right]$$

$$=\frac{\Gamma(\alpha_1+n\tau_1+n\tau_2+n\tau_3)\,\Gamma(\beta_1+n\tau_1)\,\Gamma(\beta_2+n\tau_2)\,\Gamma(\beta_3+n\tau_3)\,\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_3)}{\Gamma(\gamma_1+n\tau_1)\,\Gamma(\gamma_2+n\tau_2)\,\Gamma(\gamma_3+n\tau_3)\,\Gamma(\alpha_1)\,\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\beta_3)}$$

$$R_A^{\tau_1,\tau_2,\tau_3}(\alpha_1+n\tau_1+n\tau_2+n\tau_3,\beta_1+n\tau_1,\beta_2+n\tau_2,\beta_3+n\tau_3;\gamma_1+n\tau_1,\gamma_2+n\tau_2,\gamma_3+n\tau_3;x,y,z) \quad \dots (3.3)$$

Similar differentiation formulae can be obtained for other functions also.

References

[1] Al-Shammery, A.H. and Kalla, S.L. (2000) : An extension of some hypergeometric functions of two variables, Rev. Acad. Canar. Cienc. XII (Nums 1 – 2) 189 – 196.

[2] Erdelyi, A. (1939) : Transformations of hypergeometric integrals by means of fractional integration by parts, Quart. J. Math. Oxford Ser. 10, 176 – 189.

[3] Garg, Mridula, Mishra, Rupakshi and Kalla, S.L. (2004) : Extensions of Some Horn Functions, Rev. Acad. Canar. Cienc. XVI (Nums 1 – 2), 99 – 111.

[4] Castle, J.A. (2006) : Some results on the generalized functions of Appell. The University of Zulia, Maracaibo, Venezuela, Vol. 29. No.1.

[5] Srivastava, Krishna ji (1956) : On certain hypergeometric integrals, Ganita, 13 – 28.

[6] Mathai, A.M. (1993) : Hypergeometric Functions of Several Matrix Arguments, Centre for Mathematical Sciences, Thiruvanthapuram, v.25, 21 – 26.

[7] Saran, Shanti (1954) : Hypergeometric functions of three variables, Ganita, 5(2), 77 – 91.

[8] Srivastava, H.M. and Singhal, J.P. (1964) : A note on certain Hypergeometric Functions of two and three variables, Ganita 15, 99 – 108.

Ganita Sandesh Vol. 26 No. (2) (*December* 2012) pp. 145 - 157 **ISSN : 0970-9169**

An EOQ Inventory Model with Two Level Credit-Linked Demands Under Permissible Delay in Payments for Weibull Deteriorating Items

Vipin Kumar

Department of Mathematics, BKBIET, PILANI (Raj.) INDIA

Email: vipinbkniet@yahoo.co.in

C.B.Gupta

Department of Mathematics, BITS, PILANI (Raj.) INDIA

Email: cbbits@gmail.com

S.R.Singh

Department of Mathematics, D.N.College, MEERUT (UP) INDIA

Email: shivrajpundir@yahoo.com

Abstract

The supplier offers a fixed credit period to the retailer but the retailer does not offer any credit period to its customers. In real practice, retailer may offer a credit period to its customer in order to boost his own demand. In this paper, the impact of credit period on demand is studied when units of inventory deterioration follows the Weibull distribution. A deterministic inventory model is developed to determine the optimal credit-period and replenishment policy for the retailer.

KEY WORDS: Inventory, deteriorating items, Weibull distribution, credit-linked, demand.

1. Introduction

The concept of trade-credit was first introduced by Haley and Higgins (1973).They developed inventory model to determine Economic Order Quantity with known constant deterministic demand under condition of permissible delay in payments, no shortage and zero lead time. Goyal (1985) excluded penalty cost due to late payment in Heley and Higgins model. Chung (1989) developed inventory model under permissible delay in payments using the discounted cash flows approach. Shah and Aggarwal and Jaggi extended the Gopal's model to incorporate deterioration of units in the inventory. Jamal et.al.(2000) generalized the model to allow shortages. Dye developed inventory model for stock-dependent demand for deteriorating items when partial back-logging is allowed and trade-credit is offered. Dye and Chang (2003) formed a replenish-ment policy with liner demand , deterioration ,shortage and permissible delay in payment. Chang et.al. developed the optimal inventory model for deteriorating items with instantaneous stock dependent demand and time value of money when permissible delay in payments is offered. Teng et al. (2005) developed the optimal pricing and lot sizing under permissible delay in payments by considering the difference between selling price and purchase quantity and demand to be price sensitive. Goyal et. al.(2007) formulated an EOQ model for a retailer when supplier offers a progressive interest scheme to make the decision. Shah and Soni(2008) computed optimal ordering policy for stock dependent demand under scenario of progressive payments.

In most of the business, the supplier offers a credit period to the retailer and retailer; in turn posses on some credit period to customers. Huang introduced an inventory model when retailer offers a credit period to its customer which is smaller than the credit period offered by the supplier, in order to boost the demand. The models discussed above consider the effect of credit period on the objective function.

The impact of credit period on demand is ignored. In practice, it is observed that demand of an item does depend upon the length of the credit period offered by the supplier to the retailer or retailer to the customer. Jaggi et al. gave idea of credit-linked demand function to determine the retailer's optimal credit and replenishment policy when both the suppliers as well as the retailer offers the credit period to stimulate the user demand. In this paper, we also consider effect of deterioration which follows Weibull distribution on optimal policy when demand is dependent upon the allowable credit.

2. Assumptions

The mathematical model is developed under the following assumptions :

1. The inventory system under consideration deals with single item.

2. The supplier offers a credit period T_1 to settle the accounts to the retailer and the retailer, in turn, offers a credit period T_2 to his customers to settle the accounts.

3. The demand rate is a function of the customer's credit period T_2 offered by the retailer.

   The demand function for $D(T_2)$ any T_2 can be represented as a difference equation.

$$D(T_2+1)-D(T_2) = r\left[R_m - D(T_2)\right];$$

   where $D(T_2)$:demand for any T_2 per unit time

   R_m : Maximum demand

   r : Rate of saturation of demand with the initial condition $D(0)=R_0$ (initial demand)

   The solution of the above difference equation is

$$D(T_2) = R_0(1-r)^{T_2} + R_m\left[1-(1-r)^{T_2}\right]$$

4. Replenishment rate is instantaneous.
5. Shortages are not allowed.
6. Lead – time is zero.
7. The deterioration of units in inventory follows the two parameters Weibull distribution

$$\theta(t) = \alpha\beta t^{\beta-1}$$

   where α = scale parameter $(0 < \alpha << 1)$, β = shape parameter $(\beta > 0)$

8. Deteriorated units can neither be repaired nor replaced during a cycle time.

3. Notations

The following notations are used in the model :

Q : Order quantity.

T : Cycle time.

$I(t)$: The inventory level at any instant of time t, $0 \le t \le T$

A : Ordering cost per order.

C_p : Unit purchase cost of an item.

S_p : Unit selling price of an item.

h : Inventory carrying cost.

I_e : The interest earned per Rs per unit time.

I_c : The interest charged per Rs per unit time.

T_1 : Retailer's credit period offered by the supplier for settling the account.

T_2 : Customer's credit period offered by the retailer for settling the account.

$P(T,T_2)$: Retailer's profit per unit time.

4. Mathematical Model

The inventory level gradually falls due to demand and deterioration. The inventory level at any time t, is governed by the differential equation

$$\frac{dI(t)}{dt} + \theta I(t) = -D(T_2) \qquad 0 \le t \le T$$

Since $\theta(t) = \alpha\beta t^{\beta-1}$

then
$$\frac{dI(t)}{dt} + \alpha\beta t^{\beta-1} I(t) = -D(T_2) \qquad 0 \le t \le T \qquad \ldots(1)$$

With the initial condition $I(0) = Q$ and boundary condition $I(T) = 0$

$$I(t) = D(T_2)\left[(T-t) + \frac{\alpha}{\beta+1}\left(T^{\beta+1} - t^{\beta+1}\right) - \alpha t^{\beta}(T-t)\right]; \quad 0 \le t \le T \qquad \ldots(2)$$

(neglecting the higher power of α)

and the order quantity is

$$Q = I(0) = D(T_2)\left[T + \frac{\alpha}{\beta+1}T^{(\beta+1)}\right] \quad \ldots(3)$$

The retailer's profit per unit time compromises of the following components

1: Sales revenue $\quad$ $$S.R. = \frac{S_p Q}{T} \quad \ldots(4)$$

2: Cost of purchasing $\quad$ $$P.C = \frac{C_p Q}{T} \quad \ldots(5)$$

3 : Cost of placing orders $\quad$ $$O.C. = \frac{A}{T} \quad \ldots(6)$$

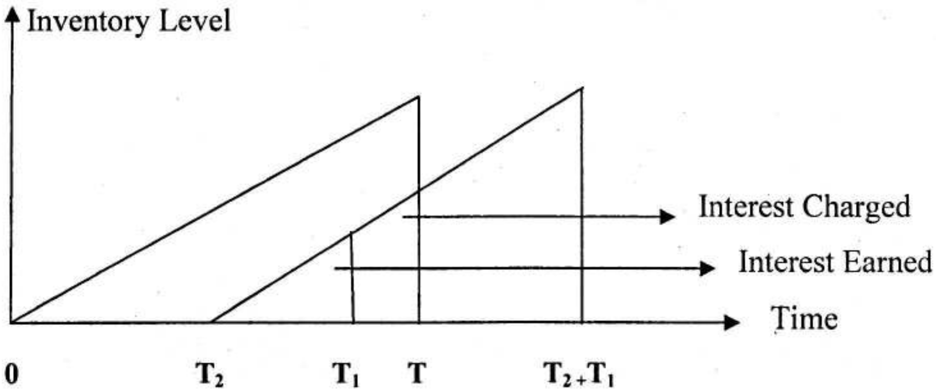
4: Inventory holding cost $\quad$ $$H.C = \frac{C_p h}{T}\int_0^T I(t).dt$$

$$= C_p h D(T_2)\left(\frac{T}{2} + \frac{\alpha\beta}{(\beta+1).(\beta+2)}T^{(\beta+1)}\right) \quad \ldots(7)$$

The calculation of interest earned and charged will depend upon T , T_1 and T_2 . The following cases arise :

Case I: When $T_2 \le T_1 \le T + T_2$.

Therefore the retailer generates revenue in time interval $[T_2, T_1]$ and earns interest on sales revenue for the time period $[T_1 - T_2]$.



Interest earned per unit time $\quad$ $$I.E_1 = \frac{C_p I_e}{T}\int_0^{T_1-T_2} D(T_2)\, t\, dt$$

$$= \frac{C_p I_e D(T_2)(T_1-T_2)^2}{2T} \quad \ldots(8)$$

Interest charges are payable to the supplier by the retailer per cycle during $[T_1, T+T_2]$ on the unsold inventory after the due time T_1 .

Interest charged per unit time is given by

$$I.C_1 = \frac{C_p I_e}{T}\int_0^{T+T_2-T_1} I(t)\,dt$$

$$= \frac{C_p I_e D(T_2)}{T}\left[\begin{array}{l}\left(\dfrac{T^2-(T_2-T_1)^2}{2}\right)+\dfrac{\alpha\beta}{(\beta+1)(\beta+2)}(T+T_2-T_1)^{\beta+2} \\ \dfrac{\alpha T(T+T_2-T_1)}{\beta+1}\left\{T^{\beta}-(T+T_2-T_1)^{\beta}\right\}\end{array}\right] \quad \ldots(9)$$

Thus the retailer 's profit per unit time is given by

$$P_1(T,T_2) = \text{S.R} - \text{P.C} - \text{O.C} - \text{H.C} - \text{I.C}_1 + \text{I.E}_1$$

$$= \frac{S_p Q}{T} - \frac{C_p Q}{T} - \frac{A}{T}$$

$$- C_p hD(T_2)\left(\frac{T}{2}+\frac{\alpha\beta}{(\beta+1)(\beta+2)}T^{(\beta+1)}\right)$$

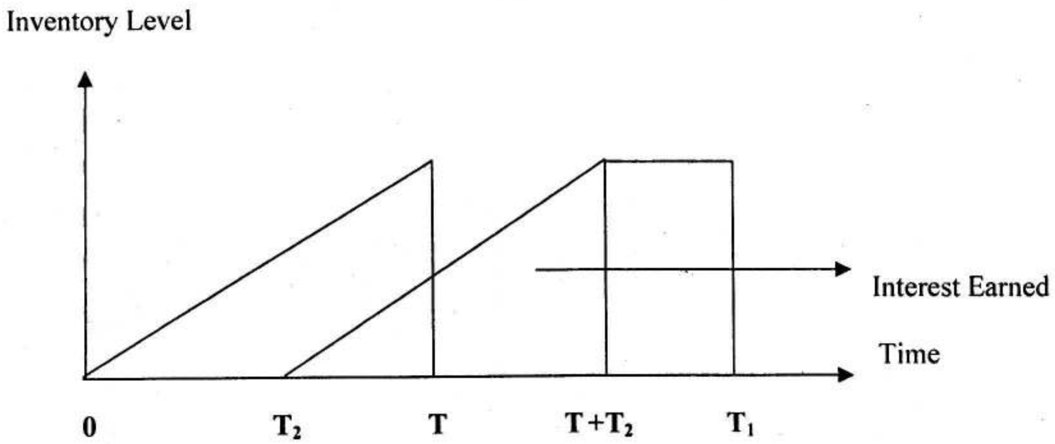
$$-\frac{C_p I_e D(T_2)}{T}\left[\begin{array}{l}\left(\dfrac{T^2-(T_2-T_1)^2}{2}\right)+\dfrac{\alpha\beta}{(\beta+1)(\beta+2)}(T+T_2-T_1)^{\beta+2} \\ \dfrac{\alpha T(T+T_2-T_1)}{\beta+1}\left\{T^{\beta}-(T+T_2-T_1)^{\beta}\right\}\end{array}\right]+\frac{C_p I_e D(T_2)(T_1-T_2)^2}{2T}$$

$$=(S_p-C_p)D(T_2)\left[1+\frac{\alpha}{\beta+1}T^{\beta}\right]-\frac{A}{T}-C_p hD(T_2)\left(\frac{T}{2}+\frac{\alpha\beta}{(\beta+1).(\beta+2)}T^{(\beta+1)}\right)$$

$$-\frac{C_p I_e D(T_2)}{T}\left[\begin{array}{l}\left(\dfrac{T^2-(T_2-T_1)^2}{2}\right)+\dfrac{\alpha\beta}{(\beta+1)(\beta+2)}(T+T_2-T_1)^{\beta+2} \\ \dfrac{\alpha T(T+T_2-T_1)}{\beta+1}\left\{T^{\beta}-(T+T_2-T_1)^{\beta}\right\}\end{array}\right]+\frac{C_p I_e D(T_2)(T_1-T_2)^2}{2T} \quad \ldots(10)$$

Case II: When $T_2 \leq T+T_2 \leq T_1$

In this case, the retailer earns interest on the revenue received during the period $[T_2, T+T_2]$ and on total sales revenue for a period of (T_1-T-T_2).



Thus **the total interest earned per unit time is given by**

$$I.E_2 = \frac{C_p I_e}{T}\int_{T_2}^{T+T_2} D(T_2).t\,dt + \frac{C_p I_e}{T} T(T_1-T-T_2)$$

$$= \frac{C_p I_e}{2} D(T_2)(2T_1-T) \qquad \ldots(11)$$

The interest charged per unit time is given by

$$I.C_2 = 0 \qquad \ldots(12)$$

Thus the retailer's profit per unit time is given by

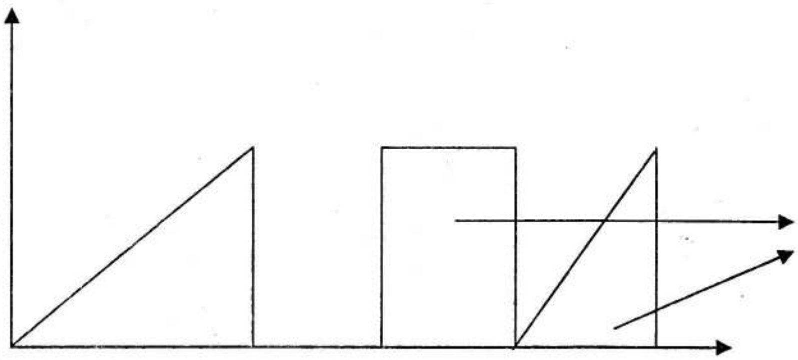
$$P_2(T,T_2) = \text{S.R} - \text{P.C} - \text{O.C} - \text{H.C} - \text{I.C}_2 + \text{I.E}_2$$

$$= \frac{S_p Q}{T} - \frac{C_p Q}{T} - \frac{A}{T} - C_p h D(T_2)\left(\frac{T}{2} + \frac{\alpha\beta}{(\beta+1).(\beta+2)}T^{(\beta+1)}\right) + \frac{C_p I_e}{2} D(T_2)(2T_1-T)$$

$$= (S_p - C_p)D(T_2)\left[1+\frac{\alpha}{\beta+1}T^{\beta}\right] - \frac{A}{T} - C_p h D(T_2)\left(\frac{T}{2} + \frac{\alpha\beta}{(\beta+1).(\beta+2)}T^{(\beta+1)}\right) + \frac{C_p I_e}{2} D(T_2)(2T_1-T) \qquad \ldots(13)$$

Case III : When $T_1 \le T_2 \le T+T_2$,

Then interest charges are payable to the supplier by the retailer per cycle during (T_1, T_2) on the inventory and during $(T_2, T+T_2)$ on the unsold inventory after the due time T_1.



Therefore **the interest charged per unit time is given by**

$$I.C_3 = \frac{C_p I_c}{T}\left[(T_2 - T_1)Q + \int_{T_2}^{T+T_2} I(T)\,dt\right]$$

$$= C_p I_c D(T_2)(T_2 - T_1)\left(1 + \frac{\alpha}{\beta+1}T^{\beta}\right)$$

$$+ C_p I_c D(T_2)\left[\begin{array}{l}\left(\frac{T}{2} - T_2\right) + \frac{\alpha}{\beta+1}\left\{T^{\beta+1} - (T+T_2)^{\beta+1} + T_2^{\beta+1}\right\} \\ \frac{\alpha\beta}{T(\beta+1)(\beta+1)}\left\{(T+T_2)^{\beta+2} - T_2^{\beta+2}\right\}\end{array}\right] \quad \ldots (14)$$

and **interest earned per unit time is given by**

$$\text{I.E}_3 = 0 \quad \ldots (15)$$

Thus **the retailer's profit per unit time is given by**

$$P_3(T,T_2) = \text{S.R} - \text{P.C} - \text{O.C} - \text{H.C} - \text{I.C}_3 + \text{I.E}_3$$

$$= \frac{S_p Q}{T} - \frac{C_p Q}{T} - \frac{A}{T} - C_p h D(T_2)\left(\frac{T}{2} + \frac{\alpha\beta}{(\beta+1).(\beta+2)} T^{(\beta+1)}\right)$$

$$- C_P I_c D(T_2)(T_2 - T_1)\left(1 + \frac{\alpha}{\beta+1} T^{\beta}\right)$$

$$- C_p I_c D(T_2)\begin{bmatrix}\left(\frac{T}{2} - T_2\right) + \frac{\alpha}{\beta+1}\left\{T^{\beta+1} - (T+T_2)^{\beta+1} + T_2^{\beta+1}\right\} \\ \frac{\alpha\beta}{T(\beta+1)(\beta+1)}\left\{(T+T_2)^{\beta+2} - T_2^{\beta+2}\right\}\end{bmatrix}$$

$$= (S_p - C_P)D(T_2)\left[1 + \frac{\alpha}{\beta+1}T^{\beta}\right] - \frac{A}{T} - C_p hD(T_2)\left(\frac{T}{2} + \frac{\alpha\beta}{(\beta+1)(\beta+2)} T^{(\beta+1)}\right)$$

$$- C_P I_c D(T_2)(T_2 - T_1)\left(1 + \frac{\alpha}{\beta+1} T^{\beta}\right)$$

$$- C_p I_c D(T_2)\begin{bmatrix}\left(\frac{T}{2} - T_2\right) + \frac{\alpha}{\beta+1}\left\{T^{\beta+1} - (T+T_2)^{\beta+1} + T_2^{\beta+1}\right\} \\ + \frac{\alpha\beta}{T(\beta+1)(\beta+2)}\left\{(T+T_2)^{\beta+2} - T_2^{\beta+2}\right\}\end{bmatrix} \quad \ldots(16)$$

Hence the retailer's profit per unit time is given by

$$P(T,T_2) = \begin{cases} P_1(T,T_2), \; T_2 \le T_1 \le T + T_2 \\ P_2(T,T_2), T_2 \le T + T_2 \le T_1 \\ P_3(T,T_2), \; T_1 \le T_2 \le T + T_2 \end{cases} \quad \ldots(17)$$

which is a function of two variables T and T_2 where T is continuous and T_2 is discrete.

Now, to determine the optimal values of T and T_2 which maximize $P(T, T_2)$.

For fixed T_2, and maximum value of $P(T, T_2)$

we have

$$\frac{\partial P_1(T,T_2)}{\partial T} = (S_p - C_p) D(T_2) \frac{\alpha\beta}{(\beta+1)} T^{\beta-1} + \frac{A}{T^2} - C_p h D(T_2)\left\{\frac{1}{2} + \frac{\alpha\beta.T^{\beta}}{\beta+2}\right\}$$

$$- C_p I_c D(T_2)\left\{\frac{1}{2} + \frac{(T_2 - T_1)^2}{2T^2}\right\} + \frac{\alpha\beta}{(\beta+1)}(T + T_2 - T_1)^{\beta+1} + \frac{\alpha(2T + T_2 - T_1)}{(\beta+1)}$$

$$\left\{T^{\beta} - (T + T_2 - T_1)^{\beta}\right\} + \frac{\alpha\beta.T(T + T_2 - T_1)}{(\beta+1)}\left\{T^{\beta-1} - (T + T_2 - T_1)^{\beta-1}\right\} - \frac{C_p I_e D(T_2)}{2T^2}$$

$$(T_1 - T_2)^2 = 0 \qquad \ldots (18)$$

provided that,

$$\frac{\partial^2 P_1(T, T_2)}{\partial T^2} > 0$$

and

$$\frac{\partial P_2(T,T_2)}{\partial T} = (S_p - C_p) D(T_2) \frac{\alpha\beta}{(\beta+1)} T^{\beta-1} + \frac{A}{T^2}$$

$$- C_p h D(T_2)\left\{\frac{1}{2} + \frac{\alpha\beta.T^{\beta}}{\beta+2}\right\} - \frac{C_p I_c D(T_2)}{2} = 0 \qquad \ldots (19)$$

provided that,

$$\frac{\partial^2 P_2(T, T_2)}{\partial T^2} > 0$$

and
$$\frac{\partial P_3(T,T_2)}{\partial T} = \left(S_p - C_p\right)D(T_2)\frac{\alpha\beta}{(\beta+1)}T^{\beta-1} + \frac{A}{T^2} - C_p\, h\, D(T_2)\left\{\frac{1}{2} + \frac{\alpha\,\beta.T^{\beta}}{\beta+2}\right\}$$

$$- C_P I_c D(T_2)(T_2 - T_1)\frac{\alpha\beta}{(\beta+1)}T^{\beta-1}$$

$$- C_p\, I_c\, D(T_2)\begin{bmatrix} \frac{1}{2} + \alpha\left\{T^{\beta} - (T+T_2)^{\beta}\right\} - \frac{\alpha\,\beta}{T^2(\beta+1)(\beta+2)} \\ \left\{(T+T_2)^{\beta+2} - T_2^{\beta+2}\right\} + \frac{\alpha\,\beta}{T(\beta+1)}(T+T_2)^{\beta+1} \end{bmatrix} \quad \ldots (20)$$

provided that,

$$\frac{\partial^2 P_3(T, T_2)}{\partial T^2} > 0$$

5. Computational Algorithm

In order to optimize T and T_2 simultaneously , we have the following steps.

Step 1: We start with $T_2 = 1$

Step 2: Find the optimal value of T using equations (18), (19), (20)

Step 3: If $0 \le T_1 - T_2 \le T$ then calculate $P_1(T,T_2)$ otherwise go to step5

Step 4: If $P_1(T,T_2) > P_1(T,T_2 - 1)$,increment T_2 by $T_2 + 1$ and go to step2 otherwise current value of T_2 is optimal. Determine Q and $P(T, T_2)$.

Step 5: If $0 \le T \le T_1 - T_2$ then calculate $P_2(T, T_2)$ otherwise go to step7

Step 6: If $P_2(T,T_2) > P_2(T,T_2 - 1)$, increment T_2 by $T_2 + 1$ and go to step2 otherwise current value of T_2 is optimal. Determine Q and $P(T, T_2)$.

Step 7: If $T_1 - T_2 \le 0 \le T$ then calculate $P_3(T, T_2)$

Step 8: If $P_3(T,T_2) > P_3(T,T_2 - 1)$, increment T_2 by $T_2 + 1$ and go to step2 otherwise current value of T_2 is optimal. Determine Q and $P(T, T_2)$.

6. Conclusion

In this paper, the effect of credit linked demand on the retailer's optimal profit is studied and it is observed that the credit period offered to its customer has positive impact on demand but deterioration negative impact on retailer's profit.

References

[1] Aggarwal, S.P. and Jaggi, C.K., Ordering policies of deteriorating items under permissible delay in payments. Journal of the Operational Research Society, 46, **(1995)** 658– 662.

[2] Chang, H. J., Huang, C.H. and Dye, C.Y. An inventory model with stock –dependent demand and time value of money when credit period is provided.
Journal of Information and Optimization Science, 25(2), **(2004)** 237 – 254.

[3] Chung, K.J. Inventory Control and Trade Credit revisited.
Journal of the Operational Research Society, 40, **(1989)** 495 – 498.

[4] Dye, C.Y., A deteriorating inventory model with stock – dependent demand rate and partial backlogging under the conditions of permissible delay in payments.Operations Research, 39, **(2002)** 189 – 198.

[5] Goyal, S.K., Economic Order Quantity under conditions of permissible delay in payments. Journal of Operational Research Society, 36, **(1985)** 335 – 338.

[6] Goyal, S.K., Teng, J.T., and Chang, C.T., Optimal ordering policies when the supplier provides a progressive interest scheme. European Journal of Operational Research, 179(2), **(2007)** 404 – 413.

[7] Haley, C.W., and Higgins, R.C., Inventory policy and trade credit financing, management science, 20, **(1973)** 464 – 471

[8] Hwang, H. and Shinn, S.W., Retailer's pricing and lot – sizing policy for exponentially deteriorating products under the condition of permissible delay in payments. Computers and Operations Research, 24, **(1997)** 539-547.

[9] Jaggi, C.K., and Aggarwal, S.P., Credit financing in economic ordering policies of deteriorating items. International Journal of Production Economics, 34, **(1994)** 151 – 155.

[10] Jaggi, C.K., Goyal, S.K. and Goel, S.K., Retailer's optimal replenishment decisions with credit – linked demand under permissible delay in payments. European Journal of Operational Research, 190, **(2008)** 130 – 135.

[11] Jamal, A.M.M., Sarkar, B.R. and Wang, S. Optimal payment time for a retailer under permitted delay of payment by the wholesaler. International Journal of Production Economics, 66, **(2000)** 59 – 66.

[12] Shah, Nita H. and Soni , H., Optimal ordering policy for stock–dependent demand under progressive payment scheme. European Journal of Operational Research, 184(1), **(2008)** 91 –100.

[13] Teng, J.T., On economic order quantity under conditions of permissible delay in payments. Journal of the Operational Research Society, 53, **(2002)** 915 – 918.

[14] Teng, J.T., Chang, C.T. and Goyal, S.K., Optimal pricing and Ordering policy under permissible delay in payments. International Journal of Production Economics, 97, **(2005)** 121 – 129.

[15] N. H. Shah, K. T. Shukla and B. J. Shah, Deteriorating Inventory Model for Two – Level Credit – Linked Demand under Permissible Delay in Payments, ASOR Bulletin, (2008) 27 -35.

Ganita Sandesh Vol. 26 No. (2) (*December* 2012) pp. 159-164 ISSN : 0970-9169

More Distribution Properties of Gauss Hypergeometric Function of Matrix Argument

Yogesh Sharma

Department of Mathematics, Jodhpur National University, JODHPUR (INDIA)

E-mail : dryogesh121@rediffmail.com

Abstract

In this paper we establish distribution properties involving gauss hypergeometric function and also find, r^{th} moment about origin ,mean ,variance, Mellin transform. Further we find their mean time to failure, and its reliability. The argument and parameter are restricted to take only those value for which the density Function are non negative and have meaning.

All the matrices considered are real positive, definite and symmetric matrices of order $p \times p$.

2010 Mathematics Subject Classification: 44A10, 44A35, 15XX, 26A33

Key words: Laplace Transform, Integral equation, Matrix argument,

1. Introduction

Laplace and Inverse laplace transforms

Laplace and inverse Laplace transforms of matrices variable are respectively given by equations

$$L_T[f(Z)] = \int_{T>0} e^{tr(-TZ)} f(T)dT = \Phi(Z) \qquad \dots(1.1)$$

And

$$\frac{2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}} \int_{\mathrm{Re}(Z)>0} e^{tr(TZ)}\, \Phi(Z)dZ = \begin{cases} f(T), & T>0 \\ 0, & elsewhere \end{cases}, \qquad \dots(1.2)$$

where $\Phi(Z)$ is complex analytic function and integral is taken over $Z = X + i\,Y$ with fixed $X > X_0$ and over the space S_p^*, S_p^* is the corresponding space of Z.

For the conditions and details the readers may see Mathai [2],[3], Mathai and Saxena [1], Seemon Thomas, Alex Thannippara and A.M. Mathai [4],

where $\int\limits_{O<U<X} = \int\limits_{O}^{X}$ means that the integral is over $U > O, X > O, X - U > O$ and O denotes the null matrix of order $p \times p$.

Sharma [5],[6],[7] obtained the Inversion formula for fractional integral operator of matrix variable involving Jacobi polynomials, and generalized Stieltjes transform of Matrix argument ,Integral Transform of Matrix Variable involving Gauss hypergeometric Function.

2. Main Result

In this paper we define function

$$f(X) = |X|^{\alpha - \frac{(p+1)}{2}}$$

$$\frac{\Gamma_P(-\gamma)\Gamma_P\left(\nu+\frac{(p+1)}{2}-\alpha\right)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}\right)\Gamma_P\left(\nu+\frac{p+1}{2}\right){}_2F_1\left(-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};-X\right)}{\Gamma_P(\alpha)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha\right)\Gamma_P(-\gamma-\alpha)} \quad \ldots (2.1)$$

serve probability density function.

provided $\int\limits_{X>O} f(X) = I$

3. Computation of mean time to failure

The distribution $F(T)$ is given as

$$F(T) = P(0 < X \leq T) = \int_0^T f(X)dX,$$

where $f(T)$ is called probability density function or it is known as failure density function.

$F(T) =$

$$\int_0^T \frac{\Gamma_P(-\gamma)\Gamma_P\left(\nu+\frac{(p+1)}{2}-\alpha\right)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}\right)\Gamma_P\left(\nu+\frac{p+1}{2}\right){}_2F_1\left(-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};-X\right)}{\Gamma_P(\alpha)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha\right)\Gamma_P(-\gamma-\alpha)}$$

$$|X|^{\alpha - \frac{(p+1)}{2}} dX \quad \ldots (2.2)$$

Let $X = V\ T$ then $dX = |T|^{\frac{p+1}{2}} . dV$ for fixed $T, |U| = |T|\,|V|$ and respective regions $O < U < T,\ O < V < I$ and using Mathai and Saxena [1,pg114,5.3.9], we get

$$f(T) = \Gamma_p(-\gamma)|T|^{\alpha}\Gamma_p\left(\nu+\frac{(p+1)}{2}-\alpha\right)\Gamma_p\left(\nu+\delta+\gamma+\frac{p+1}{2}\right)$$

$$\frac{\Gamma_p\left(\nu+\frac{p+1}{2}\right)\Gamma_p\left(\frac{p+1}{2}\right){}_3F_2\left(\alpha,-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};\alpha+\frac{p+1}{2}-X\right)|X|^{\alpha-\frac{(p+1)}{2}}dX}{\Gamma_p\left(\alpha+\frac{p+1}{2}\right)\Gamma_p\left(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha\right)\Gamma_p(-\gamma-\alpha)}, \quad \ldots(2.3)$$

where $\Gamma_p\left(\nu+\frac{(p+1)}{2}-\alpha\right), \Gamma_p\left(\nu+\delta+\gamma+\frac{p+1}{2}\right) > \frac{p+1}{2}$

4. Reliability

The reliability of a component is the probability that it does not fail till t and it is defined as

$$R(T) = P(X > T) = I - P(0 < X \leq T) = I - F(T)$$

$$= I-\Gamma_p(-\gamma)|T|^{\alpha}\Gamma_p\left(\nu+\frac{(p+1)}{2}-\alpha\right)\Gamma_p\left(\nu+\delta+\gamma+\frac{p+1}{2}\right)$$

$$\frac{\Gamma_p\left(\nu+\frac{p+1}{2})\Gamma_p(\frac{p+1}{2}\right){}_3F_2\left(\alpha,-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};\alpha+\frac{p+1}{2}-X\right)|X|^{\alpha-\frac{(p+1)}{2}}dX}{\Gamma_p\left(\alpha+\frac{p+1}{2}\right)\Gamma_p\left(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha\right)\Gamma_p(-\gamma-\alpha)} \quad \ldots(2.4)$$

5. Failure rate functions (Hazard rate function)

The failure rate function of the component denoted $h(T)$ is defined as

$$h(T) = \frac{f(T)}{I-F(T)} = \frac{f(T)}{R(T)}$$

6. The moment of the distribution

The r^{th} moment about origin of the matrix variable X with p.d.f (2.1) is given by

$$E[|X|^r] = \int_{X>0} |X|^r f(X)dX \quad \ldots(2.5)$$

$$E[|X|^r] = f(X) = \Gamma_P(-\gamma)\Gamma_P(\alpha+r)\Gamma_P\left(\nu+\frac{(p+1)}{2}-\alpha\right)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}\right)$$

$$\frac{\Gamma_P\left(\nu+\frac{p+1}{2}\right){}_3F_1\left(\alpha+r-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};-X\right)}{\Gamma_P(\alpha)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha\right)\Gamma_P(-\gamma-\alpha)}, \quad \ldots(2.6)$$

where $\mathrm{Re}(\alpha) > \frac{p+1}{2}$, $\mathrm{Re}\left(\gamma+\nu+\delta+\frac{(p+1)}{2}\right) > \frac{p-1}{2}$, $\mathrm{Re}\,B > 0$

=0 , elsewhere

7. Mean and Variance of the distribution

$$E[|X|^1] = \int_{X>0} |X|^1 f(X)dX$$

$$E[|X|^1] = \frac{\Gamma_P(-\gamma)\Gamma_P(\alpha+1)\Gamma_P\left(\nu+\frac{(p+1)}{2}-\alpha\right)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}\right)\Gamma_P\left(\nu+\frac{p+1}{2}\right){}_3F_1\left(\alpha+1-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};-X\right)}{\Gamma_P(\alpha)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha\right)\Gamma_P(-\gamma-\alpha)} \quad \ldots (2.7)$$

And

$$E[|X|^2] = \frac{\Gamma_P(-\gamma)\Gamma_P(\alpha+2)\Gamma_P\left(\nu+\frac{(p+1)}{2}-\alpha\right)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}\right)\Gamma_P\left(\nu+\frac{p+1}{2}\right){}_3F_1\left(\alpha+2-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};-X\right)}{\Gamma_P(\alpha)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha\right)\Gamma_P(-\gamma-\alpha)} \quad \ldots (2.8)$$

By definition, variance $= E\left(|X^2|\right) - [E(|X|)]^2$, we get

$$\text{Variance} = \frac{\Gamma_P(-\gamma)\Gamma_P(\alpha+2)\Gamma_P\left(\nu+\frac{(p+1)}{2}-\alpha\right)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}\right)\Gamma_P\left(\nu+\frac{p+1}{2}\right){}_3F_1\left(\alpha+2-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};-X\right)}{\Gamma_P(\alpha)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha\right)\Gamma_P(-\gamma-\alpha)} -$$

$$\Gamma_P(-\gamma)\Gamma_P(\alpha+1)\Gamma_P\left(\nu+\frac{(p+1)}{2}-\alpha\right)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}\right)$$

$$\left[\frac{\Gamma_P\left(\nu+\frac{p+1}{2}\right){}_3F_1\left(\alpha+1-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};-X\right)}{\Gamma_P(\alpha)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha\right)\Gamma_P(-\gamma-\alpha)}\right]^2 \quad \ldots (2.9)$$

The various parameter are given by same restriction given by (2.1).

8. Mellian transform of the distribution

By definition of Mellin transform

$$M[f(X)] = \int_{X>0} |X|^{\delta-\frac{(p+1)}{2}} f(X)dX$$

$$\Gamma_P(-\gamma)\Gamma_P\left(\alpha+\delta-\frac{p+1}{2}\right)\Gamma_P\left(\nu+\frac{(p+1)}{2}-\alpha\right)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}\right)$$

$$= \frac{\Gamma_P\left(\nu+\frac{p+1}{2}\right){}_3F_1\left(\alpha+\delta-\frac{p+1}{2}-\gamma,\nu+\delta+\gamma+\frac{p+1}{2};\nu+\frac{p+1}{2};-X\right)}{\Gamma_P(\alpha)\Gamma_P\left(\nu+\delta+\gamma+\frac{p+1}{2}-\alpha\right)\Gamma_P(-\gamma-\alpha)} \quad \ldots (2.10)$$

Particular Case : If we take matrix of order 1×1 i.e if we take $p = 1$, we get

$$f(x) = |x|^{\alpha-1} \frac{\Gamma_P(-\gamma)\Gamma_P(\nu+1-\alpha)\Gamma_P(\nu+\delta+\gamma+1)\Gamma_P(\nu+1)_2F_1(-\gamma,\nu+\delta+\gamma+1;\nu+1;-x)}{\Gamma_P(\alpha))\Gamma_P(\nu+\delta+\gamma+1-\alpha)\Gamma_P(-\gamma-\alpha)} \quad \ldots (2.11)$$

References

[1] Mathai, A.M. and Saxena, R.K: The H –Function with Applications in Statistics and Other Disciplines, John –Wiley & sons, New York **(1978)**.

[2] Mathai, A.M: Jacobians of Matrix Transformations and Function of Matrix Argument; World Scientific Publishing Co. Pvt Ltd; Singapore **(1997)**.

[3] Mathai, A.M. : Fractional integrals in the matrix-variate cases and connection to Statistical distributions. *Integral Transforms and Special Functions*, 20(12), **(2009)** 871-882.

[4] Seemon Thomas, Alex Thannippara and A.M. Mathai : On a matrix-variate generalized type -1 Dirichlet Model, Journal of Probability and Statistical Science, 6(2), **(2008)** 187-200.

[5] Sharma Yogesh: Generalized Stieltjes transform of Matrix variable, Indian Academy of Mathematics, vol 27(2)**(2005)**289-293.

[6] Sharma Yogesh : Integral Transform of Matrix Variable involving Gauss hypergeometric Function Vol. 53 No.3 July **(2010)**, 47-51 Vijnana Parishad Anusandhan Patrika, Allahabad.

[7] Sharma Yogesh: Inversion Formula for Fractional Integral Operator of Matrix Argument involving Jacobi polynomials. Indian Academy of Mathematics vol.35 ,no1 **(2013)**.

Ganita Sandesh Vol. 26 No. (2) (*December* 2012) pp. 165-170 ISSN : 0970-9169

An Algorithm for Minimizing Quadratic Programming Problem Using Method of Least Squares

K. C. Sharma

Department of Mathematics, Govt. M.S.J. College BHARATPUR (RAJ.) INDIA

Email: sharma_kc08@yahoo.co.in

Jitendra Singh

Department of Mathematics, Govt. College BARAN (RAJ.) INDIA

Email: jsambesh@gmail.com

Abstract

Quadratic programming is a technique, familiar in Operations Research, for optimizing a quadratic objective function subject to linear inequality or equality constraints. A number of techniques are available to solve these problems. In this paper, a new algorithm is proposed to solve quadratic programming problem (QPP) which is based on least squares. In this algorithm, there is no need to transform QPP into an equivalent linear programming problem.

Keywords: *Convex Quadratic Programming, Convexity and Concavity, Cholesky Decomposition.*

1. Introduction

Non-linear programming arises in the mathematical modeling of several problems in real world applications. Some of the problems may be formulated as quadratic programming (QP) with a quadratic objective function and a set of linear equality or inequality constraints. The algorithms for quadratic programming have been developed in the last few decades and different methods are available for solving the convex quadratic programming problems such as extensions of the simplex method, gradient projection method, conjugate gradient method, augmented Lagrangian method, active set method, interior point method and dual method etc. Wolfe [9] has given an algorithm for this kind of problem that is based on fairly simple modification of simplex method and converges in a finite number of iterations. Terlaky [8] proposed an algorithm which does not require the enlargement of the basic table as Frank-Wolfe [3] method does. Here we proposed a new method based on the least squares for the solution of a convex quadratic programming problem. The method of least-squares for optimization of quadratic functions is widely practiced. Boot [1] discussed the least-squares method of optimization problem in several restricted cases. Let X is an n x n fixed design matrix, $x \in \Re^n$ parameter vector and consider the linear model $Y = Xx + \varepsilon$. The method for minimizing $\|Y - Xx\|^2$ subject to various linear inequality and equality constraints has been proposed by Judge and Takayama [7] which uses a simplex algorithm to obtain the optimal solution x^*. Liew [5] used the principle pivoting algorithm of Dantzig and Cottle and provided an approximate covariance matrix for x^*. During this paper, we extend this work and propose an algorithm to obtain the optimal solution for minimizing quadratic programming without transforming into equivalent linear programming problem.

2. Cholesky Decomposition

Cholesky decomposition is a fundamental tool in matrix computations. Let us consider $Q \in R^{nxn}$ be a hessian matrix of quadratic objective function of the proposed problem that can be Cholesky factorized as $Q = L^TL$ in which L is upper triangular with positive diagonal elements. According to condition of proposed problem the hessian matrix Q to be positive definite and its Cholesky decomposition exists and unique.

3. Problem Formulation

Consider the following quiadratic programming problem as:

$$\begin{aligned} &\textit{Minimize } f(x) = \tfrac{1}{2}x^T Q x + \gamma^T x + d \\ &\textit{s.t.} \qquad Ax \geq \beta \\ &\qquad\qquad x \geq 0 \end{aligned} \qquad \text{...(1)}$$

where $x, \gamma \in \Re^n$, $\beta \in \Re^m$, $A \in \Re^{mxn}$ and $Q \in \Re^{nxn}$ is positive definite matrix. The matrix Q may be Cholesky decomposed as $Q = L^TL$, where L is an upper triangular matrix with positive diagonal elements.

4. The Least Squares Problem

The linear regression model with non negative constraints has the following form

$$\begin{aligned} Y &= Xx + \varepsilon \\ \text{s. t. } & x \geq 0 \end{aligned} \qquad \text{...(2)}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^T$ is error term, $Y \in \Re^n$, $x \in \Re^n$ and $X \in \Re^{nxk}$ is design matrix. The estimation method is least squares, in which we choose the coefficients $x = (x_0, x_1, . \, x_k)^T$ which minimize the residual sum of squares. The residual sum of squares is often called the sum of squares of the errors about the regression line and denoted by SSE. This minimization procedure for estimating the parameters is called the method of least squares. Let us consider following problem:

$$\text{Minimize (SSE)} = \sum_{i=1}^{n}(\varepsilon^2) = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}(y_i - x_0 - x_1 x_i)^2 \qquad \text{...(3)}$$

$$\text{s. t.} \qquad x \geq 0$$

for k = 2, we have $\hat{y} = x_0 - x_1 x_i$, $x = (x_0, x_1)^T$ and given a set of regression data $\{(x_i, y_i) ; i=1,2,\ldots n\}$, and a fitted model, $\hat{y} = x_0 - x_1 x_i$, the i^{th} residual ε_i is given by

$$\varepsilon_i = y_i - \hat{y}_i, \quad i = 1, 2, \ldots\ldots\ldots n$$

The regression model with linear inequality constraints and non negative variables in the following form:

$$Y = Xx + \varepsilon, \text{ s. t. } Ax \geq \beta, x \geq 0 \qquad \ldots(4)$$

$\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^T$ is error term and $Y = (y_1, y_2, \ldots, y_n)$.

The ordinary least squares estimator is used. This is found by the following procedure:

$$Y = Xx + \varepsilon \Rightarrow \varepsilon = Y - Xx$$

$$\Rightarrow \varepsilon^T \varepsilon = (Y - Xx)^T (Y - Xx)$$

The vector x that yields a ($\varepsilon^T \varepsilon$) minimum is the least squares estimates of x. Now we obtain general regression programming with the help of least squares method as follows:

$$\text{Minimize } f(x) = (Y - Xx)^T (Y - Xx), \quad \text{Subject to } a^T_i x \geq \beta_i, \; x_i \geq 0, \qquad \ldots(5)$$

where $X = L$, $Y = -\frac{1}{2} (L^T)^{-1} \gamma$

5. Algorithm

The algorithm for the solution of minimizing quadratic programming problem has following steps

Step 1. *First we find out the matrices X and Y by Cholesky decomposition of positive definite matrix of given quadratic programming problem and for this construct unconstrained problem as: Minimize* $f(x) = (Y - Xx)^T (Y - Xx)$.

Step 2. *Find the critical point* x^* *of above unconstrained problem and if "the critical point is inside the feasible region of quodratic programming problem then this critical point is optimal solution and process terminates. else go to the next step.*

Step 3. *Create an index set* $P_l = \{i: a^T_i x < \beta_i\} \neq \Phi$ *for which the (single)constraint violate at the infeasible point and find out following equality constrained problem*

$$\text{Minimize } f(x) = (Y - Xx)^T (Y - Xx), \text{ s.t. } a^T_i x = \beta_i, \; i \in P_l.$$

solving this problem and denote the solution by $x^{(i)}$.

Step 4. *Check the feasibility condition by creating an index set* $R_l = \{i: a^T_i x^{(i)} \geq \beta_i, i \in P_l\}$ *and check whether* R_l *is a non empty set. If "yes", then* $x^{(i)}$, $i \in R_l$ *is the optimal solution of the QPP and stop. If "no" then again creating a set* $P_l - R_l = \{i : a^T_i x^{(i)} < \beta_i, \; i \in P_l\}$ and increase by r= r+1 and create the index set $P_r = \{(i_1, i_2, \ldots i_r)$: $a^T_{ir} x^{(i1, \ldots\ldots ir-1)}$

$< \beta_i,\ (i_1,\ldots,i_{r-1}) \in P_{r-1} - R_{r-1}\}$. Solve the problem with linear constraint $a^T_{ir}\, x^{(i1,\ldots\ldots ir-1)} = \beta_i$, $(i_1,\ldots,i_{r-1}) \in P_r$ and denote the solution of this problem by $x^{(i1,\ldots\ldots ir)}$.

Step 5. *Again check the feasibility condition as* $J_r = \{(i_1,i_2,..i_r):\ a^T_{ir}\, x^{(i1,..ir)} \geq \beta_i,\ (i_1,\ldots,i_r) \in P_r\}$ is non empty if yes then above solution is optimal and if not then *repeat this procedure until* $R_r \neq \emptyset$, *and find out the optimal solution.*

Numerical Illustration:

Consider a quadratic programming problem as:

$$\text{Minimize } f(x) = x_1^2 + x_2^2 - 8x_1 - 10x_2,\ \textit{Subject to } -3x_1 - 2x_2 \geq -6,\ x_1, x_2 \geq 0$$

The algorithm is as follows:

Step 1. *Find X and Y as:*

$$A = (-3\ \ -2),\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \gamma = \begin{pmatrix} -8 \\ -10 \end{pmatrix},\ \textit{and } \beta = -6$$

$$X = Cholesky(Q) = \begin{pmatrix} l_{11} & l_{12} \\ 0 & l_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad Y = -\frac{1}{2}(L^T)^{-1}\gamma = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

The unconstrained quadratic optimization problem becomes

$$\textit{Minimize } f(x) = (Y - Xx)^T(Y - Xx) = (4 - x_1)^2 + (5 - x_2)^2$$

Step 2. *Find the critical point* $x^* = (4\quad 5)^T$ *and this critical point is outside of feasible region of quadratic programming problem then this critical point (infeasible point) is not optimal solution of the problem.*

Step 3. *Now we create an index set* $P_1 = \{1\} \neq \Phi$ *for which the constraint is violated at this critical point and find out following equality constrained problem:*

$$\textit{Minimize } f(x) = (4 - x_1)^2 + (5 - x_2)^2$$
$$\textit{subject to}\quad -3x_1 - 2x_2 = -6,$$

and the solution of this problem is $x^{(i)} = (0.307692,\ 2.538462)^T$.

Step 4. *Now we check the feasibility condition such that* $R_1 = \{1\} \neq \Phi$ *then this solution is optimal solution and optimum value is -21.3077.*

Computational analysis of this problem

Input Data					
	x_1	x_2	Total	Sign	Limits
Objective Function	QP	QP	-21.3077		
Constraint	3	2	6	<=	6
Non-Negativity	>=	>=			
Output Results					
	x_1	x_2	$f(x)$		
	0.307692	2.538462	-21.3077		

Results

Target function (Minimum)

Name	Original Value	Final Value
QP	-21.30769225	-21.3076922

Adjustable variables

Name	Original Value	Final Value
x_1	0.307692317	0.307692317
x_2	2.538461514	2.538461514

Constraints

Name	Variable Value	Status	Slack
Constraint First Total	5.999999978	Binding	0
x_1	0.307692317	Not Binding	0.307692317
x_2	2.538461514	Not Binding	2.538461514

Sensitivity Analysis

Adjustable variables

Name	Final Value	Reduced Gradient
x_1	0.307692317	0
x_2	2.538461514	0

Constraints

Name	Final Value	Lagrange Multiplier
Constraint First Total	5.999999978	-2.461537167

Limits of the variables

Obj. function	Value
QP	-21.3076922

Variables	Value	L. Limit	Target Result	Upper Limit	Target Result
x_1	0.307692317	0	-18.94082828	0.307692324	-21.30769231
x_2	2.538461514	0	-2.366863972	2.538461525	-21.30769231

6. Conclusion

The aim of this study is to present an algorithm for finding an optimal solution of convex quadratic programming. The principle of this algorithm is based on critical point of least squares problem. If the critical point of least squares lies in the feasible solution then this critical point is also optimal solution and it does not lie in the feasible solution then we construct an equality constraint problem with violation of constraint and solve it. This algorithm is efficient as compare with the other method and requires less time to solve the problem. We therefore, hope that this algorithm may be used as an effective tool for solving convex QP problems in which the diagonal elements of positive definite matrix are positive and hence time and labor may be saved.

References

[1] Boot, John C. G., *Quadratic Programming*, Chicago, Rand McNally and Company, **(1964)** 1-54, 198-200.

[2] Filzmoser P., *Linear and Nonlinear Methods for Regression and Classification and applications in R*, Institut f. Statistik u. Wahrscheinlichkeitstheorie 1040 Wien, Wiedner Hauptstr. 8-10/107 Austria **(2008)**.

[3] Frank M and Wolfe P., *An Algorithm for Quadratic Programming*, Naval Research Logistic Quarterly, 3 **(1956)**, 95-220.

[4] Judge, G.G. and et al, *Inequality Restrictions in Regression Analysis*, Journal of the American Statistical Association 61 **(1966)**, 166-181.

[5] Liew, C. K., *Inequality Constrained Least-Squares Estimation*. Journal of the American Statistical Association. 71 **(1976)**, 746-751.

[6] Gupta O. K., *Applications of Quadratic Programming, Journal of Information and Optimization Sciences*, 16(1) **(1995)**, 177-194.

[7] Takayama and et al, *Spatial Equilibrium and Quadratic Programming* J. Farm Ecom. 44 **(1964)**, 67-93.

[8] Terlaky T., *A New Algorithm for Quadratic Programming* EJOOR, 32 **(1987)**, 294-301, North-Holland.

[9] Wolfe P., *The Simplex method for Quadratic Programming*, Econometrica, 27 **(1959)**, 382-398.

Ganita Sandesh Vol. 26 No. (2) (*December* 2012) pp. 171-176 ISSN : 0970-9169

General Multi-Level Programming Problem via Goal Programming

Bhupendra Sharma
BSACET, MATHURA (UP), INDIA
Email: bhsharma.math@gmail.com
Shifali Bhargava,
B.S.A. College, MATHURA (UP), INDIA
Email: shifalibhargava@gmail.com

Abstract

In this paper an algorithm is given to solve a multi-level programming problem using a linear preemptive goal programming model. A goal programming problem is formulated which is equivalent to given multi-level programming problem.

Keywords: Multi-Level Programming, Goal Programming,

1. Introduction

The general multi level programming problem has been an area of active research for many years and there have been a number of successful practical applications of the problem in areas like government, autonomous institutions, agriculture, military, maintenance, management, networks, schools, hospitals, banks etc. A multi-level programming problem (MLLP) is characterized by presence of multiple linear objective functions subject to the usual linear constraints. One of the important characteristics of multi-level programming problem is that a planner at a certain level of hierarchy may have their objective function and decision space is determined partially by other levels. As a class of MLPP, most of the developments focus on bi-level linear programming [2, 3, 9, 11]. The basic concept of the MLPP technique is that the first level decision maker (DM) sets his goal and then asks each subordinate level of the organization for their optimal solution. The lower level decision makers are then submitted and modify first DM in consideration of the over all benefit for the organization, the process continues until a compromise solution is reached.

General linear multi-level programming problem in which we have leader problem with n follower problems is defined as:

MLPP

$$\underset{x_1}{Max}\, F(x_1, x_2, \ldots, x_n) = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\underset{x_2}{Max}\, F(x_1, x_2, \ldots, x_n) = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\underset{x_n}{Max}\, F(x_1, x_2, \ldots, x_n) = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

$$\text{Subject to} \qquad C_1x_1 + C_2x_2 + \cdots + C_nx_n \leq r$$

$$x_1, x_2, \ldots, x_n \geq 0$$

$a_{11}, a_{21}, \ldots, a_{n1}, x_1 \in R^{n_1}$, $a_{21}, a_{22}, \ldots, a_{n2}, x_2 \in R^{n_2}, \ldots. a_{n1}, a_{n2}, \ldots, a_{nn}, x_n \in R^{nn}$ and $C_1, C_2, \ldots, C_n$ are the matrices $m \times n_1, m \times n_2, \ldots, m \times n_n$ respectively.

Let $S = \{(x_1, x_2, \ldots, x_n) : C_1x_1 + C_2x_2 + \cdots + C_nx_n \leq r\}$ denote the constraint region of MLPP. For given $\overline{x_n}$, let $S(\overline{x_n}) = \{(x_1, x_2, \ldots x_{n-1}) : C_1x_1 + C_2x_2 + \cdots + C_{n-1}x_{n-1} \leq r - C_nx_n\}$ denote the (n-1)th follower's solution space.

Let $X(\overline{x_n})$ denote the set of optimal solutions to the (n-1)th follower's problem .

The leader's solution space is defined as

$$P(x_1, x_2, \ldots, x_n) : \{(x_1, x_2, \ldots, x_n) : (x_1, x_2, \ldots, x_n) \in S,\ x_1 \in X(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})\}$$

where $X(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$ is the set of optimal solution to the first follower's problem..

The MLPP can be expressed as

$$\underset{x_i}{Max}\{a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n : (x_1, x_2, \ldots, x_n) \in S,\ x_i \in X(x_i)\} \quad \forall i = 1, 2, \ldots, n.$$

with notions of feasibility and optimality defined as follows:

Definition 1. A point $(x_1, x_2, \ldots, x_n)$ is said to be feasible to the MLPP if $(x_1, x_2, \ldots, x_n) \in P$.

Definition 2. A point $(x_1^*, x_2^*, \ldots, x_n^*)$ is said to be an optimal solution of the MLPP if

$$\sum_{i=1}^{n} a_{1i}x_i^* \geq \sum_{i=1}^{n} a_{1i}x_i \qquad \forall i = 1, 2, \ldots, n.$$

It is assumed that S and P are bounded and non empty as it generates the existence of optimal solution of the MLPP. This problem then becomes maximizing the degree of attainment of these goals called goal programming (GP). GP was introduced by Charnes and Cooper and then developed. [7, 8] The main idea behind GP is to minimize the distance between the objective function Z and aspiration level $\overline{Z}$. The aspiration level $\overline{Z}$ is determined by the decision maker or the decision analyst.

Now consider the linear multi-objective model

$$\text{(MP)} \qquad Max\, Z = \sum Ax_i$$

$$\text{Subject to} \quad Cx \leq q$$

$$x \geq 0$$

where $x \in R^N, Z = (z_1, z_2, ..., z_k)^T$ is the vector of objectives, A is a $K \times N$ matrix of objectives and C is $M \times N$ matrix and $q \in R^M$. One of the approaches to solve the multi-objective programming is GP approach. This approach minimizes the distance between the objective function vector Z and an aspiration level vector Z^*. The aspiration level is either determined by the decision maker or is taken as $Z^* = (z_1^*, z_2^*, \ldots, z_k^*)$, where z_k^* is the optimal value of z_k subject to the set of constraints in MP.

General preemptive GP model to solve MP is given by

$$\text{(GP)} \qquad Min\, Z = \left\{ \sum_{k \in p_i} w_k g_k(n_k, p_k),\ i = 1, 2, ..., n \right\}$$

$$\text{Subject to} \qquad Cx \le q$$

$$A_k x + n_k - p_k = Z_k^*$$

$$x \ge 0$$

$n_k,\ p_k \ge 0, n_k p_k = 0,\ k = 1, 2, ..., K$.

$n_k,\ p_k$ are deviational variables and w_k are their weights and $g_k(n_k, p_k) = p_k$ in case of minimizing z_k and $g_k(n_k, p_k) = n_k + p_k$ when $z_k = z_k^*$ is required. A_k is the k[th] row vector of matrix A, I is the number of priority levels, $k \in P_i$ means that the k[th] goal is in the i[th] priority level.

Since MLPP is NP hard, it is not so easy to solve it. Here an algorithm to solve MLP problem using preemptive goal programming model is presented.

2. Formulation of Goal Programming Problem Equivalent to Multi-Level Programming Problem

Phase I. In this phase the MLPP is converted into an equivalent GP problem. We consider the problem P_1 given by

$$(P_1) \qquad \underset{x_1}{Max}\, F(x_1, x_2, \ldots, x_n) = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\underset{x_2}{Max}\, F(x_1, x_2, \ldots, x_n) = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\underset{x_n}{Max}\, F(x_1, x_2, \ldots, x_n) = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

$$\text{Subject to} \quad C_1x_1 + C_2x_2 + \cdots + C_nx_n \le r$$

$$x_1, x_2, \ldots, x_n \ge 0$$

Let $F_1^{(1)}$ be the maximum value of F corresponding to the points $(x_{1i}^{(1)}, x_{2i}^{(1)}, ..., x_{ni}^{(1)}) \in S,\ \ i = 1, 2, ..., n^1$. Therefore

$$F_1^{(1)} = F_1\left(x_{1i}^{(1)}, x_{2i}^{(1)}, \dots, x_{ni}^{(1)}\right) = a_{11}x_{1i}^{(1)}, a_{12}x_{2i}^{(1)}, \dots, a_{1n}x_{ni}^{(1)}$$

$$F_2^{(1)} = F_2\left(x_{1i}^{(1)}, x_{2i}^{(1)}, \dots, x_{ni}^{(1)}\right) = a_{21}x_{1i}^{(1)}, a_{22}x_{2i}^{(1)}, \dots, a_{2n}x_{ni}^{(1)}$$

$$\dots \qquad \dots \qquad \dots$$

$$F_n^{(1)} = F_n\left(x_{1i}^{(1)}, x_{2i}^{(1)}, \dots, x_{ni}^{(1)}\right) = a_{n1}x_{1i}^{(1)}, a_{n2}x_{2i}^{(1)}, \dots, a_{nn}x_{ni}^{(1)}, \qquad \forall i = 1, 2, \dots, n^{(1)}$$

Calculate the value of lower level objective functions $f(x_1, x_2, \dots, x_n)$ at these points $\left(x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}\right)$.Let $f_i^{(1)} = f(x_{1i}^{(1)}, x_{2i}^{(1)}, \dots, x_{ni}^{(1)})$; $i = 1, 2, \dots, n^{(1)}$. Arrange $f_i^{(1)}$, $i = 1, 2, \dots, n^{(1)}$ in decreasing order.Let

$$f(x_{11}^{(1)}, x_{21}^{(1)}, \dots, x_{n1}^{(1)}) \geq f(x_{12}^{(1)}, x_{22}^{(1)}, \dots, x_{n2}^{(1)}) \geq \cdots\cdots\cdots \geq f(x_{1n}^{(1)}, x_{2n}^{(1)}, \dots, x_{nn}^{(1)})$$

$$i.e., \qquad f_1^{(1)} \geq f_2^{(1)} \geq \cdots\cdots\cdots \geq f_n^{(1)}$$

Then the preemptive GP model of MLPP at the point $\left(x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}\right)$ can be formulated in (P_2) given by

$$(P_2) \quad Min\, P_1(d_1^- + d_1^+)$$

$$Min\, P_2(d_2^- + d_2^+)$$

$$\dots \quad \dots$$

$$Min\, P_n(d_n^-)$$

Subject to $x_1 + d_1^- - d_1^+ = f_1^{(1)}$

$$x_2 + d_2^- - d_2^+ = f_2^{(1)}$$

$$\dots \quad \dots \quad \dots$$

$$x_n + d_n^- = f_n^{(1)}$$

$$C_1x_1 + C_2x_2 + \cdots + C_nx_n \leq r$$

$$x_1, x_2, \dots, x_n \leq 0$$

$$d_q^-, d_q^+ = 0; \quad d_q^-.d_q^+ = 0, \; q = 1, 2, \dots, n$$

The first (n-1) objectives are considered absolute.

Phase II. In this phase, on solving problem (P_2) the iteration methodology for GP problem [8] can be used. The solution of problem (P_2) is either feasible or infeasible, where feasible means objective function values of goals at priority P_1 and P_2 are zero.

In case (i) $d_n^- = 0$, $d_n^+ = 0$, by the choice of f as $f_n^{(1)}$ and hence $(x_{1n}^{(1)}, x_{2n}^{(1)}, \dots, x_{nn}^{(1)})$ is the solution of MLPP.

In case (ii) $d_{n-1}^+ > 0$ and $d_n^- \geq 0$, $d_n^+ > 0$.

If $d_n^- > 0$, then $(x_{1n}^{(1)}, x_{2n}^{(1)}, \dots, x_{nn}^{(1)})$ is a solution of MLPP.

If $d_n^- = 0$ *and* $d_n^+ = 0$, even then $(x_{1n}^{(1)}, x_{2n}^{(1)}, \ldots, x_{nn}^{(1)})$ is a solution of MLPP.

If $d_n^+ > 0$, then $(x_{1n}^{(1)}, x_{2n}^{(1)}, \ldots, x_{nn}^{(1)})$ is not a solution of MLPP.

Then we repeat the process of phase II with the next alternate solution $(x_{1s}^{(1)}, x_{2s}^{(1)}, \ldots, x_{ns}^{(1)})$ taking $F^{(n)} = F^{(s)}$ and $f_n^{(1)} = f_s^{(1)}$ where $s = (n-1)$ continuing this process till either some $(x_{1i}^{(1)}, x_{2i}^{(1)}, \ldots, x_{ni}^{(1)})$ $\forall i = 1, 2, \ldots, n$ turns out to be the solution of MLPP or none of $(x_{1i}^{(1)}, x_{2i}^{(1)}, \ldots, x_{ni}^{(1)})$ is the solution of MLPP. In the former case we stop as the solution is attained and in later case find the next best solution of problem P_1. Let the next best value of F as $F^{(s)}$ at the points $(x_{1i}^{(2)}, x_{2i}^{(2)}, \ldots, x_{ni}^{(2)}) \in S \quad \forall i = 1, 2, \ldots, n^{(2)}$ i.e. $F(x_{1i}^{(2)}, x_{2i}^{(2)}, \ldots, x_{ni}^{(2)}) = F^2 \quad \forall i = 1, 2, \ldots, n$

Find the value of f at these points and arrange them in a decreasing order.

Let $f(x_{1i}^{(2)}, x_{2i}^{(2)}, \ldots, x_{ni}^{(2)}) = f_2^{(2)} \quad \forall i = 1, 2, \ldots, n$ and $f_1^{(2)} \geq f_2^{(2)} \geq \cdots \geq f_n^{(2)}$.

Repeat phase II with the point $(x_{1i}^{(2)}, x_{2i}^{(2)}, \ldots, x_{ni}^{(2)})$, $i = 1, 2, \ldots, n$.

Continue this procedure till some extreme points turn out to be the solution of the given multi-level programming problem. Since, the set of extreme points is finite, the process converges in a finite steps.

3. Goal Programming Algorithm

Step 1: Solve the linear programming problem P_1 with the leader's objective function. Let $(x_{1i}^{(1)}, x_{2i}^{(1)}, \ldots, x_{ni}^{(1)})$, $\forall i = 1, 2, \ldots, n$ be its optimal solutions. Let $F(x_{1n}^{(1)}, x_{2n}^{(1)}, \ldots, x_{nn}^{(1)}) = F^n$ and go step 2.

Step 2: Find the value of the lower level objective function f at these points and arrange than in descending order. Let f_n be the maximum value of f at the point $(x_{1n}^{(1)}, x_{2n}^{(1)}, \ldots, x_{nn}^{(1)})$. Formulate corresponding goal programming problem P_2 and solve it.

Step 3: If $(x_{1n}^{(1)}, x_{2n}^{(1)}, \ldots, x_{nn}^{(1)})$ is the feasible solution of goal programming problem P_2 then $(x_{1n}^{(1)}, x_{2n}^{(1)}, \ldots, x_{nn}^{(1)})$ is the solution of MLPP otherwise go to next step.

Step 4: Since the problem is infeasible there are arises two cases.

(i) If $d_n^- > 0$, $d_n^+ = 0$ and $d_3^- \geq 0$ and then $(x_{1n}^{(1)}, x_{2n}^{(1)}, \ldots, x_{nn}^{(1)})$ will be solution of MLPP.

(ii) If $d_{n-1}^- > 0$ and $d_n^+ > 0$ then go to next step.

Step 5: Starting with the point $(x_{1n}^{(1)}, x_{2n}^{(1)}, \ldots, x_{nn}^{(1)})$ go to step 3. If $(x_{1n}^{(1)}, x_{2n}^{(1)}, \ldots, x_{nn}^{(1)})$ is not the solution of MLPP, repeat the step with other values of $(x_{1n}^{(1)}, x_{2n}^{(1)}, \ldots, x_{nn}^{(1)})$. If none of these give the solutions of MLPP then go to step 6.

Step 6: Find the next best solution of problem P1 and process further.

Hence we find the best solution for a multi-level programming problem from this algorithm.

4. Conclusion

We have obtained feasible solution of multi-level programming problem which satisfies the constraints. The GP approach for this MLP problem is simple and practical which gives the best solution of MLPP.

References

[1] Ayed, O.B., *Bilevel linear programming*, Computers and Operations Research, 20 (5) **(1993)** 485-501.

[2] Bard, J.F. and Falk, J.E., *An explicit solution to the multilevel programming problem*, Computers and Operations Research, 9 **(1982)** 77-100.

[3] Bard, J.F. *Optimality conditions for the bilevel programming problem*, Naval Research Logistics Quarterly, 31 **(1984)** 13-26.

[4] Bard, J.F., *An algorithm for solving the general bilevel programming problem*, Mathematics of operations Research, 8 (2) **(1983)** 260-272.

[5] Bialas, W.F. and Karwan, M.H., *On two level optimization*, IEEE Transaction on Automatic Control, 27 (2) **(1982)** 11-14.

[6] Bialas, W.F. and Karwan, M.H., *Two level linear programming*, Management Science, 30 (8) **(1984)** 1004-1020.

[7] Candler, W. and Townsley, R.J. *A linear two level programming problem*, Computers and Operations Research, 9 (1) **(1982)** 60-70.

[8] Ijiri, Y. *Management goals accounting for control*, North-Holland, Amsterdram **(1965)**.

[9] Lee, S.M. *Goal programming for decision analysis*, Auerbach Publishers, Philadelphia, PA **(1972)**.

[10] Sinha, S.B. and Sinha, S. *A linear programming approach for linear multi-level programming problems*, Journal of the Operational Research Society, 55 **(2004)** 312-316.

[11] Steuer, R.E. *Multiple Criteria Optimization – Theory*, Computation and Application, John Wiley and Sons **(1986)**.

[12] Vicent, L. N. and Calamai, P.H. *Bilevel and multilevel programming: A bibliography review*, Journal of Global Optimization, 5 **(1994)** 291-306.

[13] Wen, U.P. and Lin, S.F. *Finding an efficient solution to linear bi-level programming problem: An efficient approach*, Journal of Global Optimization, 8 **(1996)** 295-306.

Estd. 1987 Reg. No. 74 / Ajmer / 1988-89, Dated 23/12/1988

राजस्थान गणित परिषद्

RAJASTHAN GANITA PARISHAD

Executive Committee - 2014

Form IV (See Rule 8)

1. Place of Publication : AJMER (*Rajasthan*) INDIA
2. Periodicity of Publication : Half Yearly (*June and December*)
3. Printer's Name : Mahendra Singh Rawat
   Nationality : Indian
   Address : Bhawani Offset Printers, Topdara, AJMER (*Rajasthan*)
4. Publisher's Name : Dr. P.K. Mishra
   Nationality : Indian
   Address : General Secretary, Rajasthan Ganita Parishad, Department of Mathematics, Shobhasaria Engg.College, SIKAR (*Rajasthan*)
5. Editor's Name : Dr. K.C. Sharma
   Nationality : Indian
   Address : Department of Mathematics, M.S.J. Govt. (PG) College,Bharartpur (*Rajasthan*)
6. Name and Addresses of individuals who own the newspaper and partners or shareholders holding more than 1% of the total capital. : Rajasthan Ganita Parishad, Regd. Head Office, *Department of Mathematics,* Government College, AJMER (*Rajasthan*) INDIA

I, Dr. P.K. Mishra, hereby declare that the particulars given above are true to the best of my knowledge and belief.

Dated : 18 April, 2014

Dr. P.K. Mishra

Volume 26, No. 2 (*December 2012*)

GAṆITA SANDESH

गणित संदेश

The Sequence

Anjali Sisodia, Bijendra Singh and B.K.Joshi	Effect of constant influx of toxicant in a single species model following generalized logistic law	97 - 104
Anju Rani and Shalu Garg	Minimum shannon entropy for prescribed harmonic mean and second order moment	105 - 127
Shanu Sharma and Nini Verma	On certain generalized hypergeometric functions of three variables	129 - 144
Vipin Kumar, C.B.Gupta and S.R.Singh	An EOQ inventory model with two level credit-linked demands under permissible delay in payments for Weibull deteriorating items	145 - 157
Yogesh Sharma	More distribution properties of Gauss hypergeometric function of matrix argument	159 - 164
K.C.Sharma, Jitendra Singh	An algorithm for minimizing quadratic programming problem using method of least squares.	165 - 170
Bhupendra Sharma, Shifali Bhargava	General multi-level programming problem via Goal Programming	171- 176

Published by General Secretary, Rajasthan Ganita Parishad

Courtesy : Partially by Prof. G.R.Verma, Kingston (U.S.A)