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Meeting the 21st Century International Demands on Writing: Mathematics

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Abstract

We address the two contemporary international demands -- professionalism (2000) and explicit principles (2002) — for a good mathematics exposition. We combine the two demands and investigate the explicit principles of professional writing: Mathematics. At the outset, we cite the characteristics of (i) professionalism (ii) principles. We claim that professionalism is essentially elegance, euphony, and etiquettes in writing. We report one principle for each of (i) elegance (ii) euphony, and (iii) etiquettes in mathematical writing. We demonstrate how one principle can be illustrated at high school, college, and university levels. This ensures the universality—a distinguishing quality— of a principle.

Recently the first book on principles of professional mathematical writing was published [Radhakrishna 2013]. It lists 142 principles (on writing), which could be profitably pursued as a handbook for new mathematics professionals— serious in teaching and research.

1. Introduction

Mathematics-writing progresses along with sophistications in mathematics. As per the Subject Classification Scheme of American Mathematical Society (2010), there are 63 principal subjects of research and 7000 subsidiary subjects of research in mathematics. Mathematical writing is classified under the Principal subject 97 (Mathematics education). The doubling period for mathematics is 10 years. While mathematics has evolved from a cottage industry in the 20th century to a world-wide industry— practiced by 'an army of *professionals*' [Sir Michael Atiyah 2000]— in the 21st century, mathematical writing has not become *professional*, due to the absence of a guide. Soon after,

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"IMS Sponsor Lecture" delivered by Prof. L.Radhakrishna, Bangalore during 26th annual conference of Rajasthan Ganita Parishad.

an additional demand on writing was put forth [R. P. Boas 2002]: There are no explicit principles for good mathematics composition in the literature. These two contemporary demands— professionalism and principles for the standardization of mathematics presentation — have prompted the author to investigate *explicit principles of professional writing in mathematics.*

2. Characteristics of Professionalism and Principles

Professionalism is characterized by (i) passionate pleasantness (ii) ability to take care of minute details (iii) earning livelihood through authentic services — which need no supervision. The Item (i) needs clarification. Here pleasantness to professional colleagues is taken care of by *etiquettes*, which enhance the authenticity of written communication, besides readability and publishability (of research). Pleasantness to listeners implies *euphony* (in writing), which promotes amiability of the script — for the pleasantness to the ears of mathematicians— in an auditorium/ classroom. *Elegance* in writing facilitates immediacy of comprehension and clarity of the text, ensuring pleasantness to the eye and the mind of readers. Essentially elegance, euphony and etiquettes in writing constitute professionalism in presentation. There is no book catering to these three qualities of writing! In the 20th century, the high-priests of writing have not included these three qualities [Chaundy et al 1954, Halmos 1970, Knuth et al. 1989, Jerzy Trzeciak 1995, Higham 1998, Krantz 1998, and Swanson et al. 1999].

Principles should be (i) universal (same for all levels: High school, college and university) (ii) succinct (brief, clear, and in one sentence (iii) explicit. The discovery of principle based writing is the contemporary need for the researchers/ teachers, in the context of mathematics. Most of the features developed here for the sake of mathematics, hold good for other science disciplines.

3. Some Principles of Professional Writing: Mathematics

The 'abbreviation' of mathematical words:

Statement of the Principle 1 [Radhakrishna 2013]:

Write the abbreviation of a mathematical word

- without capital
- In lower case Roman
- without period at the end.

Examples of professional writing: (i) Abbreviations at high school level:

lim, sin, tan, log, exp.

(ii) Abbreviations at college level:

grad, div, curl , max, dim.

(iii) Abbreviations at university level:

hom, ker, ext, tor, lim sup.

Exception to the principle: If Log z is the principal value for the complex variable z, and log z the general value, then

$$\text{Log } z = \log z \pm 2n\,\pi i, \quad (n = 1, 2, \ldots, \text{to } \infty).$$

Casual writing in Indian educational system (text books, research journals):

[1] The various alternatives erroneously used by certain teachers/ authors as abbreviations for 'limit' are:

Lt, lt, Lim, LmT.

For instance the text prescribed for Class X [AP, 1998] uses 'Lt' for 'limit' throughout the book about 200 times! Generations of students are misled about the correct abbreviation! It follows that the high school teachers/ authors have to be helped to distinguish 'professional writing' and 'casual writing' via orientation courses.

[2] The television lessons on trigonometry broadcast to the nation by Indira Gandhi National Open University, mention often

$$\text{Sin}^2\,\theta + \text{Cos}^2\,\theta = 1,$$

which is not correct writing. The correct presentation is $\sin^2\theta + \cos^2\theta = 1$. Many of the compact discs—commercially produced— contain this mistake.

Abbreviation of words in English

A caution: Abbreviations of words in English literature follow principles, different from the Principle 1 for mathematics, mentioned above. We are aware that there is no mathematical writing without English and a professional teacher must respect all the niceties/ rules of the language. A whole chapter in *Write Mathematics Right* titled "Syntax" is devoted to English errors in mathematical writing. Two types of abbreviations are prevalent in English:

Type 1 (Contractions): Abbreviations formed from retaining the first letter and the last letter of the word Principle of good English writing: (i) keep the first letter capital (ii) no full stop at the end. Example: 'Dr' is the contraction (abbreviation) of 'doctor'. This professional demand escapes the notice of staff of many in Indian universities. [Similarly 'Rs' is the contraction of 'rupees', vide Oxford English Dictionary, 10th revised Edition, Oxford University Press, 2003, Page 1700 in Appendix 8: Guide to Good English].

Type 2 (Shortenings): Abbreviations formed from keeping the first few letters of the word: Principle of good English writing: (i) keep the first letter capital (ii) insert full stop at the end (of the few letters). Example: "Prof." is the shortening (abbreviation) of "professor", and "Dept. of Maths." Is the correct abbreviation of

"Department of mathematics".

A Principle on Euphony in Mathematical Writing

Principle 2: I*n a mathematical statement, a condition must appear as fore-thought.*

Professional example: If a series is uniformly convergent, we may integrate it term by term.

Casual example: We may integrate an infinite series term by term, if the series is convergent. [Here the condition appears as an afterthought]. Look for 14 more such principles, professional/ casual examples in Chapter 4 of *Write Mathematics Right.*

A Principle on Elegance in Mathematical Writing

Principle 3: For *an elegant presentation, display and label important formulae.*

Professional writing: Let N_0 denote the cardinal number of the countable set $N = \{1, 2, 3 \ldots \text{to } \infty\}$.

Then we have the strange arithmetic

$$N_0 + N_0 = N_0 \qquad \ldots\text{(i)}$$

$$N_0 N_0 = N_0 \qquad \ldots\text{(ii)}$$

$$n + N_0 = N_0, \quad n \in N \qquad \ldots\text{(iii)}.$$

Here Equations (i), (ii). (iii) are formulae in display!

Casual writing: Let N_0 denote the cardinal number of the set $\mathrm{N} = \{1, 2, 3 \ldots \text{ to } \infty\}$.

Then we have the strange arithmetic $N_0 + N_0 = N_0$, $N_0 N_0 = N_0, n + N_0 = N_0 \quad n \in N$.

Remark : Here 'Then we have' is in display (incorrectly), and all the three formulae are incorrectly presented, '*in text*'.

Note: Some authors/ referees/ editors of Indian journals of research papers are not aware of 'professional presentations' and consequently at the end of 'display' (in a separate line) of formulas / equations, they do not insist on 'full stops'. One can come across instances where more than 15 full stops are missing on one page of a thesis/ journal! It is necessary that teachers must be exposed to "professional" writing, right in the beginning of their career.

4. Accomplishments of "Write Mathematics Right"

Three types of presentations are considered in the book "*Write Mathematics Right: ... .* ;

- Written presentations (142 explicit principles spread over Chapters 1, 2, 3, 4)
- Oral (Power Point) presentations (5 guidelines on technicalities of typesetting for visibility and legibility

via Appendix B)

- Print presentations (20 guidelines via *Formula Cosmetics* and 31 manuscript markings/ proof-correction symbols— as part of 110 'traffic lights' in mathematical writing— in Appendixes A and C).

The numbers in brackets ( ) represent the number of accomplishments. Contemporary researchers and teachers are expected to be sufficiently efficient in all the three types of presentations when one masters 142 + 5 + 51 items of the bulleted list.

- Appreciation of mathematics (metamathematics, when 'meta' means about') is described with following topics:
  - Humor items (From history and philosophy of mathematics): 150 (see Humor index)
  - Thrill items (From history and philosophy of mathematics): 250 (See Thrill index)
  - The three standard philosophies and the latest philosophy of 21st century.
  - Ten definitions of mathematics from Aristotle to Einstein, including the 21st century definition.

5. Opinions on 'Write Mathematics Right'

British assessment: "Important database" — *Alpha Science International (Oxford U.K.)*

American assessment: *"A significant reasoned guidance on reporting of research."*

— American Mathematical Society.

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References

[1] AP, 1998, Mathematics Class X, *Government of Andhra Pradesh*, Hyderabad

[2] Boas R.P.2002, Editorial: *The Amateur Professor,* Notices of American Mathematical Society, P.1053

[3] Michael Atiyah, 2000, Editorial in Arnold et al.: *Mathematics: Frontiers and Prospects,* American Mathematical Society, P.viii

[4] Radhakrishna L. 2013, Write Mathematics Right: Principles of Professional Presentation, Exemplified with Humor and Thrills, *NAROSA (Delhi),* Pages 194; See www.narosa.com, www.alphasci.com

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Spatially homogeneous anisotropic Bianchi Type I inflationary universe for barotropic fluid distribution with variable bulk viscosity

by

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Abstract

Spatially homogeneous anisotropic Bianchi Type I inflationary scenario for barotropic fluid distribution with variable bulk viscosity is investigated. To get the deterministic and realistic model, we have assumed $\rho \; \alpha \; 3H^2$, $\zeta \; \alpha \; \rho^{1/2}$ *as considered by Barrow (1988) and Gron (1990) where* ρ *is the matter density,* ζ *the bulk viscosity, H the Hubble parameter and conservation equation* $T^j_{i;j} = 0$ *is taken into account. We find that bulk viscosity prevents the matter density to vanish. The model in general represents anisotropic universe but at late time, it isotropizes. The spatial volume increases with time representing inflationary scenario. The energy conditions as given by Kolassis et al. (1998), Chatterjee and Banerjee (2004) are discussed. The energy condition* $\rho + p \geq 0$ *is violated due to the presence of scalar field* (ϕ) *in inflationary universe.*

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1. Introduction

The anisotropic models provide a systematic way to obtain cosmological models more general than Friedmann-Robertson-Walker (FRW) model. But FRW model are unstable near the singularity (Patridge and Wilkinson (1967)) and fail to describe early universe. Therefore, spatially homogeneous and anisotropic Bianchi Type I metric is undertaken to study the universe in its early stages of evolution of universe. The existence of anisotropic universe that approaches the isotropic phase is pointed out by Land and Magueijo (2005). The matter distribution is satisfactorily described by perfect fluid due to large scale distribution of galaxies in our universe. However, a realistic treatment of the problem requires the consideration of material distribution other than perfect fluid. This is supported by the fact that when neutrino decoupling occurred, the matter behaved like viscous fluid in early stages of evolution of universe. Misner (1967, 1968) studied the effect of viscosity on the evolution of universe and suggested that strong dissipation due to the neutrino viscosity, may considerably reduce the anisotropy. Anisotropic Bianchi Type I viscous cosmological models have been investigated by Belinski and Khalatnikov (1976). They pointed out that Bianchi Type I universe with viscous fluid will approach asymptotically isotropic steady state model. Heller (1978) analyzed viscous isotropic and anisotropic homogeneous cosmological models in relation to the weak, dominant and strong energy conditions.

Bali (1984, 1985) investigated solutions with viscous magneto hydrodynamic matter sources using the ansatz $\zeta\theta$ = constant. Barrow (1988) has considered a radiation dominated model with constant coefficient of bulk viscosity and positive spatial curvature. Gron (1990) investigated viscous inflationary models and pointed out that bulk and shear viscosities cause exponential decay of anisotropy while non-linear viscosity causes power-law decay of anisotropy. Zimdahl (1996) investigated that sufficiently large viscous pressure leads to inflationary behaviour. The effect of viscosity upon the expansion of the universe in an inflationary era has been investigated by many authors viz. Waga et al. (1986), Padmanabhan and Chitre (1987), Barrow (1987), Lima et al. (1988), Chimento et al. (1997), Maartens and Mendez (1997), Brevik et al. (2011). The effect of bulk viscosity on cosmological models is also investigated by Saha (2005), Singh et al. (2000), Bali and Singh (2008), Bali et al. (2012), Ram et al. (2012), Brevik and Gron (2013).

Inflation is the extremely rapid expansion of the early universe by a factor of 10^{78} in volume driven by negative pressure vacuum energy density. Guth (1981) introduced the concept of inflation while investigating the problem of why we do not see magnetic monopoles today. He found that a positive energy false vacuum generates an exponential expansion of space according to general relativity. As pointed out by Guth (1981), inflationary models provide a potential solution to the problem of structure formation in Big-Bang cosmology like horizon problem, isotropy problem, flatness problem and magnetic monopole problems. Inflationary scenario for homogeneous and isotropic space-time (FRW model) has been studied by many authors viz. Linde (1982), Burd and Barrow (1998), La and Steinhardt (1989). Rothman and Ellis (1986) have pointed out that we can have solution of isotropic problem if we work with anisotropic metric that isotropizes in special case. Keeping in view of these investigations, Bali and Jain (2002), Bali (2011) investigated some inflationary cosmological models for flat potential in Bianchi Type I space-time. Recently Bali and Singh (2014) investigated LRS (Locally Rotationally Symmetric) Bianchi Type I inflationary model for stiff fluid distribution with variable bulk viscosity.

In the present investigation, we have investigated spatially homogeneous anisotropic Bianchi Type I inflationary model for barotropic fluid distribution with variable bulk viscosity. To get the realistic scenario, we have assumed the barotropic condition $p = (\gamma-1)\rho$, $1 \le \gamma \le 2$, $\rho = 3H^2, \zeta \alpha \rho^{1/2}$ as considered by Barrow (1988) and Gron (1990) and conservation equation $T^{j}_{i;j} = 0$ is taken into account. We find that spatial volume increases with time representing inflationary scenario. The deceleration parameter $(q) < 0$ which shows that the model represents accelerating universe which matches with the result as obtained by Riess et al.(1998) and Perlmutter etal. (1999).

2. Metric and Field Equations

We consider spatially homogeneous Bianchi Type I metric in the form

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2 \qquad \ldots(1)$$

where A, B, C are metric potentials and functions of *t* alone.
Einstein field equation is taken in the form

$$R_{ij} - \frac{1}{2} R\, g_{ij} = -T_{ij} \qquad \ldots(2)$$

(in geometrized unit $8\pi G = 1 = c$)
with

$$T_{ij} = (\rho + p) v_i v_j + p\, g_{ij} - \zeta\theta(g_{ij} + v_i v_j) + \partial_i\phi\,\partial_j\phi - \left[\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi + V(\phi)\right] g_{ij} \qquad \ldots(3)$$

and

$$\frac{1}{\sqrt{-g}}\partial_i[\sqrt{-g}\,\partial^i\phi] = -\frac{dV}{d\phi} \qquad \ldots(4)$$

where ρ is the matter density, p the isotropic pressure, ζ the coefficient of bulk viscosity, θ the expansion in the model, g_{ij} the metric tensor, v^i the flow vector satisfying $g_{ij}v^i v^j = -1$, ϕ the Higgs field and V the potential.

The **Higgs field** is an invisible energy field, that exists everywhere in the universe. The field is accompanied by fundamental particle called Higgs-Boson, which it uses to continuously interact with other particles. As the particles pass through the field, they are endowed with the property of mass.

Einstein field equation (2) together with (3) and (4) for the metric (1) leads to

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} = -p + \zeta\theta - \frac{\dot{\phi}^2}{2} + V(\phi) \qquad \ldots(5)$$

$$\frac{A_{44}}{A}+\frac{C_{44}}{C}+\frac{A_4C_4}{AC}=-p+\zeta\theta-\frac{\dot{\phi}^2}{2}+V(\phi) \qquad \dots(6)$$

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}=-p+\zeta\theta-\frac{\dot{\phi}^2}{2}+V(\phi) \qquad \dots(7)$$

$$\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}+\frac{B_4C_4}{BC}=\rho+\frac{\dot{\phi}^2}{2}+V(\phi) \qquad \dots(8)$$

Equation (4) leads to

$$\ddot{\phi}+\dot{\phi}\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)=\frac{dV}{d\phi} \qquad \dots(9)$$

3. Solution of Field Equations

Equations (5), (6) and (7) lead to

$$\frac{A_{44}}{A}-\frac{B_{44}}{B}+\frac{C_4}{C}\left(\frac{A_4}{A}-\frac{B_4}{B}\right)=0 \qquad \dots(10)$$

and

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}+\frac{A_4}{A}\left(\frac{B_4}{B}-\frac{C_4}{C}\right)=0 \qquad \dots(11)$$

Equation (10) leads to

$$\left(\frac{A_4}{A}-\frac{B_4}{B}\right)_4+\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)\left(\frac{A_4}{A}-\frac{B_4}{B}\right)=0 \qquad \dots(12)$$

Thus we have

$$\frac{\left(\frac{A_4}{A}-\frac{B_4}{B}\right)_4}{\left(\frac{A_4}{A}-\frac{B_4}{B}\right)}=-3H \qquad \dots(13)$$

Similarly equation (11) leads to

$$\frac{\left(\frac{B_4}{B}-\frac{C_4}{C}\right)_4}{\left(\frac{B_4}{B}-\frac{C_4}{C}\right)}=-3H \qquad \ldots(14)$$

where
$$\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}=3H \qquad \ldots(15)$$

H being Hubble parameter.
Now conservation equation

$$T^{\nu}_{\mu;\nu}=0$$

leads to

$$\frac{\partial}{\partial t}(T^4_4)+\frac{\partial}{\partial t}(\log\sqrt{-g})T^4_4-\frac{1}{2}\frac{\partial}{\partial t}(g_{\alpha\alpha})T^{\alpha\alpha}=0 \qquad \ldots(16)$$

Thus we have

$$\dot{\rho}+(\rho+p)\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)+\dot{\phi}\,\ddot{\phi}+\dot{\phi}^2\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)$$

$$-\zeta\theta\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)=0 \qquad \ldots(17)$$

Stein-Schabes (1987) has pointed out that inflation will take place if potential $V(\phi)$ has flat region. To find inflationary scenario, we consider flat region so that $V(\phi)$ is constant. Therefore equation (9) leads to

$$\dot{\phi}\,\ddot{\phi}+\dot{\phi}^2\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)=0 \qquad \ldots(18)$$

To get the deterministic solution, we assume that universe is filled with barotropic fluid distribution $p=(\gamma-1)\rho$ and $\rho=3H^2, \zeta=a\,\rho^{1/2}$ as considered by Barrow (1998) and Gron (1990). Using the above conditions, now equations (17) with (18) and

$$\zeta = \frac{1}{3}\rho^{1/2} \qquad (\text{assuming } a = \frac{1}{3}) \qquad \ldots(19)$$

lead to

$$6\,H\,\dot{H} + 9H^3(\gamma - \frac{1}{\sqrt{3}}) = 0 \qquad \ldots(20)$$

which leads to

$$H = \frac{2}{\alpha t + \beta} \qquad \ldots(21)$$

where

$$\alpha = \sqrt{3}(\sqrt{3}\gamma - 1) \qquad \ldots(22)$$

Equation (14), (15) and (22) lead to

$$\frac{A_4}{A} - \frac{B_4}{B} = \beta_1(\alpha t + \beta)^{-6/\alpha} \qquad \ldots(23)$$

$$\frac{B_4}{B} - \frac{C_4}{C} = \beta_2(\alpha t + \beta)^{-6/\alpha} \qquad \ldots(24)$$

where β_1, β_2 are constants.

From equations(23), (24) and (15), we have

$$A = \beta_3(\alpha t + \beta)^{2/\alpha} \exp\left[\frac{2\beta_1 + \beta_2}{3(\alpha - 6)}(\alpha t + \beta)^{\frac{\alpha - 6}{\alpha}}\right] \qquad \ldots(25)$$

$$B=\beta_4(\alpha t+\beta)^{2/\alpha}\exp\left[\frac{\beta_2-\beta_1}{3(\alpha-6)}(\alpha t+\beta)^{\frac{\alpha-6}{\alpha}}\right] \quad ...(26)$$

$$C=\beta_5(\alpha t+\beta)^{2/\alpha}\exp\left[-\frac{2\beta_2+\beta_1}{3(\alpha-6)}(\alpha t+\beta)^{\frac{\alpha-6}{\alpha}}\right] \quad ...(27)$$

where β_3,β_4,β_5 are constants and $1<\alpha<6$.

The metric (1) leads to the form

$$ds^2=-dt^2+\beta_3^2(\alpha t+\beta)^{4/\alpha}\exp\left[2\left\{\frac{2\beta_1+\beta_2}{3(\alpha-6)}(\alpha t++b)^{\frac{\alpha-6}{\alpha}}\right\}\right]dx^2$$

$$+\beta_4^2(\alpha t+\beta)^{4/\alpha}\exp\left[2\left\{\frac{\beta_2-\beta_1}{3(\alpha-6)}(\alpha t+\beta)^{\frac{\alpha-6}{\alpha}}\right\}\right]dy^2$$

$$+\beta_5^2(\alpha t+\beta)^{4/\alpha}\exp\left[-2\left\{\frac{(2\beta_2+\beta_1)}{3(\alpha-6)}(\alpha t+\beta)^{\frac{\alpha-6}{\alpha}}\right\}\right]dz^2 \quad ...(28)$$

After suitable transformation of coordinates, the metric (28) leads to

$$ds^2=-\frac{dT^2}{\alpha^2}+T^{4/\alpha}\exp\left[2\left\{\frac{2\beta_1+\beta_2}{3(\alpha-6)}T^{\frac{\alpha-6}{\alpha}}\right\}\right]dX^2+T^{4/\alpha}\exp\left[2\left\{\frac{\beta_2-\beta_1}{3(\alpha-6)}T^{\frac{\alpha-6}{\alpha}}\right\}\right]dY^2$$

$$+T^{4/\alpha}\exp\left[-2\left\{\frac{2\beta_2+\beta_1}{3(\alpha-6)}\right\}T^{\frac{\alpha-6}{\alpha}}\right]dZ^2 \quad ...(29)$$

4. Physical and Geometrical Features

The matter density (ρ), the isotropic pressure (p), the bulk viscosity (ζ), the expansion (θ), the shear (σ), the spatial volume (R^3), the deceleration parameter (q) for the model (29) are given by

$$H = \frac{2}{T} \qquad \ldots(30)$$

$$\rho = \frac{12}{T^2} \qquad \ldots(31)$$

$$p = (\gamma - 1)\rho = \frac{12(\gamma - 1)}{T^2} \qquad \ldots(32)$$

where $1 \leq \gamma \leq 2$

$$\zeta = \frac{\rho^{1/2}}{3} = \frac{2}{\sqrt{3}\,T} \qquad \ldots(33)$$

$$\theta = \frac{6}{T}$$

$$\ldots(34)$$

$$\sigma = \frac{(\beta_1^2 + \beta_2^2 + \beta_1\beta_2)}{\sqrt{3}(6-\alpha)T^{6/\alpha}} \qquad \ldots(35)$$

$$R^3 = ABC = \ell\, T^{6/\alpha} \qquad \ldots(36)$$

$$q = -\frac{\alpha(2\alpha - 1)}{2} < 0 \text{ as } \alpha > 1. \qquad \ldots(37)$$

The Higgs field (ϕ) is given by equation (18) as

$$\phi = \frac{L\,\alpha T^{\frac{\alpha-6}{\alpha}}}{(\alpha - 6)} + M \qquad \ldots(38)$$

where L and M are constants.

5. Energy Conditions

Following Kolassis et al. (1988), Chatterjee and Banerjee (2004), we discuss briefly weak, dominant and strong energy conditions in the context of inflationary scenario with the bulk viscosity for the model (29). We have

$$T_{00} = \rho + \frac{\dot{\phi}^2}{2} + k,\ T_{11} = p - \sqrt{3}H^2 + \frac{\dot{\phi}^2}{2} - k$$

$$= T_{22} = T_{33} \qquad \ldots(39)$$

where $V(\phi) = k$ (constant).

In the locally Minkowskian frame, the roots of matrix equations

$$|T_{ij} - rg_{ij}| = \text{diag}\left[\left(\rho + \frac{\dot{\phi}^2}{2} + k - r, r + p - \sqrt{3}H^2 + \frac{\dot{\phi}^2}{2} - k,\right.\right.$$

$$\left.\left. r + p - \sqrt{3}H^2 + \frac{\dot{\phi}^2}{2} - k, r + p - \sqrt{3}H^2 + \frac{\dot{\phi}^2}{2} - k\right)\right]. \qquad \ldots(40)$$

give the eigen values r of our energy-momentum tensor as

$$r_0 = \rho + \frac{\dot{\phi}^2}{2} + k,\ r_1 = -p + \sqrt{3}H^2 - \frac{\dot{\phi}^2}{2} + k = r_2 = r_3$$

The energy conditions for our model are

5.1 Weak Energy Conditions

$$r_0 \geq 0 \text{ leads to } \rho + \frac{\dot{\phi}^2}{2} + k \geq 0, r_0 - r_1 \geq 0 \text{ leads to}$$

$$\gamma\rho + \dot{\phi}^2 \geq \sqrt{3}H^2 \quad \text{as } p = (\gamma - 1)\rho \qquad \ldots(41)$$

5.2 Dominant Energy Conditions

$$r_0 \geq 0 \text{leads to } \rho + \frac{\dot{\phi}^2}{2} + k \geq 0, -r_0 \leq -r_1 \leq r_0 \text{ leads to}$$

$$\rho \geq p - \sqrt{3}H^2 \ + 2k \quad \dots(42)$$

5.3 Strong Energy Conditions

$$r_0 - \Sigma r_i \geq 0 \text{leads to } \rho + 3p + 2\dot{\phi}^2 \geq \sqrt{3}H^2 + 2k \quad \dots(43)$$

If we group (32) and (34) then we have

$$\rho + 3p + 2\dot{\phi}^2 \geq 3\sqrt{3}H^2 + 2k$$

The reality condition $\rho + p \geq 0$ is violated for inflationary universe due to the presence of scalar field ϕ.

6. Conclusion

The spatial volume increases with time representing inflationary scenario. The matter density is initially large but decreases with time. The bulk viscosity prevents the matter density to vanish. Initially the model starts with a big-bang at T = 0 and the expansion decreases with time. Later on, it represents accelerating universe which matches with the result as obtained by Riess et al. (1998) and Perlmutter et al. (1999). The decelerating expansion at the initial epoch provides obvious provision for the formation of large structure of universe. The formation of structures in the universe is better supported by decelerating expansion. Also the late time acceleration is in agreement with the observations of 16 type Ia supernovae made by Hubble Space Telescope (HST) (Riess et al. 2004). Since $\frac{\sigma}{\theta} \neq 0$ in general, therefore anisotropy is maintained. However, the model isotropizes at late time. The Higgs field evolves slowly but the universe expands. The presence of bulk viscosity tends to increase the inflationary phase. The energy conditions as given by Kolassis et al. (1988), Chatterjee and Banerjee (2004) are discussed. The model (29) has Point Type singularity at T = 0 (MacCallum (1971)). We also observed that the stiff fluid case ($\gamma = 2$), dust distribution ($\gamma =$ 1) and radiation dominated model ($\gamma = 4/3$) are having similar results.

References

[1] Bali, R. and Jain, V.C. Pramana – J. Phys. **59**. 1 (2002).

[2] Bali, R. and Singh, J.P. Int. J. Theor. Phys. **47**. 3288 (2008).

[3] Bali, R. and Singh, S. Canad. J. Phys. **92**. 1 (2014).

[4] Bali, R. Astrophys. Space Sci. **107**. 155 (1984).

[5] Bali, R. Astrophys. Space Sci. **116**. 81 (1985).

[6] Bali, R. Int. J. Theor. Phys.**50**. 3043 (2011).

[7] Bali, R., Singh, P. and Singh, J.P. Astrophys. Space Sci. **341**. 701 (2012).

[8] Barrow, J.D. Nucl. Phys.B **310**. 743 (1988).

[9] Barrow, J.D. Phys. Lett. **B187**. 12 (1987).

[10] Barrow, J.D. Phys. Letters **B183**. 285 (1987).

[11] Belinski, V.A. and Khalatnivkov, I.M. Sov. Phys. J.E.T.P.**42**. 205 (1976).

[12] Brevik, I. and Gron, O. In Recent Advances in Cosmology. Nova Science Publ. New York 2013. p.97.

[13] Brevik, I., Elizalde, E., Najiri, S. and Odintsov, S.D. Phys. Rev. **D84**. 103508 (2011).

[14] Burd, A.B. and Barrow, J.D. Nucl. Phys. **B 308**. 929 (1998).

[15] Chatterjee, S. and Banerjee, A. Gen. Relativ. Gravit. **36**. 303 (2004).

[16] Chimento, L.P., Jakubi, A.S., Mendez, V. and Maartens, R. Class. Quant. Gravity. **14**. 3363 (1997).

[17] D.La and Steinhardt, P.J. Phys. Rev. Lett.**62**. 367 (1989).

[18] Gron, O. Astrophys. Space Sci. **173**.191 (1990).

[19] Guth, A.H. Phys. Rev. **D23**. 347 (1981).

[20] Heller, M. Acta Cosmoligica. **7**. 7 (1978).

[21] Kolassis, A., Santos, N.O. and Tsonbeli, D. Class. Quant. Gravity.**5**. 1329 (1988).

[22] Land, K. and Magueijo, J. Phys. Rev. Lett. **95**.071301 (2005).

[23] Lima, J.A.S. Portugal, R. and Waga, I. Phys. Rev. **D37**. 2755 (1988).

[24] Linde, A.D. Phys. Lett. **B108**. 389 (1982).

[25] Maartens, R. and Mendez, V. Phys. Rev. **D55**. 1937 (1997).

[26] MacCallum, M.A.H. Com. Math. Phys. **20**. 57 (1971).

[27] Misner, C.W. Astrophys. J. **151**. 431 (1968).

[28] Misner, C.W. Nature. **214**. 40 (1967).

[29] Padmanabhan, T. and Chitre, S.M. Phys. Letters **A120**. 433 (1987).

[30] Patridge, R.B. and Wilkinson, D.T. Phys. Rev. Lett. **18**.557 (1967).

[31] Perlmatter, S. et al. Astrophys. J. **517**. 565(1999).

[32] Ram, S., Singh, M.K. and Verma, M.K. J. Mod. Phys. **3**. 9 (2012).

[33] Riess, A.G. et al. Astron. J. **116**. 1009 (1998).

[34] Riess, A.G. et al. Astrophys. J. **607**, 665 (2004).

[35] Rothman, T. and Ellis, G.F.R. Phys. Lett. **B 180**. 19 (1986).

[36] Saha, B. Mod. Phys. Lett. A **20**. 2127 (2005).

[37] Singh, C.P., Kumar, S. and Pradhan, A. Int. J. Mod. Phys. **D9**. 393 (2000).

[38] Stein-Schabes, J.A. Phys. Rev. **D 35**. 2345 (1987).

[39] Waga, I. Falcao, R.C. and Chandra, R. Phys. Rev. **D.33**. 1839 (1986).

[40] Zimdahl, W. Phys. Rev. **D.53**.54831 (1996).

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Distributed Control System For Bottle Filling Plant In MATLAB / Graphical User Interface

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Abstract

Distributed control system (DCS) is a control system or a manufacturing system, process or any kind of dynamic system, in which the controller elements are not central in location but are distributed throughout the system with each component sub-system controlled by one or more controllers. The entire system is connected by networks for communication and monitoring. DCS is a very important term used in all type of industries, to monitor and control different stages of process. DCS is a computer control software application which has been designed to work on the computer for the process by providing with all the devices. It will reduce the process cost, labor cost, time to monitor and control process. This paper aims to develop graphical user interface for the subsystem of bottle filling plant. Bottle filling plant includes four main parts which have been simulated in MATLAB software.

Keywords : Distributed control system,Process control management,Bottle filling plant,Matlab/simulation,PID.

1. Introduction

Generally, the concept of automatic control includes two major operations, which are the transmission of signals back and forth the calculation of control actions [10]. Carrying out these operations in real plant requires a set of hardware and instrumentation that serve as the platform for these tasks [10']. Distributed control system (DCS) is the most modern control platform. All the parts of DCS are connected by networks for communication and monitoring. These networks can be LAN, MAN, WAN.Now a days, DCS is a very important term used in all kind of industries, to monitor and control distributed equipments.

A DCS is a processor or a controller which uses both proprietary interconnections and communications protocol for communication. The main parts of DCS are input module, output module and memory unit. A DCS has a CPU which receives information from input modules and sends information to output modules. The input modules receive information from input instruments in the process (or field) and transmit instructions to the output instruments in the field. Buses connect the processor and I/O modules through multiplexer or de-multiplexers. Buses also connect the distributed controllers with the central controller.

A lot of work has been done in the field of distributed control system. Khin Nway Oo et. al implemented DCS using MATLAB [1]. Ling Wang et. al developed DCS for PLC based application [2]. Hla Myo Tun implemented DCS for vehicle spare parts [3]. Bottle filling plant have four main parts which are PLC remote device, Conveyor belt system, Hydraulic motor, Bottle filling robot [1]. Two windows have been developed using GUI feature of MATLAB which are main window and process control window. Whole operations can be controlled with these two windows. All the parts are simulated in MATLAB and analysis is done by using transient response characteristics of all the system.

Elements of Distributed control system- DCS consists of following elements [10].

- Local Control Unit-This is denoted as local computer. This unit can handle 8 to 16 individual PID loops, with 16 to 32 analog input lines, 8 to 16 analog output signals and some a limited number of digital inputs and outputs.
- Data Acquisition Unit-This unit may contain 2 to 16 times as many analog input/output channels as the LCU. Digital (discrete) and analog I/O can be handled. Typically, no control functions are available.
- Batch Sequencing Unit-Typically, this unit contains a number of external events, timing counters, arbitrary function generators, and internal logic.
- Local Display-This device usually provides analog display stations, analog trend recorder, and sometime video display for readout.
- Bulk Memory Unit-This unit is used to store and recall process data. Usually mass storage disks or magnetic tape are used.
- General Purpose Computer - This unit is programmed by a customer or third party to perform sophisticated functions such as optimization, advance control, expert system, etc.
- Central Operator Display - This unit typically will contain one or more consoles for operator communication with the system, and multiple video color graphics display units.
- Data Highway - A serial digital data transmission link connecting all other components in the system may consist of coaxial cable. Most commercial DCS allow for redundant data highway to reduce the risk of data loss.

- Local area Network (LAN) -Many manufacturers supply a port device to allow connection to remote devices through a standard local area network.

Bottle filling plant- bottle filling plant can be divided in three main parts which are input module, output module and PLC. Input module includes IR sensors and level sensors. IR sensors are used to detect bottles at the I/O positions and for filling and capping operation. Output of sensors cannot given directly to PLC they are given through relay circuit. All the control operations (filling and capping) are done using PLC. Whole process is automated by feeding the necessary condition into the PLC using ladder logic. Transportation of bottles is done by conveyor belt system.

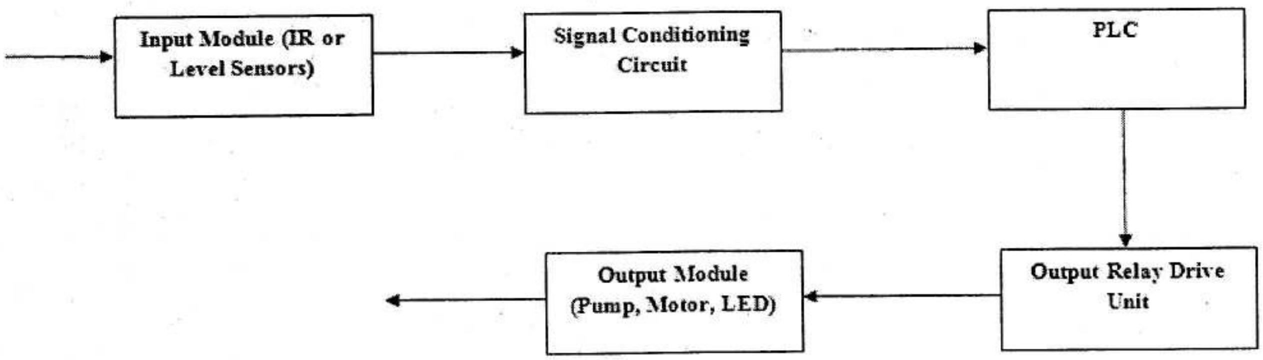

Fig.1: Bottle filling plant

2. Implementation and Simulation

1. SIMULINK model of PLC remote device :

PLC consists of two parts i.e. PID controller and PLC remote device [1]. PID is used for stability analysis. Rate limiter is used to limit response after controlled with PID for PLC plant.

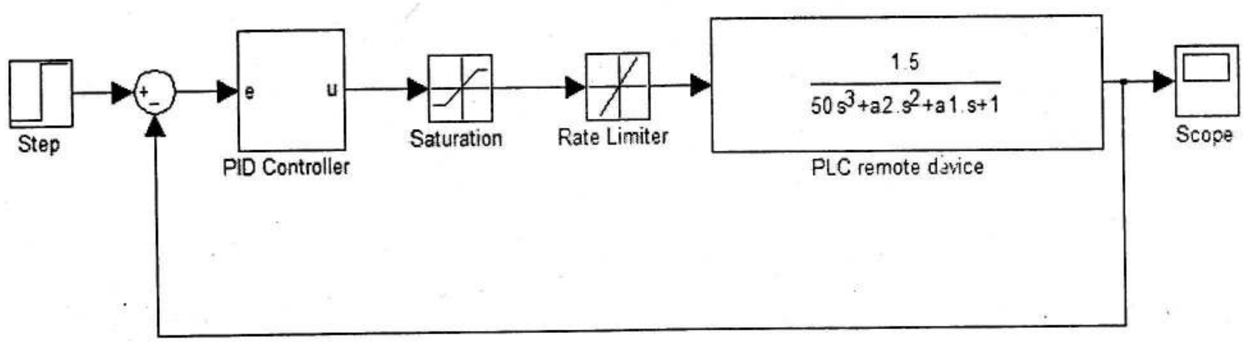

Fig.2: PLC remote device

The saturation block is used for supporting the rate limit input to control the process. Simulation result for PLC remote device has been implemented in MATLAB. It has two uncertain parameters a_2 and a_1 in the transfer function of PLC remote device. First of all, initial values of a_2 and a_1 are defined and PID is tuned for these values. K_P, K_I, K_D can be obtained by using SISO tool in the MATLAB. These values can be optimized using different optimization techniques.

2. SIMULINK model of Conveyor system :

Conveyor system is used for the transportation of the bottles. The main part of the conveyor is dc servo motor [1]. The controller for the dc servo motor is PID controller which is tuned to gain high accuracy. First of all, the gain of pre amplifier is chosen to stabilize the output of conveyor belt system and then PID is tuned using Z-N method in the SISO tool MATLAB. Integrator is used to get stable response.

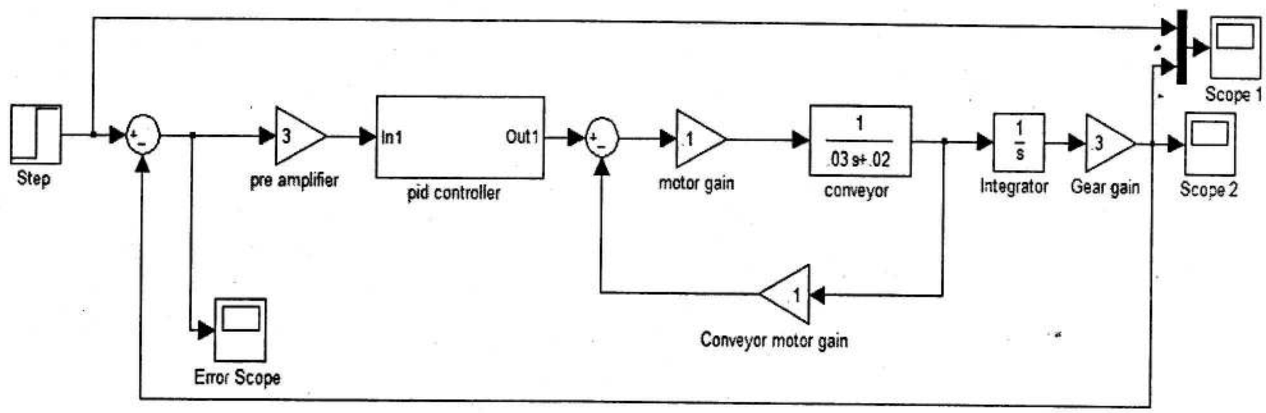

Fig.3: Conveyor belt system

3. SIMULINK model of Hydraulic motor :

SIMULINK model consists of two main parts which are valve and hydraulic motor [1]. Forward path gain is 10 for unit step input and lower feedback gain is to limit the feedback signal from the hydraulic motor.

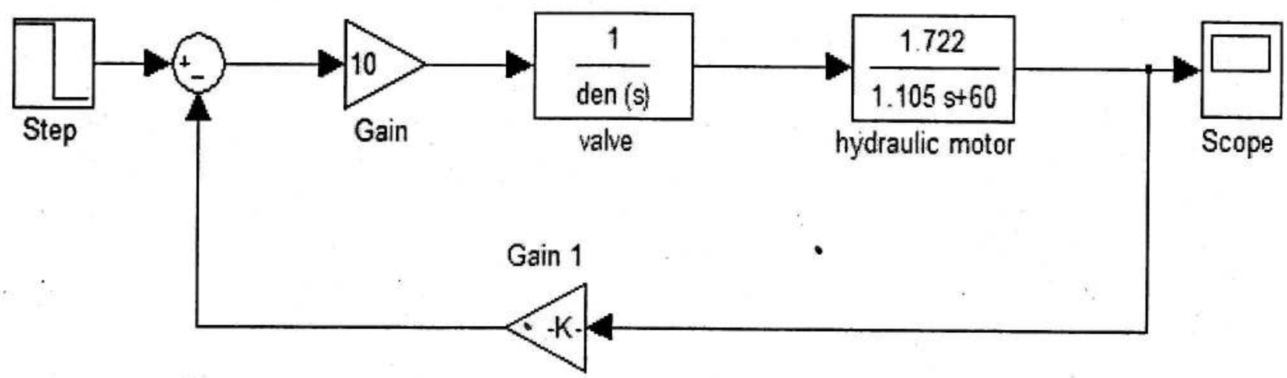

Fig.4: Hydraulic motor

4. SIMULINK model of Bottle filling robot-

There are two main transfer functions to control the bottle filling robot which are robot plant and controller stages [1]. The robot plant can be represented by rotational dynamic of the mechanical arm and controller is for compensation of the robot plant for bottle filling robot. The disturbance is also considered as Td(s).

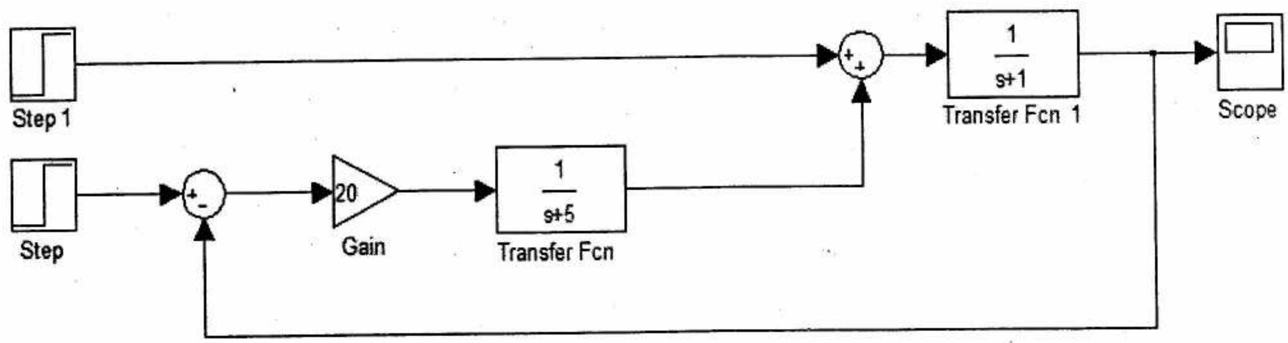

Fig.5: Bottle filling robot

3. GUI & RESULTS

The main GUI window consists of open and close button to control the filling system process control. The open button is used to open the second GUI window which includes actual distribute remote devices for the process management. The close button will exit the whole process. When open button is activated then process control management window is displayed.

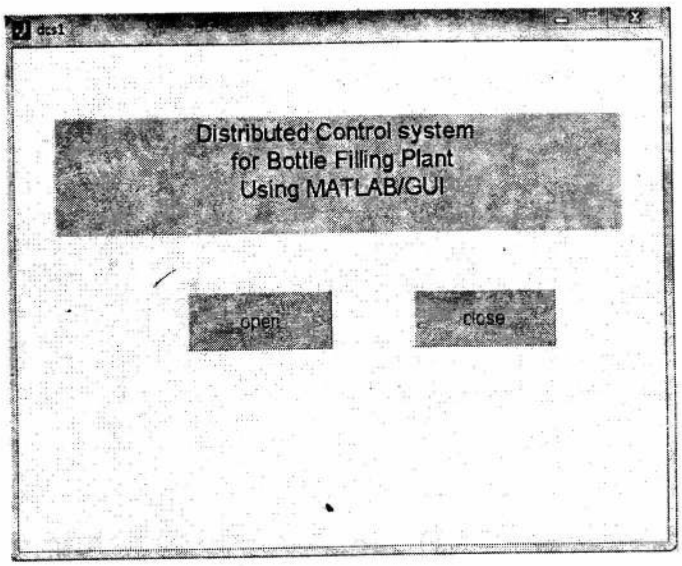

Fig.6: Main GUI window

Simulation results for Process control management

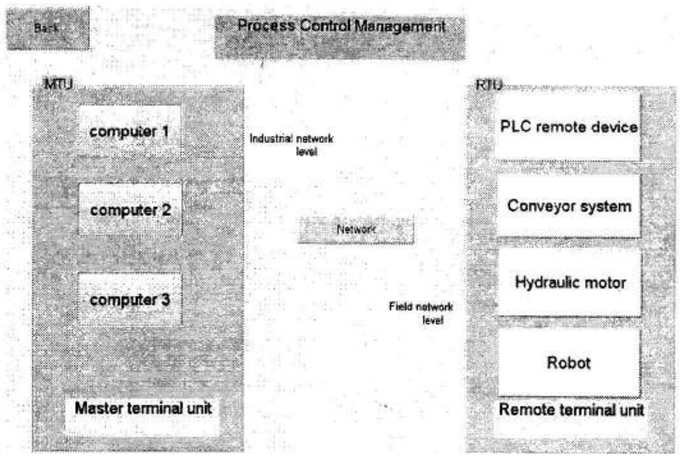

Fig.7: Process control window

There are two part of process control management window which are remote terminal unit and master terminal unit. Master terminal unit has three virtual computer connected to it and remote terminal unit includes four push button which are used to control the process stages of the bottle filling plant. When any button is pressed then it will open that simulation model and then control can be done from there. The three virtual computers are connected through industrial network which can be connected to the RTU through the network button when it is pressed. There is another push button to go back to the main GUI window.

Simulation result of PLC remote device

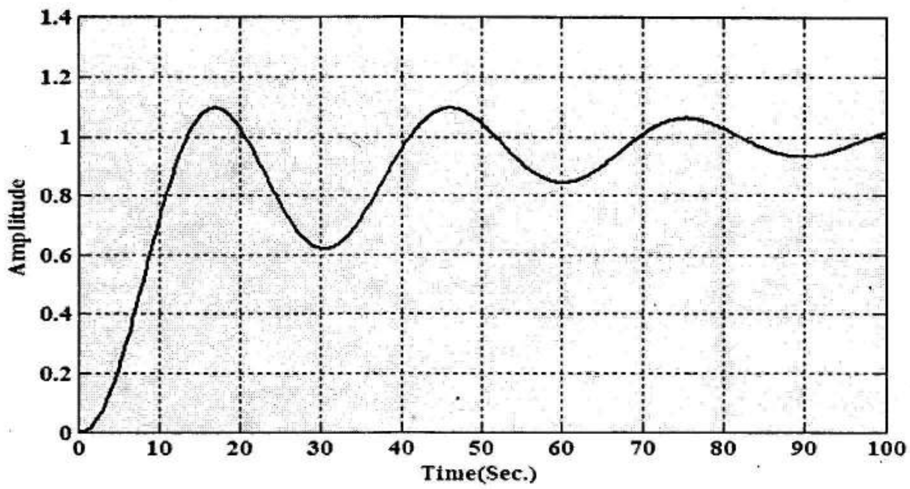

Fig.8: Step response of PLC remote device

Simulation result for PLC remote device is shown in Fig.8 and it will approaches to one at 120 milliseconds. This duration of process time is very accurate for PLC.

Table 1: Transient performance characteristics of PLC remote device

Rise time	Settling time	% overshoot
8.2	120	6.68

Simulation result of Conveyor belt system

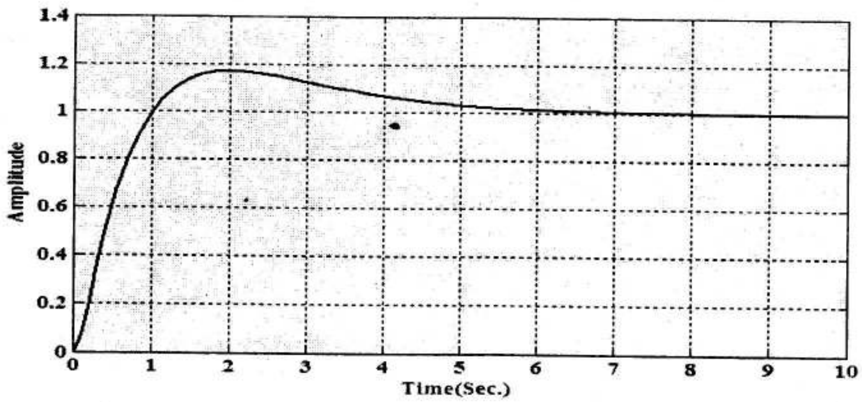

Fig.9: Step response of Conveyor belt system

The step response for the conveyor system is shown in Fig.9, and it will be stable after 7 milliseconds after evaluating transfer function of conveyor.

Table 2: Transient performance characteristics of Conveyor belt system

Rise time	Settling time	% overshoot
0.8	7.5	8

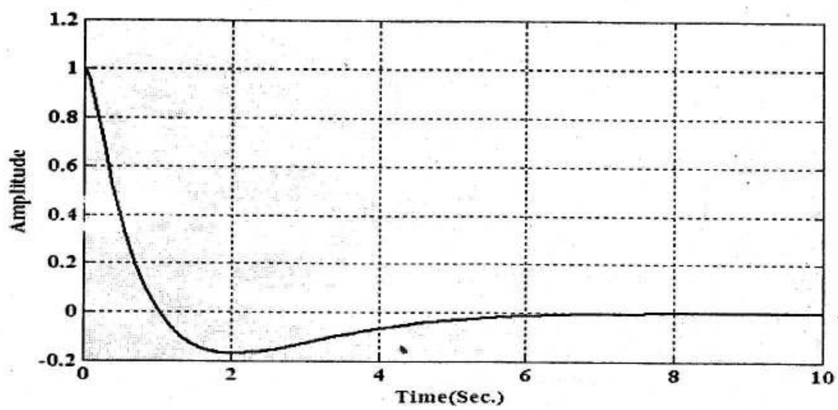

Fig.10: Error response of Conveyor belt system

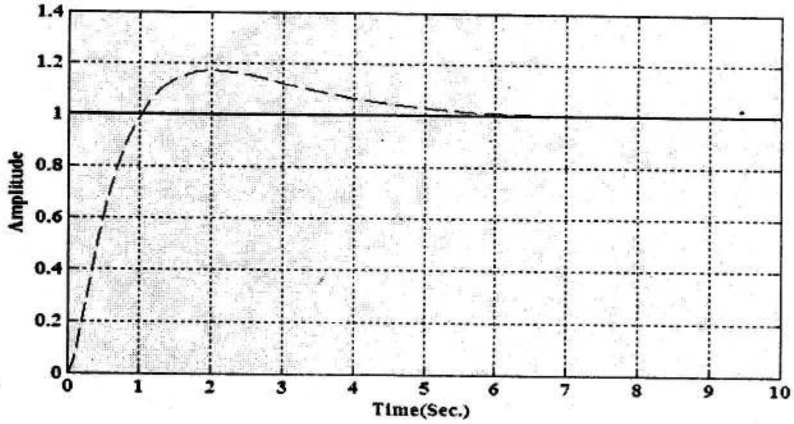

Fig.11: Comparison response system of Conveyor belt

Fig.10 shows the error response of conveyor belt and it will be zero after 7 MS. The comparison response of conveyor belt input and output is shown in Fig.11.

Simulation result of Hydraulic motor

Simulation result for hydraulic motor is shown in Fig.13. The dynamic stable response is starting from 4 milliseconds. The peak value is approximately 0.38 at time 0.633 millisecond and it is very convenient.

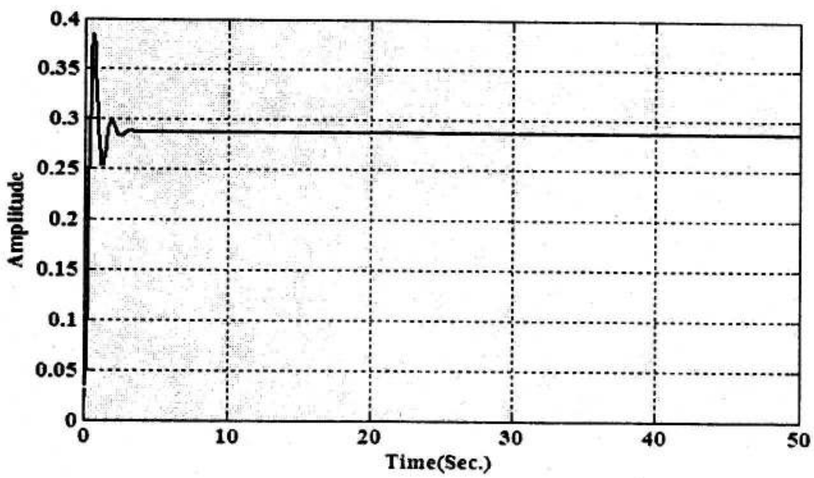

Fig.12: Step response of Hydraulic motor

Table 3: Transient performance characteristics of Hydraulic motor

Rise time	Settling time	% overshoot
0.255	2.11	34.3

Simulation result of Bottle filling robot

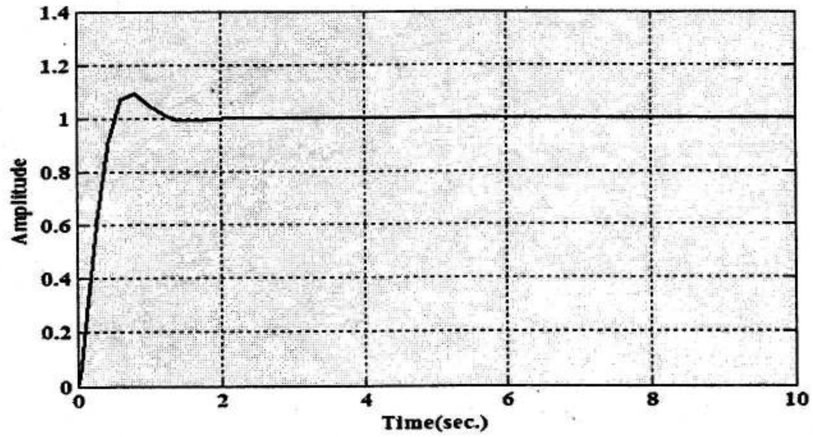

Fig.13: Step response of Bottle filling robot

Simulation response for the step input is shown in Fig.13which shows that percentage overshoot is very less. Peak amplitude is 1.1 at time 0.742 milliseconds.

Table 4: Transient performance characteristics of Bottle filling robot

Rise time	Settling time	% overshoot
0.362	1.15	9.7

4. Conclusions

Paper aims to develop distribution control system for bottle filling plant in simulation/MATLAB. The main purpose of this work is to develop the DCS for bottle filling plant which can control the devices from anywhere by monitoring the whole process. Distributed control system can access with any parts of system by using GUI. This system is a low cost data acquisition, processing and monitoring system which will reduce the labor, monitoring, and controlling cost. In this workhydraulic motor, IR or level sensors, PLC, pumps, LED, actuators and other devices are used and their simulation model is created using MATLAB software. Graphical user interface is used to create main GUI and process control management window. This can control all the devices from anywhere in the plant. In this work, the conveyor is used to send the bottle from one place to another to the desired location. Robotic arm is used to pickup the bottle and sent it to the desired location or for the filling purpose. And then robotic arm waits for some time and pick up new bottle for filling. When filling process is completed bottle is transported for the capping and labeling. After it whole product will be stored to packaging station. All these movements are done by conveyor system from robotic arm to the packaging station. All the subsystems have been implemented by using MATLAB/SIMULATION software and their simulation results have been plotted and analysis is done by using stability characteristics of all the system.

References

[1] Khin Nway Oo, Zaw Min Naing, Hla Myo Tun" Implementation Of Distributed Control System In Process Control Management Using MATLAB"international journal of scientific & technology research, volume 3, issue 6, june 2014

[2] Ling Wang and Hong-Ying Wei, "Development of a Distributed Control System for PLC-Based Applications", College of Electronic and Information Engineering, Hebei University, 2010.

[3] Hla Myo Tun, "Distributed Control System For Vehicle Spare Parts Manufacturing Plant" Ph.D Thesis, Mandalay Technological University, Mandalay, Myanmar, 2008.

[4] Gareth Talamini, "Operator Interface Design for Industrial Control", University of Queensland, October, 1997.

[5] Tsourveloudis,N; Ioannidis S.& Valavanis, K, "FuzzySurplus Based Distributed Control of Manufacturing Systems" USA, APEM Journal, Pp-5-12, 2006.

[6] Cole, E., R. Krutz, and J. Connelly,"The Network Security Bible" New York: John Wiley & Sons 2005.

[7] Edward W. Kamen, "Introduction to Industrial Controls and manufacturing", School of Electrical and Computer Engineering Georgia Institute of Technology, 1994.

[8] Beum, H. "Technology Update: Cyber Security Guidance- Interface Technologies," Control Engineering, June 1, 2004.

[9] Anonymous, "Distributed Control Systems" Chemical Engineering Department, King Saud University, 2002.

[10] Ogunnaike, B. and Ray, W., "Process Dynamics, Modeling and Control" Oxford University Press, UK, 1994.

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Generalized Fibonacci Sequence and its properties

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Abstract

One of the simplest and more studied integer sequences is the Fibonacci sequences. Fibonacci sequence is famous for possessing wonderful and amazing properties. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions and others by preserving the recurrence relation. In this paper, generalized Fibonacci sequence is introduced and defined by $U_n = pU_{n-1} + qU_{n-2}, n \geq 2$, $U_0 = a$ *and* $U_1 = 2a+1$, *where a, p and q are integers. Further some standard identities and determinant identities of generalized Fibonacci sequence are presented.*

Keywords: Generalized Fibonacci sequence, Generating function, Binet's Formula

Mathematics Subject classification 2010: 11B37, 11B39

1. Introduction

It is well-known that the Fibonacci and Lucas sequences are most prominent example of second order recursive sequences. The Fibonacci sequence is a sequence of numbers starting with integer 0 and 1, where each next term of the sequence calculated as the sum of the previous two. The Fibonacci sequence [7] is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}\ ,\ n \geq 2 \ \text{ with } F_0 = 0, F_1 = 1 \qquad \ldots(1.1)$$

The similar interpretation also exists for Lucas sequence. Lucas sequence [7] is defined by the recurrence relation:

$$L_n = L_{n-1} + L_{n-2} \ , n \geq 2 \ \text{ with } L_0 = 0 \, , L_1 = 1 \qquad \ldots(1.2)$$

The second order recurrence sequence has been generalized in two ways mainly first by preserving the initial conditions and second by preserving the recurrence relation.

Horadam [3] defined generalized Fibonacci sequence $\{H_n\}$ by

$$H_{n+2} = H_{n+1} + H_n \ , H_0 = q \text{ and } H_1 = p, \ n \geq 0 \text{ where } p, q \text{ are arbitrary integers.} \qquad \ldots(1.3)$$

Horadam [4] introduced and studied properties of another generalized Fibonacci sequence $\{w_n\}$ and defined generalized Fibonacci sequence $\{w_n\}$ by the recurrence relation:

$$w_n = pw_{n-1} - qw_{n-2} \ , n \geq 2 \text{ with } w_0 = a \, , w_1 = b \text{ where } a, b, p \text{ and } q \text{ are arbitrary integers.} \qquad \ldots(1.4)$$

Waddill and Sacks [9] extended the Fibonacci numbers recurrence relation and defined the sequence $\{P_n\}$ by the recurrence relation:

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} \ , n \geq 3 \qquad \ldots(1.5)$$

where P_0, P_1 and P_2 are not all zero given arbitrary algebraic integers.

Jaiswal [5] defined generalized Fibonacci sequence $\{T_n\}$ by

$$T_{n+1} = T_n + T_{n-1} \ , n \geq 1 \text{ with } T_1 = a \text{ and } T_2 = b \qquad \ldots(1.6)$$

Falcon and Plaza [2] introduced k^{th} Fibonacci sequence $\{F_{k,n}\}_{n \in N}$ and defined it by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \ , n \geq 1 \text{ with } F_{k,0} = 0, F_{k,1} = 1 \qquad \ldots(1.7)$$

Edson and Yayenie [1] studied a new generalization $\{q_n\}$, with initial condition $q_0 = 0$ and $q_1 = 1$ which is generated by the recurrence relation (when n is even) or $q_n = bq_{n-1} + q_{n-2}$ (when n is odd), where are non zero real numbers.

Singh, B, Bhatnagar, S. and Sikhwal, O. [8] defined generalized Fibonacci sequence $\{G_n\}$ by the recurrence relation:

$$G_n = pG_{n-1} + qG_{n-2}, n \ge 2 \text{ With } G_0 = a, G_1 = a+b \quad \ldots(1.8)$$

In this paper, we introduce generalized Fibonacci sequence and present identities consisting even and odd terms. Further we defined some specific identities and some determinant identities.

2. Generalized Fibonacci sequence

Generalized Fibonacci sequence $\{U_n\}_{n=0}^{\infty}$ is introduced and defined by the recurrence relation

$$U_n = pU_{n-1} + qU_{n-2}, n \ge 2\ ,\ U_0 = a \text{ and } U_1 = 2a+1, \text{ where } a, p \text{ and } q \text{ are integers.} \quad \ldots(2.1)$$

The first few terms are as follows:

$$U_1 = 2a+1,$$

$$U_2 = (2a+1)p + aq,$$

$$U_3 = (2a+1)p^2 + apq + (2a+1)q,$$

$$U_4 = (2a+1)p^3 + ap^2q + aq^2 + 2(2a+1)pq,$$

$$U_5 = (2a+1)p^4 + ap^3q + 3(2a+1)p^2q + 2apq^2 + (2a+1)q^2, \ldots$$

The characteristic equation of recurrence relation (2.1) is $t^2 - pt - q = 0$, which has two real roots

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2} \text{ and } \beta = \frac{p-\sqrt{p^2+4q}}{2}$$

Also $\alpha\beta = -q, \alpha+\beta = p, \alpha-\beta = \sqrt{p^2+4q}, \alpha^2+\beta^2 = p^2+2q$. ...(2.2)

Generating function of generalized Fibonacci sequence is

$$\sum_{n=0}^{\infty} U_n t^n = U(t) = U_n(t) = \frac{a+(2a+1-ap)t}{1-pt-qt^2} \quad \ldots(2.3)$$

Hypergeometric representation of generating function of generalized Fibonacci sequence

$$\sum_{n=0}^{\infty} \frac{U_n}{n!} t^n = \left[a+(2a+1-ap)t\right] e^{pt}\, {}_2F_1(n+1,1;1;qt^2)$$

By generating function (2.3), we have

$$\sum_{n=0}^{\infty} U_n t^n = \frac{a+(2a+1-ap)t}{1-pt-qt^2} = [a+(2a+1-ap)t][1-(p+qt)t]^{-1}$$

$$= [a+(2a+1-ap)t]\sum_{n=0}^{\infty}(p+qt)^n t^n$$

$$\sum_{n=0}^{\infty} U_n t^n = [a+(2a+1-ap)t]\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n}\binom{n}{k} p^{n-k}(qt)^k$$

$$= [a+(2a+1-ap)t]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!\,n-k!}p^{n-k}q^k t^{n+k}$$

$$= [a+(2a+1-ap)t]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!\,n!}p^n q^k t^{n+2k}$$

$$= [a+(2a+1-ap)t]\sum_{n=0}^{\infty}\frac{(pt)^n}{n!}\sum_{k=0}^{\infty}\frac{n+k!}{k!}q^k t^{2k}$$

$$= [a+(2a+1-ap)t]e^{pt}\sum_{k=0}^{\infty}\frac{n+k!}{k!}(qt^2)^k$$

$$\sum_{n=0}^{\infty}\frac{U_n}{n!}t^n = [a+(2a+1-ap)t]e^{pt}\sum_{k=0}^{\infty}\frac{n+k!}{n!}\frac{(qt^2)^k}{k!}$$

$$\sum_{n=0}^{\infty}\frac{U_n}{n!}t^n = [a+(2a+1-ap)t]e^{pt}\sum_{k=0}^{\infty}(n+1)_k\frac{(1)_k}{(1)_k}\frac{(qt^2)^k}{k!}$$

Hence,
$$\sum_{n=0}^{\infty}\frac{U_n}{n!}t^n = [a+(2a+1-ap)t]e^{pt}\,{}_2F_1\left(n+1,1;1;qt^2\right) \qquad \ldots(2.4)$$

Binet's formula of generalized Fibonacci sequence is defined by

$$U_n = A\alpha^n + B\beta^n = A\left(\frac{p+\sqrt{p^2+4q}}{2}\right)^n + B\left(\frac{p-\sqrt{p^2+4q}}{2}\right)^n \qquad \ldots(2.5)$$

Here, $A=\dfrac{(2a+1)-a\beta}{\sqrt{p^2+4q}}$ and $B=\dfrac{a\alpha-(2a+1)}{\sqrt{p^2+4q}}$

Also, $AB=\dfrac{2a^2p+a^2q+ap-(2a+1)^2}{(\alpha-\beta)^2}$,

$$A\beta+B\alpha=ap-(2a+1)\ A\beta^2+B\alpha^2=ap^2-(2a+1)p+aq.$$

Generalized Fibonacci sequence generate many classical sequence on the basis of value of a, p and q, like Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, modified associated Pell sequence, Jacobthal sequence, Jacobsthal-Lucas sequence, Fibonacci-Like sequence, Chebyshev first kind sequence, Chebyshev second kind sequence etc.

3. Identities of Generalized Fibonacci sequence

Now some identities of generalized Fibonacci sequence are present using generating function and Binet's formula.

Theorem 1. (Explicit Sum Formula) Let U_n be the n^{th} term of generalized Fibonacci sequence. Then

$$U_n=a\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}p^{n-2k}q^k+\left(\frac{2a+1}{p}-a\right)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}p^{n-2k}q^k \qquad ...(3.1)$$

Proof. By generating function (2.3), we have

$$\sum_{n=0}^{\infty}U_nt^n=\frac{a+(2a+1-ap)t}{1-pt-qt^2}\quad=\left[a+(2a+1-ap)t\right]\left[1-(p+qt)t\right]^{-1}$$

$$=\left[a+(2a+1-ap)t\right]\sum_{n=0}^{\infty}(p+qt)^nt^n$$

$$=\left[a+(2a+1-ap)t\right]\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\binom{n}{k}p^{n-k}(qt)^k$$

$$=\left[a+(2a+1-ap)t\right]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!n-k!}p^{n-k}q^kt^{n+k}$$

$$= [a+(2a+1-ap)t]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n+k!}{k!n!}p^n q^k t^{n+2k}$$

$$= [a+(2a+1-ap)t]\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!n-2k!}p^{n-2k} q^k t^{n}$$

$$= \sum_{n=0}^{\infty}\left[a\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!n-2k!}p^{n-2k} q^k\right]t^{n} + \sum_{n=0}^{\infty}\left[\left(\frac{2a+1}{p}-a\right)\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n-k!}{k!n-2k!}p^{n-2k+1} q^k\right]t^{n+1}$$

Equating the coefficient of t^n **,we obtain**

$$U_n = a\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}p^{n-2k}q^k + \left(\frac{2a+1}{p}-a\right)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\binom{n-k-1}{k}p^{n-2k}q^k$$

By taking different values of a, p **and** q in above identity, explicit formulas also can be obtained for Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal-Lucas, Fibonacci-Like sequences.

Theorem 2. (Sum of First n terms) sum of first n terms of generalized Fibonacci sequence is given by

$$\sum_{k=0}^{n-1}U_k = \frac{U_n + qU_{n-1} - a - (2a+1-ap)}{p+q-1} \tag{3.2}$$

Proof. Using the Binet's formula (2.5) we have

$$\sum_{k=0}^{n-1}U_k = \sum_{k=0}^{n-1}\left[A\alpha^k + B\beta^k\right] = A\left[\frac{1-\alpha^n}{1-\alpha}\right] + B\left[\frac{1-\beta^n}{1-\beta}\right]$$

$$= \frac{(A+B)-(A\beta+B\alpha)-(A\alpha^n+B\beta^n)+\alpha\beta(A\alpha^{n-1}+B\beta^{n-1})}{1-(\alpha+\beta)+\alpha\beta}$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=0}^{n-1}U_k = \frac{U_n + qU_{n-1} - a - (2a+1-ap)}{p+q-1}$$

Theorem 3. (Sum of First n terms with odd indices) Sum of first n terms (with odd indices) of generalized Fibonacci sequence is given by

$$\sum_{k=0}^{n-1} U_{2k+1} = \frac{U_{2n+1} - q^2 U_{2n-1} - (2a+1) + (2a+1-ap)q}{p^2 - q^2 + 2q - 1} \qquad (3.3)$$

Proof. Using the Binet's formula (2.5), we have

$$\sum_{k=0}^{n-1} U_{2k+1} = \sum_{k=0}^{n-1}\left[A\alpha^{2k+1} + B\beta^{2k+1}\right] = A\left[\frac{\alpha(1-\alpha^{2n})}{1-\alpha^2}\right] + B\left[\frac{\beta(1-\beta^{2n})}{1-\beta^2}\right]$$

$$= \frac{(A\alpha + B\beta) - \alpha\beta(A\beta + B\alpha) - (A\alpha^{2n+1} + B\beta^{2n+1}) + (\alpha\beta)^2(A\alpha^{2n-1} + B\beta^{2n-1})}{1-(\alpha^2+\beta^2)+(\alpha\beta)^2}$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=0}^{n-1} U_{2k+1} = \frac{U_{2n+1} - q^2 U_{2n-1} - (2a+1) + (2a+1-ap)q}{p^2 - q^2 + 2q - 1}$$

Theorem 4. (Sum of First *n* terms with even indices) Sum of first n terms (with even indices) of generalized Fibonacci sequence is given by

$$\sum_{k=0}^{n-1} U_{2k} = \frac{U_{2n} - q^2 U_{2n-2} - \left[(2a+1)p - a(p^2+q)\right] - a}{p^2 - q^2 + 2q - 1} \qquad ...(3.4)$$

Proof. Using the Binet's formula (2.5), we have

$$\sum_{k=0}^{n-1} U_{2k} = \sum_{k=0}^{n-1}\left[A\alpha^{2k} + B\beta^{2k}\right] = A\left[\frac{(1-\alpha^{2n})}{1-\alpha^2}\right] + B\left[\frac{(1-\beta^{2n})}{1-\beta^2}\right]$$

$$= \frac{(A+B) - (A\beta^2 + B\alpha^2) - (A\alpha^{2n} + B\beta^{2n}) + (\alpha\beta)^2(A\alpha^{2n-2} + B\beta^{2n-2})}{1-(\alpha^2+\beta^2)+(\alpha\beta)^2}$$

Using subsequent results of Binet's formula, we get

$$\sum_{k=0}^{n-1} U_{2k} = \frac{U_{2n} - q^2 U_{2n-2} - \left[(2a+1)p - a(p^2+q)\right] - a}{p^2 - q^2 + 2q - 1}$$

Theorem 5. (Catalan's Identity) Let U_n be the n^{th} term of generalized Fibonacci sequence. Then

$$U_n^2 - U_{n+r}U_{n-r} = \frac{(-q)^{n-r}}{(2a+1)^2 - a(2a+1)p - a^2q}\left[(2a+1)U_r - aU_{r+1}\right]^2, n > r \geq 1. \qquad \ldots(3.5)$$

Proof. Using the Binet's formula (2.5), we have

$$U_n^2 - U_{n+r}U_{n-r} = (A\alpha^n + B\beta^n)^2 - (A\alpha^{n+r} + B\beta^{n+r})(A\alpha^{n-r} + B\beta^{n-r}) = AB(\alpha\beta)^n\left(2 - \alpha^r\beta^{-r} - \alpha^{-r}\beta^r\right)$$

$$= -AB(-q)^{n-r}\left(\alpha^r - \beta^r\right)^2 = \frac{\left[(2a+1)^2 - a(2a+1)p - a^2q\right]}{(\alpha-\beta)^2}(-q)^{n-r}\left(\alpha^r - \beta^r\right)^2$$

Using subsequent results of Binet's formula, we get

$$U_n^2 - U_{n+r}U_{n-r} = \left[(2a+1)^2 - a(2a+1)p - a^2q\right](-q)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha - \beta}\right)^2$$

Since $\dfrac{\alpha^r - \beta^r}{\alpha - \beta} = \dfrac{(2a+1)U_r - aU_{r+1}}{(2a+1)^2 - a(2a+1)p - a^2q}$, we obtain

$$U_n^2 - U_{n+r}U_{n-r} = \frac{(-q)^{n-r}}{(2a+1)^2 - a(2a+1)p - a^2q}\left[(2a+1)U_r - aU_{r+1}\right]^2, n > r \geq 1.$$

Corollary 6. (Cassini's Identity) Let U_n be the n^{th} term of generalized Fibonacci sequence. Then

$$U_n^2 - U_{n+1}U_{n-1} = (-q)^{n-1}\left[(2a+1)^2 - a(2a+1)p - a^2q\right], n \geq 1. \qquad \ldots(3.6)$$

Taking $r = 1$ in the Catalan's identity (3.5), the required identity is obtained.

Theorem 7. (d'Ocagne's Identity) Let U_n be the n^{th} term of generalized Fibonacci sequence. Then

$$U_mU_{n+1} - U_{m+1}U_n = (-q)^n\left[(2a+1)U_{m-n} - aU_{m-n+1}\right], m > n \geq 0. \qquad \ldots(3.7)$$

Proof. Using the Binet's formula (2.5), we have

$$U_m U_{n+1} - U_{m+1} U_n = (A\alpha^m + B\beta^m)(A\alpha^{n+1} + B\beta^{n+1}) - (A\alpha^{m+1} + B\beta^{m+1})(A\alpha^n + B\beta^n)$$

$$= AB(\alpha^m \beta^{n+1} + \alpha^{n+1}\beta^m - \alpha^n \beta^{m+1} - \alpha^{m+1}\beta^n) = AB(\alpha\beta)^n\left[\beta\left(\alpha^{m-n} - \beta^{m-n}\right) - \alpha\left(\alpha^{m-n} - \beta^{m-n}\right)\right]$$

$$= AB(-q)^n(\alpha - \beta)\left(\alpha^{m-n} - \beta^{m-n}\right)$$

$$= \frac{\left[(2a+1)^2 - a(2a+1)p - a^2 q\right]}{(\alpha - \beta)^2}(-q)^n(\alpha - \beta)\left(\alpha^{m-n} - \beta^{m-n}\right)$$

Since $\dfrac{\alpha^{m-n} - \beta^{m-n}}{\alpha - \beta} = \dfrac{(2a+1)U_{m-n} - aU_{m-n+1}}{(2a+1)^2 - a(2a+1)p - a^2 q}$, we obtain

$$U_m U_{n+1} - U_{m+1} U_n = (-q)^n\left[(2a+1)U_{m-n} - aU_{m-n+1}\right],\ m > n \geq 0.$$

Theorem 8. (Generalized Identity) Let U_n be the n^{th} term of generalized Fibonacci sequence. Then

$$U_m U_n - U_{m-r} U_{n+r} = (-q)^{m-r}\frac{\left[(2a+1)U_r - aU_{r+1}\right]\left[(2a+1)U_{n-m+r} - aU_{n-m+r+1}\right]}{\left[(2a+1)^2 - a(2a+1)p - a^2 q\right]},\ n > m \geq r \geq 1. \quad ...(3.8)$$

Proof: Using the Binet's formula (2.5), we have

$$U_m U_n - U_{m-r} U_{n+r} = (A\alpha^m + B\beta^m)(A\alpha^n + B\beta^n) - (A\alpha^{m-r} + B\beta^{m-r})(A\alpha^{n+r} + B\beta^{n+r})$$

$$= AB(\alpha^r - \beta^r)\left[\frac{\alpha^m\beta^n}{\alpha^r} - \frac{\alpha^n\beta^m}{\beta^r}\right] = AB\frac{(\alpha^r - \beta^r)}{(\alpha\beta)^r}\left(\alpha^m\beta^{n+r} - \alpha^{n+r}\beta^m\right)$$

$$U_m U_n - U_{m-r} U_{n+r} = -AB(-q)^{m-r}(\alpha^r - \beta^r)\left(\alpha^{n-m+r} - \beta^{n-m+r}\right)$$

$$= \frac{\left[(2a+1)^2 - a(2a+1)p - a^2 q\right]}{(\alpha - \beta)^2}(-q)^{m-r}\left(\alpha^r - \beta^r\right)\left(\alpha^{n-m+r} - \beta^{n-m+r}\right)$$

$$\frac{\alpha^r-\beta^r}{\alpha-\beta}=\frac{(2a+1)U_r-aU_{r+1}}{(2a+1)^2-a(2a+1)p-a^2q} \text{ and } \frac{\alpha^{n-m+r}-\beta^{n-m+r}}{\alpha-\beta}=\frac{[(2a+1)U_{n-m+r}-aU_{n-m+r+1}]}{[(2a+1)^2-a(2a+1)p-a^2q]}, \text{ we obtain}$$

$$U_mU_n-U_{m-r}U_{n+r}=(-q)^{m-r}\frac{[(2a+1)U_r-aU_{r+1}][(2a+1)U_{n-m+r}-aU_{n-m+r+1}]}{[(2a+1)^2-a(2a+1)p-a^2q]}, n>m\geq r\geq 1.$$

The identity (3.8) provides Catalan's and d'Ocagne's and other identities:

* If $m=n$, the Catalan's identity (3.5) is obtained.
* If $m=n$ and $r=1$ in identity (3.8), the Cassini's identity (3.6) is obtained.
* If $n=m$, $m=n+1$ and $r=1$ in identity (3.8), the d'Ocagne's identity (3.7) is obtained.
* For different values of of a, p and q, we obtained all above results for Fibonacci, Lucas, Pell-Lucas, Modified Pell, Jacobsthal, Jacobsthal-Lucas, Fibonacci-Like sequence.

4. Determinant Identities

There is a long tradition of using matrices and determinants to study Fibonacci numbers. Koshy [9] explained two chapters on the use of matrices and determinants in Fibonacci numbers. In this section, some determinant identities are presented.

Theorem 1. For any integers $n\geq 0$ prove that

$$\begin{vmatrix} U_{n+1} & U_{n+2} & U_{n+3} \\ U_{n+4} & U_{n+5} & U_{n+6} \\ U_{n+7} & U_{n+8} & U_{n+9} \end{vmatrix}=0$$

Proof. Let $\Delta=\begin{vmatrix} U_{n+1} & U_{n+2} & U_{n+3} \\ U_{n+4} & U_{n+5} & U_{n+6} \\ U_{n+7} & U_{n+8} & U_{n+9} \end{vmatrix}$

Applying $C_1\to C_1(q)$ and $C_2\to C_2(p)$ we get

$$\Delta=\begin{vmatrix} qU_{n+1} & pU_{n+2} & U_{n+3} \\ qU_{n+4} & pU_{n+5} & U_{n+6} \\ qU_{n+7} & pU_{n+8} & U_{n+9} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we get

$$\Delta = \begin{vmatrix} U_{n+3} & pU_{n+2} & U_{n+3} \\ U_{n+6} & pU_{n+5} & U_{n+6} \\ U_{n+9} & pU_{n+8} & U_{n+9} \end{vmatrix}$$

Since two columns are identical, thus we obtained required results.

Theorem 2. For any integers $n \geq 0$, prove that

$$\begin{vmatrix} U_n - U_{n+1} & U_{n+1} - U_{n+2} & U_{n+2} - U_n \\ U_{n+1} - U_{n+2} & U_{n+2} - U_n & U_n - U_{n+1} \\ U_{n+2} - U_n & U_n - U_{n+1} & U_{n+1} - U_{n+2} \end{vmatrix} = 0$$

Proof. Let $\Delta = \begin{vmatrix} U_n - U_{n+1} & U_{n+1} - U_{n+2} & U_{n+2} - U_n \\ U_{n+1} - U_{n+2} & U_{n+2} - U_n & U_n - U_{n+1} \\ U_{n+2} - U_n & U_n - U_{n+1} & U_{n+1} - U_{n+2} \end{vmatrix}$

By applying $C_1 \to C_1 + C_2 + C_3$ and expanding along first row, we obtained required result.

Theorem 3. For any integers $n \geq 0$, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ qU_n & pU_{n+1} & U_{n+2} \\ pU_{n+1} + U_{n+2} & qU_n + U_{n+2} & qU_n + pU_{n+1} \end{vmatrix} = 0$$

Proof. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ qU_n & pU_{n+1} & U_{n+2} \\ pU_{n+1} + U_{n+2} & qU_n + U_{n+2} & qU_n + pU_{n+1} \end{vmatrix}$

applying $R_3 \to R_3 + R_2$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ qU_n & pU_{n+1} & U_{n+2} \\ 2U_{n+2} & 2U_{n+2} & 2U_{n+2} \end{vmatrix}$$

Taking common out $2U_{n+2}$ from third row, we get

$$\Delta = 2U_{n+2}\begin{vmatrix} 1 & 1 & 1 \\ qU_n & pU_{n+1} & U_{n+2} \\ 1 & 1 & 1 \end{vmatrix}$$

Since two rows are identical, thus we obtained required results.

Theorem 4. For any integers $n \geq 0$, prove that

$$\begin{vmatrix} U_n & U_n + U_{n+1} & U_n + U_{n+1} + U_{n+2} \\ 2U_n & 2U_n + 3U_{n+1} & 2U_n + 3U_{n+1} + 4U_{n+2} \\ 3U_n & 3U_n + 6U_{n+1} & 3U_n + 6U_{n+1} + 12U_{n+2} \end{vmatrix} = 3U_n U_{n+1} U_{n+2}$$

Proof. Let $\Delta = \begin{vmatrix} U_n & U_n + U_{n+1} & U_n + U_{n+1} + U_{n+2} \\ 2U_n & 2U_n + 3U_{n+1} & 2U_n + 3U_{n+1} + 4U_{n+2} \\ 3U_n & 3U_n + 6U_{n+1} & 3U_n + 6U_{n+1} + 12U_{n+2} \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get

$$\Delta = \begin{vmatrix} U_n & U_n + U_{n+1} & U_n + U_{n+1} + U_{n+2} \\ 0 & U_{n+1} & U_{n+1} + 2U_{n+2} \\ 0 & 3U_{n+1} & 3U_{n+1} + 9U_{n+2} \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_2$ and expanding along first row, we obtained required result.

Theorem 5. For any integers $n \geq 0$, prove that

$$\begin{vmatrix} 0 & U_n U_{n+1}^2 & U_n U_{n+2}^2 \\ U_n^2 U_{n+1} & 0 & U_{n+1} U_{n+2}^2 \\ U_n^2 U_{n+2} & U_{n+2} U_{n+1}^2 & 0 \end{vmatrix} = 2U_n^3 U_{n+1}^3 U_{n+2}^3$$

Proof. Let $\Delta = \begin{vmatrix} 0 & U_n U_{n+1}^2 & U_n U_{n+2}^2 \\ U_n^2 U_{n+1} & 0 & U_{n+1} U_{n+2}^2 \\ U_n^2 U_{n+2} & U_{n+2} U_{n+1}^2 & 0 \end{vmatrix}$

Taking common out $U_n^2, U_{n+1}^2, U_{n+2}^2$ from C_1, C_2, C_3 respectively, we get

$$\Delta = U_n^2 U_{n+1}^2 U_{n+2}^2 \begin{vmatrix} 0 & U_n & U_n \\ U_{n+1} & 0 & U_{n+1} \\ U_{n+2} & U_{n+2} & 0 \end{vmatrix}$$

Taking common out U_n, U_{n+1}, U_{n+2} from R_1, R_2, R_3 respectively and expanding along first row, we obtained required result.

5. Conclusions

In this paper, generalized Fibonacci sequence is introduced. Some standard identities of generalized Fibonacci sequence have been obtained and derived using generating function and Binet's formula. Also some determinant identities have been established and derived.

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References

[1] Edson, M. and Yayenie, O. (2009). A new generalization of Fibonacci sequence and extended Binet's formula. Integers, 9, 639-654.

[2] Falcon, S. and Plaza, A. (2007). On the Fibonacci *K*- numbers. Chaos, Solutions & Fractals, 32 (5), 1615-1624.

[3] Horadam, A. F. (1961). A generalized Fibonacci sequence. American Mathematical Monthly, 68 (5), 455-459.

[4] Horadam, A. F. (1965). Basic properties of a certain generalized sequence of numbers. The Fibonacci Quarterly, 3 (3), 161-176.

[5] Jaiswal, D. V. (1969). On a generalized Fibonacci sequence. Labdev J. Sci. Tech., Part A 7, 67-71.

[6] Koshy, T. (2001). Fibonacci and Lucas numbers with applications. New York: Wiley- Interscience Publication.

[7] Singh, B., Sikhwal, O. and Bhatnagar, S. (2010). Fibonacci-Like sequence and its properties. Int. J. Contemp. Math. Sciences, 5 (18), 859-868.

[8] Singh, B., Bhatnagar, S. and Sikhwal, O. (2013). Generalized Fibonacci sequence and its properties. Open Journal of Mathematical Modeling, 1 (6), 194-202.

[9] Waddill, M. E. and Sacks, L. (1967). Another generalized Fibonacci sequence. The Fibonacci Quarterly, 5 (3), 209-222.

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Comparison of Solutions of Field Equation for Spherically Symmetric vacuum field

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Abstract

In this paper, we have compared the solution obtained by higher order field equation to Einstein Equation by taking generalized metric for spherically symmetric isotropic and static field in vacuum and summarized the result.

I. Introduction

The general 4-dimension spherically symmetric metric form;

$$ds^2 = \alpha(t,r)dt^2 + 2\beta(t,r)\,dtdr + \gamma(t,r)\,dr^2 + \delta(t,r)\,(d\theta^2 + Sin^2\theta d\varphi^2) \qquad ...(A.1)$$

Here in the consequence of the 2-spheres being subspaces of the Riemann space, the signature of $d\theta^2$ and $d\varphi^2$ must be the same. We consider time like space and put α as +ve sign and β, γ, δ as –ve sign.

Further, variables of subspaces *r* and *t* can be carried out arbitrary nonsingular coordinate transformation;

$$t = f(t',r'), \quad r = g(t',r')$$

Where f and g are arbitrary functions subject to the condition that $\frac{\partial(t,r)}{\partial(t',r')} \neq 0$. Assume $\alpha(t,r) - \gamma(t,r) = \frac{\delta(t,r)}{r^2}$ and $\beta(t,r) = 0$.

By choosing appropriate transformation in *r* and *t*, one can digonolized the metric (A.1)

$$ds^2 = A(t,r)\,dt^2 - B(t,r)\,dr^2 - r^2(d\theta^2 + Sin^2 d\varphi^2) \qquad ...(A.2)$$

This general isotropic metric is specified by two functions of *r* and *t*, namely *A*(*r*, *t*) and *B*(*r*, *t*) and describes the geometry of 2-spheres. This line element shows such a surface has surface area Since *B*(*r*, *t*) is not necessarily equal to unity. We cannot assume that *r* is the radial distance.

Consider general spatially isotropic metric,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + Sin^2 d\varphi^2) \qquad ...(A.3)$$

Schwarzschild solution for the empty space-time outside a spherical body of mass M is

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - Sin^2\theta d\varphi^2 \qquad \text{...(A.4)}$$

Higher order Field Equation :

$$G_{\mu\nu} - \sum_{n=2}^{N} \frac{nC_n}{6}\left(l^2 R\right)^{n-1}$$

$$\left[R_{\mu\nu} - \frac{1}{2n} g_{\mu\nu} R - \frac{(n-1)}{R}\left(R_{;\mu;\nu} - g_{\mu\nu} R\right) - \frac{(n-1)(n-2)}{R^2}\left(R_{;\mu} R_{;\nu} - g_{\mu\nu} R_{;\alpha} R^{;\alpha}\right)\right] = \kappa T_{\mu\nu} \qquad \text{...(B.1)}$$

For empty space $T_{\mu\nu} = 0$. Here C_n are arbitrary dimensionless coefficients corresponding to n and l is constant for balancing the dimension.

For $n = 1$, equation (B.1) agrees with Einstein Field equation. We consider $n = 2$ for our solution and this makes equation (B.1) as

$$G_{\mu\nu} - \frac{C_2}{3}\left(l^2 R\right) R_{\mu\nu} - \frac{1}{4} g_{gm} R - \frac{1}{R}\left(R_{;\mu;\nu} - g_{\mu\nu} R\right) = 0 \qquad \text{...(B.2)}$$

C_2 and l are constants and can put equal to unity for our purpose.

Solution:

Using MATHEMATICA, on solving equation (B.2) for the general metric (A.3), we find three highly non-linear independent equations having fourth derivatives of $A(r)$ and $B(r)$ with fourth powers of $A(r)$ and $B(r)$. It is observed that

$$A(r) = \left[B(r)\right]^{-1} \qquad \text{... (C.1)}$$

And thus, equations are

$$-4 + 8r + 2r^2 + 92A^2 + A(-88 - 8r - 2r^2 - 8rA' - 48r^2 A'' + 32r^3 A''' + 8r^4 A^{iv}) + 8r^3 A'' + r^4 A''^2$$

$$+ 2rA'(4 - 2r - r^2 + 2r^2 A'') + 2r^4 A''' = 0 \qquad \text{...(C.1)}$$

$$-4 - 8r - 4A^2 + A(8 + 8r - 8rA') - 8r^3 A'' + r^4 A'' + r^4 A''^2$$

$$+2rA'(4 + 2r + r^2 + 2r^2 A'') - 2r^4 A''' = 0 \qquad \text{...(C.2)}$$

$$-4 + 8r + 2r^2 - 4A^2 - 2A(-4 + 4r + r^2 + 4rA') + 8r^3 A''$$

$$+ r^4 A''^2 + 2rA'(4 - 2r - r^2 + 2r^2 A'') + 2r^4 A''' = 0 \qquad \text{...(C.3)}$$

$$2r^4 (A'')^2 + (r^4 + 8r^3 A')A'' + 16rA'(1 - A) + A(16 - 2r^2) - 8 + 2r^2 - 8A^2 = 0 \qquad \text{...(C.4)}$$

Where prime denotes differentiation with respect to r. Since equation (C.2) contains fourth order derivative so we can leave it for further computation. From equation (C.3) and (C.4) we obtained

$$2r^4(A'')^2+\left(r^4+8r^3A'\right)A''+16rA'(1-A)+A\left(16-2r^2\right)-8+2r^2-8A^2=0 \qquad \text{... (C.5)}$$

This is a non-linear second order second degree ordinary differential equation in $A(r)$. We solved (C.5) numerically by putting initial condition that for larger distance space converges to Minkowski Space.

Graphically, we plotted equation (A.4) for Sun and compare with the solution of equation (C.5) using MATLAB and the result is shown in a figure (1) given below:

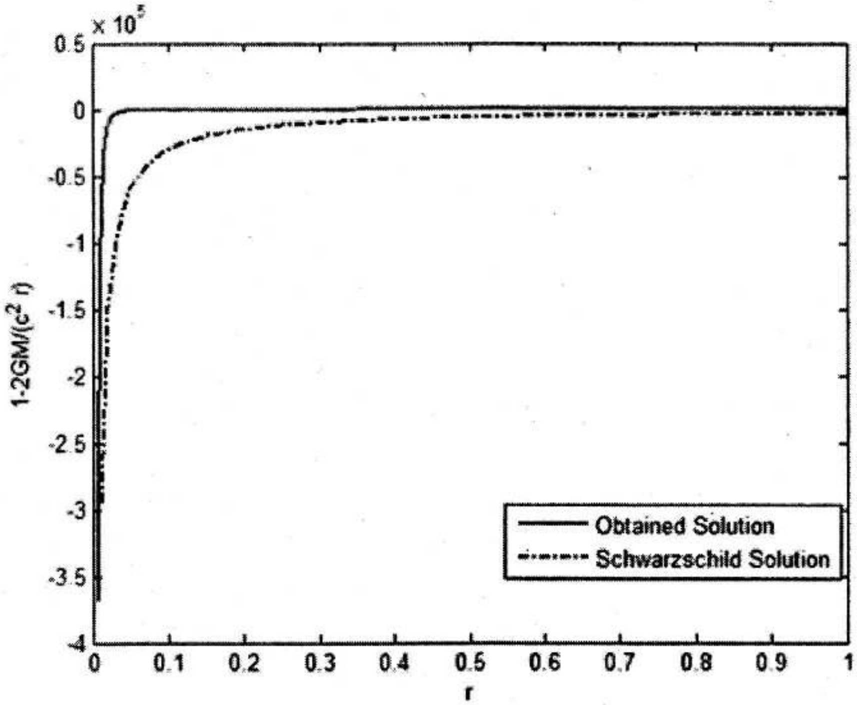

Figure (1) shows a graph between 'r' and result fromSchwarzschild solution(dotted line)and a solution obtained by equation (C.5).We took $M=1.989\times10^{30}Kg$, $G=6.67300\times10^{-11}m^3Kg^{-1}s^{-2}$ and

$$c=3\times10^8 ms^{-2}$$

II. Result

We observed that for large value of r, solution of (C.5) overlaps to Schwarzschild solution. The theory also confirms this fact and can be seen by asymptotic solution. Near $r = 0$, both solutions overlaps and goes to negative side of infinity rapidly. Between zero and large values of r, we find that solution of (C.5) has greater gradient than Schwarzschild Solution and its approach to Schwarzschild solution is sharp. It gives a possibility of existing of higher order terms in negative powers r in Schwarzschild solution. We introduce a higher term in Schwarzschild solution, *i.e.*

$$f(r)=\left(1-\frac{2GM}{c^2r}-\frac{2GM}{c^2r^2}\right) \qquad \text{.... (D.1)}$$

And compare with the solution of equation (C.5) graphically (Fig (2)). In Fig (2), curve corresponding to solution of (C.5) and plot of (D.1) match almost for all values of *r*.

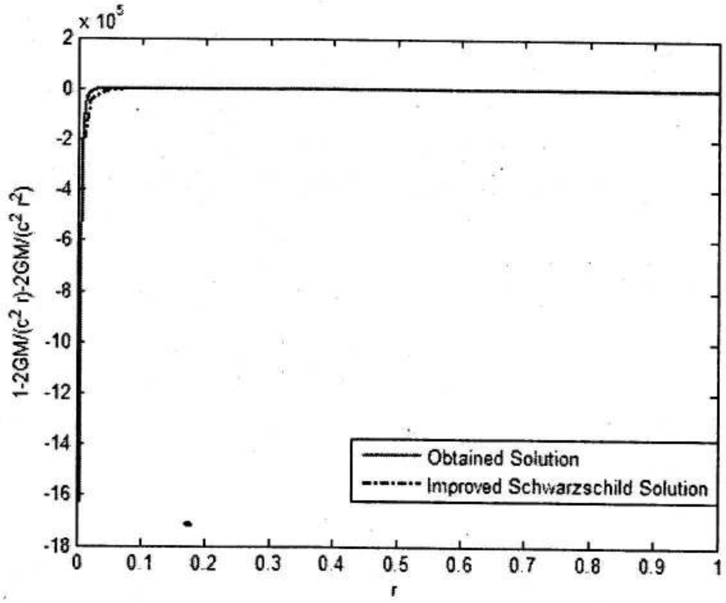

Figure (2), solid line represents the solution of (C.5) and dotted line is corresponding to equation (D.1). In this figure both curve matches almost identically. (here we took $M = 1.3487 \times 10^{28}$ Kg)

It gives a conclusion that if we consider higher orders in Einstein Field Equation then higher orders derivatives appear in the differential equation. Exterior solution of higher order field equation for spherically symmetric static field in empty space will be Schwarzschild solution with a modification that higher powers of $\frac{1}{r}$ will appear in the line element.

REFERENCES

[1] S. Capozziello, V. F. Cardone and M. Francaviglia: f(R) theories of gravity in Palatini approach matched with observations, Gen.Rel.Grav. 38 (2006), 711.

[2] S. N. Pandey: On cosmological solutions of a higher order theory of gravitation. NuovoComento, 116 B (2001), 327.

[3] S. N. Pandey: On f(R) theory of gravity, Royal Society of S. Wales, 2008.

[4] D. Lovelock, The Einstein tensor and its generalizations, J. Math. Phys. 12 (1971) 498.

[5] D. Lovelock, The four-dimensionality of space and the Einstein tensor, J. Math. Phys. 13 (1972) 874.

[6] Schwarzschild, "On the gravitational field of a point mass according to Einstein Theory",Sitzungsber. PreussAkad. Wiss., p 189, (1916a).

[7] Schwarzschild, "On the gravitational field of a sphere of incompressible fluid according to Einstein Theory",Sitzungsber. PreussAkad. Wiss., p 424, (1916b).

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Effects of Electromagnetic Field on the Dynamics of Bianchi type III Universe with Anisotropic Dark Energy

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Abstract

Spatially homogeneous Bianchi type III cosmological models are investigated in the presence of anisotropic dark energy. We examine the effects of electromagnetic field on the dynamics of the universe and anisotropic behaviorof dark energy. The law of variation of the mean Hubble parameter is used to find exact solutions of the Einstein field equations. We find that electromagnetic field promotes anisotropic behavior of dark energy which becomes isotropic for future evolution. It is concluded that the isotropic behavior of the universe model is seen even in the presence of electromagnetic field and anisotropic fluid.

Keywords: Bianchi III .Dark-Energy. Electromagnetic field

1. Introduction

The most remarkable advancement in cosmology is its observational evidence which says that our universe is in an accelerating expansion phase. Supernova Ia data [1, 2] gave the first indication of the accelerated expansion of the universe. This was confirmed by the observations of anisotropies in the cosmic microwave background (CMB) radiation as seen in the data from satellite such as WMAP [3] and large scale structure [4]. Today's one of the major concerns of cosmology is the dark energy. Recent cosmological observations [1, 2, 3, 4] suggest that our universe is (approximately) spatially flat and its cosmic inflation is due to the matter field (dark energy) having negative pressure (violating energy conditions). The composition of universe density is the following: 74% dark energy, 22% dark matter and 4% ordinary matter [5]. Though there is compelling evidence that expansion of the universe is accelerating, yet the nature of dark energy has been under consideration since the last decade [6]-[10]. Several models have been proposed for this purpose e.g. Chaplygin gas, phantoms, quintessence, cosmological constant and dark energy in brane worlds. However, none of these models can be regarded as being entirely convincing so far.

The detection of dark energy would be a new clue to an old puzzle: the gravitational effect of the zero point energies of particles and fields. The total with other energies, that are close to homogeneous and nearly independent of time, acts as dark energy. The paramount characteristic of the dark energy is a constant or slightly changing energy density as the universe expands, but we do not know the nature of dark energy very well [5] – [13]. Dark Energy has been conventionally characterized by the equation of state (EoS) parameter $\omega = p/\rho$ which is not necessarily constant. The simplest dark energy candidate is the vacuum energy $(\omega = -1)$ which is mathematically equivalent to the cosmological constant (Λ). The other conventional alternatives, which can be described by minimally coupled scalar fields, are quintessence ($\omega > -1$), phantom energy ($\omega < -1$) and quintom (that can across from phantom region to quintessence region) as evolved and have time dependent EoS parameter. Some other limits obtained from observational results coming from SN Ia data [14] and SN Ia data collaborated with CMBR anisotropy and galaxy clustering statistics [16] are $-1.67 < \omega < -0.62$ and $-1.33 < \omega < -0.79$ respectively. However, it is not at all obligatory to use a constant value of ?. Due to lack of observational evidence in making a distinction between constant and variable, usually the equation of state parameter is considered as a constant [17, 18] with phase wise value –1, 0, –1/3 and +1 for vacuum fluid, dust fluid, radiation and stiff fluid dominated universe respectively. But in general, ? is a function of time or redshift [19]-[21]. Bali and Tinkar [22] have investigated a model in the presence of bulk viscous barotropic fluid with variable G and . Unlike Robertson-Walkar metric, Bianchi type III can admit a dark energy that yields an anisotropic EoS parameter according to their characteristics. The cosmological data from the large scale structure [23] and type Ia supernova [3, 4] observations-do not rule out the possibility of anisotropic dark energy either [24, 25].

2. Metric and Field Equations

The spatially homogeneous and anisotropic Bianchi type-III model is described by the line element

$$ds^2 = -dt^2 + A^2(t)\,dx^2 + B^2(t)\,e^{2x}dy^2 + C^2(t)\,dz^2 \qquad \ldots(1)$$

where A, B and C are functions of 't' only.

The energy momentum tensor for the electromagnetic field is given as

$$T_\mu^{\nu(\text{em})} = \bar{\mu}\left[|h|^2\left(u_\mu u^\nu - \frac{1}{2}\delta_\mu^\nu\right) - h_\mu h^\nu\right] \qquad \ldots(2)$$

where u^{ν} is the four velocity vector satisfying

$$g_{\mu\nu}u^{\mu}u^{\nu}=1, \qquad \text{...(3)}$$

$\bar{\mu}$ is the magnetic permeability and h_{μ} is the four magnetic flux given by

$$h_{\mu}=\frac{\sqrt{-g}}{2\bar{\mu}}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}u^{\nu}, \qquad (\mu,\nu,\alpha,\beta=0,1,2,3) \qquad \text{...(4)}$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor, $F^{\alpha\beta}$ is the electromagnetic field tensor and $|h|^2=h_{\nu}h^{\nu}$,

we assume that magnetic field due to an electric current produced along x-axis and thus it is in yz-plane. In co-moving coordinates $u^{\nu}=(1,0,0,0)$ and hence equation (4) gives $h_1\neq 0$, $h_0=h_2=h_3=0$, using these values in equation (4), it follows that $F_{12}=F_{13}=0$, $F_{23}\neq 0$. The electric and magnetic field tensor are defined as

$$E_{\mu}=F_{\mu\nu}u^{\nu}, \quad B_{\mu}=\frac{1}{2}\varepsilon_{\mu\nu\alpha}F^{\nu\alpha} \qquad \text{...(5)}$$

Equation (5) leads to $F_{01}=F_{02}=F_{03}=0$. Thus the only non-vanishing component of electromagnetic field tensor $F_{\mu\nu}$ is F_{23}.

The Maxwell's equations

$$F_{\mu\nu;\alpha}+F_{\nu\alpha;\mu}+F_{\alpha\mu;\nu}=0, \quad F^{\mu\nu}_{;\alpha}=0 \qquad \text{...(6)}$$

are satisfied by $\quad F_{23}=K=\text{constant}. \qquad$ ...(7)

It follows from equation (4) that

$$h_1=\frac{AK}{\bar{\mu}BC}, \quad |h|^2=\frac{K^2}{\bar{\mu}^2B^2C^2} \qquad \text{...(8)}$$

From equations (2) and (8), we obtain

$$T_0^{0(em)}=\frac{K^2}{2\bar{\mu}B^2C^2}=T_1^{1(em)}=-T_2^{2(em)}=-T_3^{3(em)} \qquad \text{...(9)}$$

Thus we have

$$T_{\mu}^{\nu(em)} = diag\left[\frac{K^2}{2\bar{\mu}B^2C^2}, \frac{K^2}{2\bar{\mu}B^2C^2}, -\frac{K^2}{2\bar{\mu}B^2C^2}, -\frac{K^2}{2\bar{\mu}B^2C^2}\right] \qquad \ldots(10)$$

The energy-momentum tensor for anisotropic dark energy fluid is taken in the following form

$$T_{\mu}^{\nu} = diag\left[\rho, -p_x, -p_y, -p_z\right] \qquad \ldots(11)$$

This model of the dark energy is characterized by EoS, $p = \omega\rho$, where ω is not necessarily constant. From equation (11), we have

$$T_{\mu}^{\nu} = diag\left[1, -(\omega+\delta), -\omega, -(\omega+\gamma)\right]\rho \qquad \ldots(12)$$

where ρ is the energy density of the fluid; p_x, p_y and p_z are pressures and ω_x, ω_y and ω_z are directional EoS parameters on x, y and z axes respectively.

The deviation from isotropy is obtained by setting

$$\omega_x = \omega + \delta, \quad \omega_y = \omega, \quad \omega_z = \omega + \gamma \qquad \ldots(13)$$

where ω is the deviation free EoS parameter and δ and γ are the deviations from ω on x and z axes respectively.

The Einstein's field equations are given as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\left(T_{\mu\nu} + T_{\mu\nu}^{(em)}\right) \qquad \ldots(14)$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the energy momentum tensor for anisotropic fluid and $T_{\mu\nu}^{(em)}$ is the energy momentum tensor for the electromagnetic field. The Einstein's field equations for the anisotropic Bianchi type-III metric, lead to the following system of equations

$$\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC} - \frac{\alpha^2}{A^2} = 8\pi\rho + \frac{4\pi K^2}{\bar{\mu}B^2C^2} \qquad \ldots(15)$$

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} = -8\pi\omega\rho + \frac{4\pi K^2}{\bar{\mu}B^2C^2} \qquad \ldots(16)$$

$$\frac{A_{44}}{A}+\frac{C_{44}}{C}+\frac{A_4C_4}{AC}=-8\pi(\omega+\delta)\rho-\frac{4\pi K^2}{\bar{\mu}B^2C^2} \qquad ...(17)$$

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}=-8\pi(\omega+\gamma)\rho-\frac{4\pi K^2}{\bar{\mu}B^2C^2} \qquad ...(18)$$

$$\alpha\left(\frac{A_4}{A}-\frac{B_4}{B}\right)=0 \qquad ...(19)$$

here the sub-indices 4 in A, B, C denote differentiation with respect to 't'.

From equation (19), we have

$$A=kB \qquad ...(20)$$

where k is constant of integration.

Using equation (20) in equation (17), we get

$$\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{B_4C_4}{BC}=-8\pi(\omega+\delta)\rho-\frac{4\pi K^2}{\bar{\mu}B^2C^2} \qquad ...(21)$$

From equations (16) and (21), we get

$$\delta\rho=-\frac{K^2}{\bar{\mu}B^2C^2} \qquad ...(22)$$

Using equation (20), system of equations (15) - (18) reduces to

$$\frac{A_4^2}{A^2}+\frac{2A_4C_4}{AC}-\frac{\alpha^2}{A^2}=8\pi\rho+\frac{kK^2}{A^2C^2} \qquad ...(23)$$

$$\frac{A_{44}}{A}+\frac{C_{44}}{C}+\frac{A_4C_4}{AC}=-8\pi\omega\rho+\frac{kK^2}{A^2C^2} \qquad ...(24)$$

and

$$\frac{2A_{44}}{A}+\frac{A_4^2}{A^2}-\frac{\alpha^2}{A^2}=-8\pi(\omega+\gamma)\rho-\frac{kK^2}{A^2C^2} \qquad ...(25)$$

Equations (23)-(25) are three linearly independent equations in five unknown parameters A, C, ω, ρ and γ. To find a deterministic solution we need two extra conditions.

(i) The law of variation of Hubble's parameter that yields a constant deceleration parameter.

(ii) The shear (s) is proportional to expansion (θ) which leads to

$$B = C^n \qquad \ldots(26)$$

The motive behind assuming this condition is explained as: Referring to Thorne [27] the observations of the velocity-redshift relation for extragalactic sources suggest that the Hubble expansion of the universe is isotropic today to within 30 percent. More precisely, the red-shift studies place the limit $\frac{\sigma}{H} \leq 0.30$

The motive behind assuming this condition is explained as: Referring to Thorne [27] the observations of the velocity-redshift relation for extragalactic sources suggest that the Hubble expansion of the universe is isotropic today to within 30 percent. More precisely, the red-shift studies place the limit , where s is shear and H is Hubble constant. Collins et al. [26] have pointed out for spatially homogeneous metric; the normal congruence to the homogeneous hyper-surface satisfies the condition $\frac{\sigma}{\theta}$ is constant, where θ is expansion in the model.

The average scale factor of Bianchi type III metric is given by

$$R = \left(ABCe^{\alpha x}\right)^{\frac{1}{3}} \qquad \ldots(27)$$

We define the generalized mean Hubble's parameter as

$$H = \frac{1}{3}\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) = \frac{R_4}{R} \qquad \ldots(28)$$

Since the line element (1) is completely characterized by Hubble's parameter H. Therefore let us consider that mean Hubble parameter H is related to average scale factor R by the following relation

$$H = k_2 R^{-s} \qquad \ldots(29)$$

where $k_2 > 0$ and $s \geq 0$ are constants.

The deceleration parameter is defined as

$$q = -\frac{RR_{44}}{R_4^2} \qquad \ldots(30)$$

From equations (28) and (29), we have

$$R_4 = k_2 R^{-s+1} \qquad \text{...(31)}$$

which leads to

$$R_{44} = -k_2^2 (s-1) R^{-2s+1} \qquad \text{...(32)}$$

Using equations (31) and (32) in (30) we get

$$q = s - 1 \qquad \text{...(33)}$$

On solving equation (31), we get

$$R = (Dt + c_1)^{\frac{1}{s}} \quad \text{when } s \neq 0 \qquad \text{...(34)}$$

and $\quad R = c_2 e^{k_2 t} \qquad$ when $s = 0$ ...(35)

where c_1 and c_2 are constants of integration.

3. Solution of Field Equations

Case I. $s \neq 0$

From equations (20), (26), (27) and (34) we have

$$A = k l_1^n (c_3 t + c_1)^{\frac{3n}{s(2n+1)}} \qquad \text{...(36)}$$

$$B = l_1^n (c_3 t + c_1)^{\frac{3n}{s(2n+1')}} \qquad \text{...(37)}$$

and

$$C = l_1 k_1 (c_3 t + c_1)^{\frac{3}{s(2n+1')}} \qquad \text{...(38)}$$

where

$$l_1 = e^{-\frac{\alpha x}{2n+1}} \text{ and } k_1 = \frac{1}{k^{2n+1}}.$$

Now the metric (1) becomes

$$ds^2 = -dt^2 + k^2 l_1^{2n} (c_3 t + c_1)^{\frac{6n}{s(2n+1)}} dx^2 + l_1^{2n} (c_3 t + c_1)^{\frac{6n}{s(2n+1')}} dy^2 + l_1^2 k_1^2 (c_3 t + c_1)^{\frac{6}{s(2n+1')}} dz^2 \qquad \text{...(39)}$$

After taking suitable transformations of coordinates, we have

$$dS^2 = -\frac{dT^2}{c_3^2} + T^{\frac{6n}{s(2n+1)}} dX^2 + T^{\frac{6n}{s(2n+1')}} dY^2 + T^{\frac{6}{s(2n+1')}} dZ^2 \qquad \text{...(40)}$$

4. Some Geometrical and Physical Properties

Scalar expansion (θ) is given as

$$\theta = \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} = \frac{(n+2)l_2}{2T} \quad \text{...(41)}$$

Directional Components of Hubble Parameter are obtained as

$$H_1 = \frac{A_4}{A} = \frac{l_2}{T} \quad \text{...(42)}$$

$$H_2 = \frac{B_4}{B} = \frac{l_2}{T} \quad \text{...(43)}$$

$$H_3 = \frac{C_4}{C} = \frac{l_2}{nT} \quad \text{...(44)}$$

Hence the Hubble parameter (H) is given as

$$H = \frac{l_2(2n+1)}{3nT} \quad \text{...(45)}$$

Components of shear tensor (σ) are given as

$$\sigma_1^1 = \frac{(n-1)l_2}{3nT} \quad \text{...(46)}$$

$$\sigma_2^2 = \frac{(n-1)l_2}{3nT} \quad \text{...(47)}$$

$$\sigma_3^3 = \frac{2(n-1)l_2}{3nT} \quad \text{...(48)}$$

and $$\sigma_4^4 = 0 \quad \text{...(49)}$$

The shear tensor (σ) is given as

$$\sigma^2 = \frac{1}{2}\left[\left(\sigma_1^1\right)^2 + \left(\sigma_2^2\right)^2 + \left(\sigma_3^3\right)^2 + \left(\sigma_4^4\right)^2\right] \quad \text{...(50)}$$

For our model σ is calculated as $$\sigma = \frac{1}{\sqrt{3}}\frac{(n-1)l_2}{nT} \quad \text{...(51)}$$

Hence the ratio $\frac{\sigma}{\theta}$ is given as $$\frac{\sigma}{\theta}=\frac{2(n-1)}{\sqrt{3}n(n+2)} \quad ...(52)$$

Spatial volume (V) is evaluated as $$V=kl_1^{n+1/3}T^{n+1/s(2n+1)} \quad ...(53)$$

The isotropy parameter $\left(\overline{A}\right)$ is given as $$\overline{A}=\frac{3}{(n+2)^2} \quad ...(54)$$

which is constant.

From equation (23), matter density (ρ) is calculated as

$$\rho=\frac{1}{8\pi}\left[\frac{(n+2)l_2^2}{nT^2}-\frac{\alpha^2}{k^2l_1^{2n}T^{6n/s(2n+1)}}-\frac{4\pi K^2}{\overline{\mu}\left(l_1^nT^{3n/s(2n+1)}\right)^{n+2/n}}\right] \quad ...(55)$$

From equation (17), we have

$$\delta=-\frac{K^2}{\overline{\mu}B^2C^2\rho} \quad ...(56)$$

which gives $$\delta=-\frac{8\pi K^2}{\overline{\mu}\left(l_1^{n+2}T^{3(n+2)/s(2n+1)}\right)}\left[\frac{(n+2)l_2^2}{nT^2}-\frac{\alpha^2}{k^2l_1^{2n}T^{6n/s(2n+1)}}-\frac{4\pi K^2}{\overline{\mu}\left(l_1^{n+2}T^{3(n+2)/s(2n+1)}\right)}\right]^{-1} \quad ...(57)$$

From equation (16), we have

$$\omega=\frac{1}{8\pi\rho}\left[\frac{4\pi K^2}{\overline{\mu}\left(l_1^{n+2}T^{n+2/n}\right)}-\frac{l_2\left(l_2-c_3\right)}{T^2}-\frac{l_2\left(\frac{l_2}{n}-c_3\right)}{nT^2}-\frac{l^2{}_2}{nT^2}\right] \quad ...(58)$$

Using equation (18), we get

$$\gamma = -\frac{1}{8\pi\rho}\left[\frac{4\pi K^2}{\bar{\mu}\left(l_1^{n+2}T^{n+2/n}\right)} + \frac{2l_2(l_2 - c_3)}{T^2} - \frac{\alpha^2}{k^2 l_1^{2n} T^{6n/s(2n+1)}} + \frac{l^2_2}{nT^2} - 8\pi\omega\right] \quad \ldots(59)$$

where ρ and ω are given by equation (55) and (58).

In the absence of magnetic field ρ, δ, ω and γ are given as

$$\rho = \frac{1}{8\pi}\left[\frac{(n+2)l_2^2}{nT^2} - \frac{\alpha^2}{k^2 l_1^{2n} T^{6n/s(2n+1)}}\right] \quad \ldots(60)$$

$$\delta = 0 \quad \ldots(61)$$

$$\omega = \frac{1}{8\pi\rho}\left[-\frac{l_2(l_2 - D)}{T^2} - \frac{l_2\left(\frac{l_2}{n} - D\right)}{nT^2} - \frac{l^2_2}{nT^2}\right]c \quad \ldots(62)$$

and

$$\gamma = -\frac{1}{8\pi\rho}\left[\frac{2l_2(l_2 - D)}{T^2} - \frac{\alpha^2}{k^2 l_1^{2n} T^{6n/s(2n+1)}} + \frac{l^2_2}{nT^2} - 8\pi\omega\right] \quad \ldots(63)$$

5. Some Physical Aspects of the Model

We find that the directional Hubble parameters as well as the mean Hubble parameter are dynamical. Also the directional Hubble parameters become zero as $T \to \infty$. These coincide with mean Hubble parameter when $T \to \infty$. The volume V of the universe is zero at $T = 0$ for $n > -1$. The scalar expansion (θ) becomes large for small values of T i.e. $\theta \to \infty$ as $T \to 0$. Thus model starts with a big-bang and expansion decreases as T increases. Here A(t), B(t) and C(t) are all zero at T = 0 so model has POINT TYPE singularity for $s > 0, n > 0$. The model has CIGAR type singularity for $s > 0$ and $n < -1/2$. Components of shear tensor (σ) diverge for large values of T and becomes zero as $T \to \infty$. Since $\lim_{n\to\infty} \frac{\sigma}{\theta} \to 0$, hence the model isotropizes for large values of n. The anisotropy parameter is finite for $n \neq -2$. The quantity ρ, ω and δ are dynamical and all diverge as $T \to \infty$ for $n > 0$, $s > 0$. The deviation free EoS parameter of the DE may begin in the phantom $(\omega < -1)$ or quintessence region $(\omega > -1)$ but $\omega \to -1$ for later times of the universe. One can observe that ρ increases when ω is in phantom region and attains a constant value as $\omega \to -1$. In the absence of magnetic field δ becomes zero and ρ and ω becomes zero as $T \to \infty$.

Case II. $s = 0$

Using equations (20), (26), (27) and (35), we have

$$A = kl_3^n e^{3k_2nt/2n+1} \quad \ldots(64)$$

$$B = l_3^n e^{3k_2nt/2n+1} \quad \ldots(65)$$

and

$$C = l_3 e^{3k_2t/2n+1} \quad \ldots(66)$$

where

$$l_3 = \left[\frac{c_2^3 e^{-\alpha x}}{k}\right]^{1/2n+1}$$

Hence the metric (1) is given by

$$ds^2 = -dt^2 + k^2 l_3^{2n} e^{6k_2nt/2n+1} dx^2 + l_3^{2n} e^{6k_2nt/2n+1} dy^2 + l_3^{2n} e^{6k_2t/2n+1} dz^2 \quad \ldots(67)$$

After suitable transformations of coordinates, we have

$$dS^2 = -\frac{dT^2}{36k_2^2} + k^2 l_3^{2n} e^{nT/2n+1} dX^2 + l_3^{2n} e^{nT/2n+1} dY^2 + l_3^{2n} e^{T/2n+1} dZ^2 \quad \ldots(68)$$

6. Some Physical Properties

For the model (66) we obtain the value of scalar expansion (θ) as

$$\theta = 3k_2 \quad \ldots(69)$$

Directional Components of Hubble Parameter (H) are obtained as

$$H_1 = \frac{3k_2 n}{2n+1} \qquad \text{...(70)}$$

$$H_2 = \frac{3k_2 n}{2n+1} \qquad \text{...(71)}$$

$$H_3 = \frac{3k_2}{2n+1} \qquad \text{...(72)}$$

and the Hubble parameter (H) is given as

$$H = k_2 \qquad \text{...(73)}$$

Components of shear tensor (σ) are given as

$$\sigma_1^1 = \frac{(n-1)k_2}{2n+1} \qquad \text{...(74)}$$

$$\sigma_2^2 = \frac{(n-1)k_2}{2n+1} \qquad \text{...(75)}$$

$$\sigma_3^3 = \frac{2(n-1)k_2}{2n+1} \qquad \text{...(76)}$$

and

$$\sigma_4^4 = 0 \qquad \text{...(77)}$$

Hence the shear tensor (σ) is calculated as

$$\sigma = \frac{\sqrt{3}(n-1)k_2}{2n+1} \qquad \text{...(78)}$$

The ratio $\frac{\sigma}{\theta}$ is given as

$$\frac{\sigma}{\theta} = \frac{(n-1)}{\sqrt{3}(2n+1)} \qquad \text{... (79)}$$

Spatial volume (V) is given as

$$V = k l_3^{n+1/3} e^{T(n+1)/6(2n+1)} \qquad \text{...(80)}$$

Matter density (ρ) is calculated as

$$\rho=\frac{1}{8\pi}\left[\frac{9k_2^2n(n+2)l_2^2}{(2n+1)^2}-\frac{\alpha^2}{k^2l_3^{2n}e^{nT/(2n+1)}}-\frac{4\pi K^2}{\bar{\mu}\left(l_3^n e^{nT/2(2n+1)}\right)^{2+2/n}}\right] \quad ...(81)$$

$$\delta=-\frac{8\pi K^2}{\bar{\mu}\left(l_3^n e^{nT/2(2n+1)}\right)^{2+2/n}}\left[\frac{9k_2^2n(n+2)l_2^2}{(2n+1)^2}-\frac{\alpha^2}{k^2l_3^{2n}e^{nT/(2n+1)}}-\frac{4\pi K^2}{\bar{\mu}\left(l_3^n e^{nT/2(2n+1)}\right)^{2+2/n}}\right]^{-1} \quad ...(82)$$

The EoS parameter is evaluated as

$$\omega=\frac{1}{8\pi\rho}\left[\frac{4\pi K^2}{\bar{\mu}\left(l_3^n e^{3k_2nt/(2n+1)}\right)^{2+2/n}}-\frac{9k_2^2}{(2n+1)^2}\left(n^2+n+1\right)\right] \quad ...(83)$$

Using equation (18), we get

$$\gamma=\frac{1}{8\pi\rho}\left[\frac{4\pi K^2}{\bar{\mu}\left(l_3^n e^{nT/2(2n+1)}\right)^{2+2/n}}-\frac{27k_2^2}{(2n+1)^2}-\frac{\alpha^2}{k^2l_3^{2n}e^{nT/(2n+1)}}-8\pi\omega\right] \quad ...(84)$$

where ρ and ω are given by equation (81) and (83).

In the absence of magnetic field the physical properties are given as

$$\rho=\frac{1}{8\pi}\left[\frac{9k_2^2n(n+2)l_2^2}{(2n+1)^2}-\frac{\alpha^2}{k^2l_3^{2n}e^{nT/(2n+1)}}\right] \quad ...(85)$$

$$\delta=0 \quad ...(86)$$

$$\omega=\frac{1}{8\pi\rho}\left[-\frac{9k_2^2}{(2n+1)^2}\left(n^2+n+1\right)\right] \quad ...(87)$$

$$\gamma=\frac{1}{8\pi\rho}\left[-\frac{27k_2^2}{(2n+1)^2}-\frac{\alpha^2}{k^2l_3^{2n}e^{nT/(2n+1)}}-8\pi\omega\right] \quad ...(87)$$

7. Some Physical Aspects of the Model

In this case scalar expansion (θ) is finite and becomes zero at $k_2 = 0$. The components of Hubble parameter are dependent upon n and hence are fixed in terms of T. Similarly Hubble parameter is constant. Shear tensor (σ_i^j) is also dependent on n. Also $\lim_{n\to\infty}\frac{\sigma}{\theta} \neq 0$, so model does not approach isotropy for large value of n. Spatial volume is constant at T = 0 and as $T \to \infty$ volume increases given n > -1. Matter density (ρ) becomes finite as $T \to \infty$, then $\delta \to 0$. Eos parameter ω depends on ρ and becomes finite as $T \to \infty$. In the absence of magnetic field $\delta = 0$ further ρ and ω are finite for $T \to \infty$.

8. Summary and Conclusion

We have obtained two exact solutions of the dynamical equations for the spatially homogeneous and isotropic Bianchi type III model with magnetic field and anisotropic DE. The dark energy component is dynamical which yield anisotropic pressure. Assuming the law of variation of the mean Hubble parameter, the cosmological models are given for $s \neq 0$ and $s = 0$. The physical and geometrical properties of the models are discussed. We have found the explicit form of scale factors and have explained the nature of singularities.

The model represents uniform expansion for $s = 0$ which decreases for later times, while for $s \neq 0$, scalar expansion (θ) is constant. Our results have shown that the fluid is anisotropic which yields anisotropic EoS parameter with the electromagnetic field. It is also shown that ω is in the phantom region which tends to a constant value -1 for later times of the universe.

9. References

[1] Dunlop, J., et al.: Bianchi type III dark energy cosmological models; Nature 381(1996) 581.

[2] Spinard, H., et al.: LBDS 53W091: An Old, Red Galaxy at z = 1.552; Astrophys. J. 484(1997)581.

[3] Riess, A. G., et al.: Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant; Astron. J. 116, 1009 (1998).

[4] Permutter, S. J., et al.: Discovery of a supernova explosion at half the age of the Universe; Astrophys. J. 483 (1997) 565.

[5] Sahni, V., Starobinsky, A. A.: Statefinder—A new geometrical diagnostic of dark energy; Int. J. Mod. Phys. D 9 (2000)373.

[6] Peebles, P. J. E., Ratra, B.: The cosmological constant and dark energy; Rev. Mod. Phys. 75 (2003)559.

[7] Alam, U., et al.: Exploring the properties of dark energy using type-Ia supernovae and other datasets; J. Cosmol. Astropart. Phys. 0406 (2004) 008.

[8] Sahni, V., Starobinsky, A. A.: Reconstructing Dark Energy; Int. J. Mod. Phys. D 15 (2006) 2105.

[9] Copeland, E. J., et al.: Dynamics of Dark Energy; Int. J. Mod. Phys. D, 15 (2006) 1753.

[10] Padmanabhan, T.: Dark Energy: the cosmological challenge of the millennium; Curr. Sci. 88 (2005)1057.

[11] Carroll, S. M., Hoffmann, M.: Can the dark energy equation-of-state parameter w be less than-1? Phys. Rev. D 68 (2003) 023509.

[12] Bartelmann, M.: The dark Universe; Rev. Mod. Phys. 82 (2010).

[13] Knop, R. A.,et al.: The Astrophysical Journal Volume 598 Number 1;Astrophys. J. 598 (2003) 102.

[14] Tegmark, M., et al.: The Astrophysical Journal Volume 606 Number 2;Astrophys. J., 606, (2004)702.

[15] Kujat, J., et al.: Prospects for Determining the Equation of State of the Dark Energy: What Can Be Learned from Multiple Observables? ; Astrophys. J. 572 (2002) 1.

[16] Jimenez, R.: The value of the equation of state of dark energy; Proceedings of the 2nd CMBNet workshop on Science and Parameter Extraction, New Astron. Rev. 47 (2003) 761.

[17] Das, A., et al.: Cosmology with decaying tachyon Matter; Phys. Rev. D 72 (2005) 043528.

[18] Ratra, B., Peebles, P. J. E.: Cosmological consequences of a rolling homogeneous scalar field; Phys. Rev. D 37 (1988) 3406.

[19] Turner, M. S., White, M.: CDM models with a smooth component; Phys. Rev. D 56 (1997) R4439.

[20] Caldwell, R., et al.: Cosmological Imprint of an Energy Component with General Equation of State ; Phys. Rev. Lett. 80 (1998) 1582.

[21] Liddle, A. R., Scherer, R. J.: Classification of scalar field potentials with cosmological scaling solutions ; Phys. Rev. D 59 (1999) 023509.

[22] Bali, R., Tinkar, S.: Bianchi Type III Bulk Viscous Barotropic Fluid Cosmological Models with Variable G and Λ ;Chin. Phys. Lett. 26 (2009)029802.

[23] Tegmark, M., et al.: Cosmological parameters from SDSS and WMAP ; Phys. Rev. D 69, 103501 (2004).

[24] Bali, R., Tinkar, S.: Bianchi Type III Bulk Viscous Barotropic Fluid Cosmological Models with Variable G and Λ ; Chin. Phys. Lett. 26(2009) 029802.

[25] Koivisto, T., Mota, D. F.: Dark energy anisotropic stress and large scale structure formation; Phys. Rev. D 73 (2006) 083502.

[26] C.B. Collins, Glass, E.N., Wilkinson, D.A. : Exact spatially homogeneous cosmologies;General Relativity and Gravitation, Vol.12, Issue 10 (1980) 805-823.

[27] Thorne, K.S.: Primordial Element Formation, Primordial Magnetic Fields, and the Isotropy of the Universe; The Astrophysical Journal 148 (1967)51.

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On a new class of integrals involving generalized hypergeometric and the Wright's generalized hypergeometric function

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Abstract

In this paper, we aim at establishing two generalized integral formulae involving generalized hypergeometric and the Wright's generalized hypergeometric function .Some interesting Corollaries of our main results are also considered. The results are derived with the help of an interesting integral due to Lavoie and Trottier.

Keywords : Beta andGamma function, Generalized hypergeometric function ${}_pF_q$, Generalized (Wright) hypergeometric functions ${}_p\Psi_q$, Binomial theorem and Lavoie-Trottier integral formula.

2000 Mathematics Subject Classification. 33C05, 33C20, 33C70.

1. Introduction and Preliminaries

In recent years, many integral formulae involving a variety of special functions have been developed by many authors [1,4,5,8] for a very recent work, see also [6].Those integrals involving generalized hypergeometric functions are of great importance since they are used in applied physics and in many branches of engineering.

In present paper, we established two generalized integral formulae involving generalized hypergeometric and the Wright's generalized hypergeometric function. For this purpose we begin by recalling some known functions and earlier results.

The Wright's generalized hypergeometric function due to Srivastava and Manocha [7] is defined by

$$ {}_p\Psi_q = {}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1),\dots,(\alpha_p, A_p);z \\ (\beta_1, B_1),\dots,(\beta_q, B_q);\end{matrix}\right] = \sum_{k=0}^{\infty} \frac{\Pi_{j=1}^{p}\Gamma(\alpha_j + A_j k)\, z^k}{\Pi_{j=1}^{q}\Gamma(\beta_j + B_j k)\, k!} \qquad \dots(1.1)$$

Where the coefficients $A_1,\dots,A_q$ and $B_1,\dots,B_q$ are real positive numbers such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0$$

The function ${}_p\Psi_q[z]$ reduces to the generalized hypergeometric function ${}_pF_q$ when

$$A_i = 1(i = 1,\dots,p) \text{ and } B_j = 1(j = 1,\dots,q) \text{ in } (1.1)$$

Generalized hypergeometric series defined by Srivastava et al.[2] is

$$ {}_pF_q\left[\begin{matrix}a_1,\dots,a_p;z \\ \beta_1,\dots,\beta_q;\end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n,\dots,(\alpha_p)_n;z^n}{(\beta_1)_n,\dots,(\beta_q)_n\, n!} \qquad \dots(1.2)$$

$$= {}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z)$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in C$) by [2]

$$(\lambda)_n = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1)\dots(\lambda + n - 1) & (n \in N := \{1,2,3,\dots\}) \end{cases}$$

$$= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}\left(\lambda \in C \setminus Z_0^-\right) \qquad \dots(1.3)$$

And Z_0^- denotes the set of non positive integers.

We also recall Lavoie-Trottier integral formula [3] for our present study

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},(\Re(\alpha)>0 \text{ and } \Re(\beta)>0) \qquad ...(1.4)$$

Binomial theorem

$$(1-x)^{-\alpha}=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}x^n \qquad ...(1.5)$$

Relation between Gamma and Beta function

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad ...(1.6)$$

2. Main Results

In this section, we established twogeneralized integral formulaeinvolving generalized hypergeometric and the Wright's generalized hypergeometric function which are expressed in terms of the generalized Wright hypergeometric function , the Wright's generalized hypergeometric function and in Beta function.

Theorem 1. The following integral formula holds true: for $\rho, j\in\mathbb{C}$ and $\Re(\rho)>0, \Re(\rho+j)>0, x>0$,

$$\int_0^1 x^{\rho+j-1}(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1} {}_p\Psi_q\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2(\rho+j)}\Gamma(\rho+j),\ {}_{p+1}\Psi_{q+1}\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p),(\rho,1); \\ (\beta_1,B_1),\ldots,(\beta_q,B_q),(2\rho+j,1);\end{matrix}\ y\right] \qquad ...(2.1)$$

Proof: By applying (1.1) in the integrand of (2.1) andinterchanging the order of integral sign and summation which is verified by uniform convergence of the involved series under the given condition, we get

$$\int_0^1 x^{\rho+j-1}(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1}{}_p\Psi_q\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\sum_{k=0}^{\infty}\frac{\Pi_{j=1}^{p}\Gamma\left(\alpha_j+A_jk\right)y^k}{\Pi_{j=1}^{q}\Gamma\left(\beta_j+B_jk\right)k!}\int_0^1 x^{\rho+j-1}(1-x)^{2(\rho+k)-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{(\rho+k)-1}dx$$

Applying integral formula (1.4) and then using (1.3) we get

$$\left(\frac{2}{3}\right)^{2(\rho+j)}\Gamma(\rho+j)\cdot\sum_{k=0}^{\infty}\frac{\Pi_{j=1}^{p}\Gamma\left(\alpha_j+A_jk\right)y^k}{\Pi_{j=1}^{q}\Gamma\left(\beta_j+B_jk\right)k!}\frac{\Gamma(\rho+k)}{\Gamma(2\rho+j+k)}$$

which, upon using (1.1), yields (2.1),This completes the proof of Theorem 2.1.

Theorem 2.2. The following integral formula holds true: for $\alpha,\beta\in\mathbb{C}$ and $\Re(\beta)>0,\Re(\alpha+n)>0,\Re(\alpha)>0,x>0,$

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2(\alpha+n)-1}\left(1-\frac{x}{4}\right)^{\beta-1}(1-xy)^{-y}dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}B(\alpha,\beta)\cdot{}_2F_1\left[\begin{matrix}\alpha,\gamma; & \frac{4y}{9}\\ \alpha+\beta; & \end{matrix}\right] \qquad ...(2.2)$$

Proof: By applying $(1-xy)^{-\gamma}$ in the integrand of (2.2) andinterchanging the order of integral sign and summation which is verified by uniform convergence of the involved series under the given condition, we get

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2(\alpha+n)-1}\left(1-\frac{x}{4}\right)^{\beta-1}(1-xy)^{-y}dx$$

$$\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!}y^n\int_0^1 x^{\alpha+n-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2(\alpha+n)-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx$$

Applying integral formula (1.4) and then using (1.3) we get

$$\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n}{(\alpha+\beta)_n n!}\left(\frac{4y}{9}\right)^n$$

which, upon using (1.2) and (1.6),yields (2.2),This completes the proof of Theorem 2.2.

Corollary 2.1. Let the condition of Theorem 2.1 be satisfied ,then the following integral formula holds true:for $A_i = 1(i = 1,\ldots,p)$ and $B_j = 1(j = 1,\ldots,q)$

$$\int_0^1 x^{\rho+j-1}(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1} {}_pF_q\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)\Gamma(\rho)}{\Gamma(2\rho+j)}\cdot {}_{p+1}F_{q+1}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,\rho; \\ \beta_1,\ldots,\beta_q,2\rho+j;\end{matrix}\ y\right] \quad \ldots(2.3)$$

Corollary 2.1. Let the condition of Theorem 2.2 be satisfied ,then the following integral formula holds true.

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2(\alpha+n)-1}\left(1-\frac{x}{4}\right)^{\beta-1}(1-xy)^{-y}\,dx$$

$$=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\beta)}{\Gamma(\gamma)},\ {}_2\Psi_1\left[\begin{matrix}(\alpha,1),(\gamma,1); \\ (\alpha+\beta,1);\end{matrix}\ \frac{4y}{9}\right] \quad \ldots(2.4)$$

Proof. Same in similar manner of the proof of Theorem 2.1 andTheorem 2.2 we can prove Corollary 2.1 and 2.2 by using definition given in (1.1) and in (1.2). Therefore, we omit the details of the proof of Corollary 2.1 and 2.2.

3. Concluding Remark

Here we briefly consider another variation of the results derived in the preceding sections. generalized hypergeometric function are important special functions that arise widely in science and engineering. Certain special cases of integrals involving the generalized hypergeometric function have been investigated in the literature by a number of authors with different arguments . Therefore, the results presented in this paper are easily converted in terms of a similar type of new interesting integrals with different arguments after some suitable parametric replacements.

References

[1] F. W. L. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, NIST Handbook of Mathematical Functions, Cambridge University Press, (2010).

[2] H. M. Srivastava, J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, (2012).

[3] J. L. Lavoie, G. Trottier, On the sum of certain Appell's series, Ganita, 20 (1) (1969) 31-32.

[4] J. Choi, A. Hasanov, H. M. Srivastava, M. Turaev, Integral representations for Srivastava's triple hypergeometric functions, Taiwanese J. Math, 15 (2011) 2751-2762.

[5] M. A. Rakha, A. K. Rathie, M. P. Chaudhary, S. ALI, On A New Class of integrals involving hypergeometric function, Journal of Inequalities and Special Functions, 3 (1) (2012) 10-27.

[6] P. Agarwal, S. Jain, S. Agarwal, M. Nagpal, On a new class of integrals involving Bessel functions of thefirst kind,ISPACS,vol 2014,2014,1-7

[7] Srivastava, H.M., and H.L.Manocha (1984). A Treatise on Generating Functions. Halsted Press (Ellis Horwood,Limited, Chichester).

[8] Y. A. Brychkov, Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas, CRC

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On applications of q-Laplace transforms to the basic analogue of the generalised hypergeometric functions

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Abstract

The q-Laplace transforms of the basic analogue of H-function of two variables have been evaluated in the present paper. Applications involving the basic analogues of Fox's H-function have also been established.

Keywords : q-Laplace transforms, q-analogues of Fox's H-function of two variables.

Mathematics Subject Classification: *33D15, 44A10, 44A20*

1. Introduction

Few years ago, Yadav and Purohit [9]-[11] evaluated the q-Laplace images of a number of polynomials and generalized basic hypergeometric functions of one and more variables, including the basic analogue of Fox's H-function(due to Saxena, Modi and Kalla [7]), and Purohit, Yadav and Vyas [5]obtained q-Laplace transform of a basic analogue of the I-function (due to Saxena and Kumar [6]).Abdi [1] has investigated the fundamental properties of the q-Laplacetransforms and established several theorems related to q-images of basicfunctions.

The main object for this paper is to evaluate the q-Laplace transform of basic analogues of the H function of two variables. Interesting special cases of the main result are also discussed.

Hahn [3] defined the q-analogues of the well-known classical Laplace transforms

$$\varphi(s) = \int_0^\infty e^{-st} f(t)\,dt\ ,\ s > 0 \qquad \ldots(1.1)$$

by means of the following q-integrals

$${}_qL_s\{f(t)\} = \frac{1}{(1-q)} \int_0^{s^{-1}} E_q(qst) f(t) d(t;q) \qquad \ldots(1.2)$$

and

$${}_ql_s\{f(x)\} = \frac{1}{(1-q)} \int_0^\infty e_q(-st) f(t) d(t;q)\ , \qquad \ldots(1.3)$$

where $\mathrm{Re}(s) > 0$.

The basic integration is defined as(cf. Gasper and Rahman [2])

$$\int_0^x f(t)\,d(t;q) = x\,(1-q) \sum_{k=0}^\infty q^k f(xq^k) \qquad \ldots(1.4)$$

By virtue of the result (1.4), operator (1.2) can be expressed as

$$\varphi(s) \equiv {}_qL_s\{f(t)\} = \frac{(q;q)_\infty}{s} \sum_{j=0}^\infty \frac{q^j f(s^{-1}q^j)}{(q;q)_j} \qquad \ldots(1.5)$$

The correspondence defined in (1.2) and (1.5) shall be denoted symbolically by

$$f(t) \supset_q \varphi(s)\ ,$$

where the function $f(t)$ is called as original function, while the function $\varphi(s)$ is termed as the q-Laplace transform or q-image of the original function $f(t)$.

Particular for $f(t) = t^{\mu-1}$ (cf. Purohit and Kalla [4])

$${}_qL_s\{t^{\mu-1}\} = \frac{\Gamma_q(\mu)}{s^\mu}(1-q)^{\mu-1}, \qquad \mathrm{Re}(s) > 0 \qquad \ldots(1.6)$$

For real or complex α , and $0<|q|<1$, the q-shifted factorial is defined by

$$(q^\alpha;q)_n = \begin{cases} 1, & \text{if } n = 0 \\ (1-q^\alpha)(1-q^{\alpha+1})\ldots(1-q^{\alpha+n-1}), & \text{if } n \in N, \end{cases} \qquad \ldots(1.7)$$

also
$$(x-y)_\nu = x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right], \quad \ldots(1.8)$$

and
$$\Gamma_q(\alpha)=\frac{(q;q)_\infty (1-q)^{1-\alpha}}{(q^\alpha;q)_\infty}=\frac{(1-q)_{\alpha-1}}{(1-q)^{\alpha-1}}=\frac{(q;q)_{\alpha-1}}{(1-q)^{\alpha-1}}, \quad \ldots(1.9)$$

where $\alpha \neq 0,-1,-2,\ldots$

Following Saxena, Modi and Kalla [8], the basic analogue of the H-function of two variable is defined as:

$$H^{A,(M_1:N_1),(M_2:N_2)}_{C,D(P_1:Q_1),(P_2:Q_2)}\left[\begin{matrix} z_1 \\ z_2 \end{matrix} \; ;q \left| \begin{matrix} (c;\gamma,\gamma') \\ (a,\alpha);(a',\alpha') \\ (d;\delta,\delta') \\ (b,\beta);(b',\beta') \end{matrix} \right.\right]=$$

$$=\frac{1}{(2\pi i)^2}\int_{C_1^*}\int_{C_2^*}\frac{X_1(s,q)\,X_2(t,q)\,X_3(s,t,q)(z_1)^s\,(z_2)^t}{G(q^{1-s})G(q^{1-t})\sin\pi s \sin\pi t}\,ds\,dt, \quad \ldots(1.10)$$

where $|q|<1,\ \log q=-\omega=-(\omega_1+\omega_2)$, ω_1 *and* ω_2 *are real constant*

$$G(q^a)=\left\{\prod_{n=0}^{\infty}(1-q^{a+n})\right\}^{-1}=\frac{1}{(q^a;q)_\infty}. \quad \ldots(1.11)$$

Also
$$X_1(s,q)=\frac{\prod_{j=1}^{M_1}G(q^{b_j-\beta_j s})\prod_{j=1}^{N_1}G(q^{1-a_j-\alpha_j s})}{\prod_{j=M_1+1}^{Q_1}G(q^{1-b_j-\beta_j s})\prod_{j=N+1}^{P_1}G(q^{a_j-\alpha_j s})}, \quad \ldots(1.12)$$

$$X_2(t,q)=\frac{\prod_{j=1}^{M_2}G(q^{b_j'-\beta_j' t})\prod_{j=1}^{N_2}G(q^{1-a_j'-\alpha_j' t})}{\prod_{j=M_2+1}^{Q_1}G(q^{1-b_j'-\beta_j' t})\prod_{j=N_2+1}^{P_1}G(q^{a_j'-\alpha_j' s})}, \quad \ldots(1.13)$$

and $$X_3(s,t;q) = \frac{\prod_{j=1}^{A} G(q^{1-c_j+\gamma_j s+\gamma_j' t})}{\prod_{j=A+1}^{C} G(q^{c_j-\gamma_j s-\gamma_j' t}) \prod_{j=1}^{D} G(q^{1-d_j+\delta_j s+\delta_j' t})}. \quad \dots(1.14)$$

The coefficients γ_j and γ_j', $(1 \le j \le C)$; δ_j and δ_j' $(1 \le j \le D)$; α_j $(1 \le j \le P_1)$, $\alpha_j'(1 \le j \le P_2)$, β_j $(1 \le j \le Q_1)$, $\beta_j'(1 \le j \le Q_2)$, are positive number $A, C, D, P_1, P_2, Q_1, Q_2$, M_1, M_2, N_1 *and* N_2 are non negative integers, satisfying the following inequalities

$$0 \le A \le C,\ 0 \le M_i \le Q_i,\ 0 \le N_i \le P_i,\ D > 0; \forall i \in \{1,2\}.$$

The contour C_1^* *and* C_2^* are line parallel to $R_i(w_i s) = 0 (i = 1,2)$, with indentations, if necessary, in such a manner that all the poles of $G(q^{b_j - \beta_j s})$ for $j \in \{1,...,M_1\}$ and $G(q^{b_j - \beta_j s})$ for $j \in \{1,...,M_2\}$, lies to the right and those of $G(q^{1-c_j-\gamma_j s+\gamma_j' t})$ for $j \in \{1,\dots,A\}$, $G(q^{1-a_j+\alpha_j s})$ for $j \in \{1,\dots N_1\}$ and $G(q^{1-a_j'+\alpha_j' s})$ for $j \in \{1,\dots N_2\}$ lies to the left of the contours . An empty product is interpreted as unity. The poles of the integrand are assumed to be simple.

Further, if we set $\alpha = \alpha' = \beta = \beta' = \gamma = \gamma' = \delta = \delta' = 1$ then the definition (1.10) reduces to basic analogue of Meijer's G-function of two variable as under;

$$H_{C,D(P_1:Q_1),(P_2:Q_2)}^{A,(M_1:N_1),(M_2:N_2)}\left[\begin{matrix} z_1 \\ z_2 \end{matrix} ;q \left| \begin{matrix} (c;1,1) \\ (a,1);(a',1) \\ (d;1,1) \\ (b,1);(b',1) \end{matrix} \right.\right] = G_{C,D(P_1:Q_1),(P_2:Q_2)}^{A,(M_1:N_1),(M_2:N_2)}\left[\begin{matrix} z_1 \\ z_2 \end{matrix} ;q \left| \begin{matrix} c_1,\dots,c_C \\ a,\dots a_{P_1};a_1',\dots,a_{P_2}' \\ d_1,\dots,d_D \\ b_1,\dots b_{Q_1};b_1',\dots,b_{Q_2}' \end{matrix} \right.\right]$$

$$= \frac{1}{(2\pi i)^2} \int_{C_1^*} \int_{C_2^*} \frac{Y_1(s,q) Y_2(t,q) Y_3(s,t,q)\, (z_1)^s\, (z_2)^t}{G(q^{1-s})\, G(q^{1-t}) \sin \pi s \sin \pi t}\, ds\, dt, \quad \dots(1.15)$$

where

$$Y_1(s,q)=\frac{\prod_{j}^{M_1} G(q^{b_j-s})\prod_{j=1}^{N_1} G(q^{1-a_j+s})}{\prod_{j=M_1+1}^{Q_1} G(q^{1-b_j+s})\ \prod_{j=N+1}^{P_1} G(q^{a_j-s})}, \qquad \ldots(1.16)$$

$$Y_2(t,q)=\frac{\prod_{j=1}^{M_2} G(q^{b_j'-t})\prod_{j=1}^{N_2} G(q^{1-a_j'+t})}{\prod_{j=M_2+1}^{Q_1} G(q^{1-b_j'+t})\ \prod_{j=N_2+1}^{P_1} G(q^{a_j'-s})}, \qquad \ldots(1.17)$$

and

$$Y_3(s,t;q)=\frac{\prod_{j=1}^{A} G(q^{1-c_j+s+'t})}{\prod_{j=A+1}^{C} G(q^{c_j-s-t})\prod_{j=1}^{D} G(q^{1-d_j+s+t})}. \qquad \ldots(1.18)$$

It is interesting to observe that for $A=C=D=0$ in (1.10) the basic Fox's H-function of two variables reduces to a product of two basic Fox's H-functions of one variable as under;

$$H_{0,0\,(P_1:Q_1),(P_2:Q_2)}^{0,(M_1:N_1),(M_2:N_2)}\left[\begin{matrix} z_1 \\ z_2 \end{matrix} ;q \left| \begin{matrix} (a,\alpha);(a',\alpha') \\ (b,\beta);(b',\beta') \end{matrix} \right.\right]$$

$$=H_{P_1,Q_1}^{M_1,N_1}\left[z_1;q\left| \begin{matrix} (a,\alpha) \\ (b,\beta) \end{matrix}\right.\right] H_{P_2,Q_2}^{M_2,N_2}\left[z_2;q\left| \begin{matrix} (a',\alpha') \\ (b',\beta') \end{matrix}\right.\right], \qquad \ldots(1.19)$$

where the basic analogue of Fox's H-function of one variable due to Saxena, Modi and Kalla [7] is given by

$$H_{P_1,Q_1}^{M_1,N_1}\left[x;q\left| \begin{matrix} (a,\alpha) \\ (b,\beta) \end{matrix}\right.\right]=\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j s})\ \prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j s})\ \pi x^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j s})\ \prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j s})\ G(q^{1-s})\sin\pi s} ds, \qquad \ldots(1.19)$$

where $0 \le M_1 \le Q_1$, $0 \le N_1 \le P_1$, α_j s and β_j s are all positive integers. The contour C is a line parallel to $\text{Re}(\omega s) = 0$ with indentations if necessary, in such a manner that all the poles of $G(q^{b_j - \beta_j s})$, $1 \le j \le M_1$ are to the right, and those of $G(q^{1-a_j+\alpha_j s})$, $1 \le j \le N_1$ to the left of C. The integral converges if $\text{Re}[s\log(x) - \log\sin \pi s] <$ 0 for large values of $|s|$ on the contour C i.e. if $\left|\left\{\arg(x) - w_2 w_1^{-1}\log|x|\right\}\right| < \pi$, where $0<|q|<1$, $\log q = -w = -(w_1 + iw_2)$, w, w_1, w_2 are definite quantities. w_1 and w_2 being real.

Further, if we set $\alpha_j = \beta_i = 1, \forall\ i$ and j in (1.19), we obtain the basic analogue of Meijer's G-function, due to Saxena, Modi and Kalla [7], namely

$$H_{P_1,Q_1}^{M_1,N_1}\left[x;q\left|\begin{matrix}(a,1)\\(b,1)\end{matrix}\right.\right] \equiv G_{P_1,Q_1}^{M_1,N_1}\left[x;q\left|\begin{matrix}a_1,\ldots,a_A\\b_1,\ldots,b_B\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{M_1} G(q^{b_j-s})\prod_{j=1}^{N_1} G(q^{1-a_j+s})\ \pi x^s}{\prod_{j=m_1+1}^{Q_1} G(q^{1-b_j+s})\ \prod_{j=n_1+1}^{P_1} G(q^{a_j-s})G(q^{1-s})\sin \pi s}\,ds\,, \qquad \ldots(1.20)$$

where $0 \le M_1 \le Q_1, 0 \le N_1 \le P_1$ and $\text{Re}[s\log(x) - \log\sin \pi s] < 0$.

2. Main Results

Theorem 1: If ρ and σ being any positive integers, then the following q-Laplace of a product of two basic functions holds:

$${}_qL_p\left\{x^{\lambda-1}H_{C,D,(P_1:Q_1),(P_2:Q_2)}^{A,(M_1:N_1),(M_2:N_2)}\left[\begin{matrix}z_1x^{\rho}\\z_2x^{\sigma}\end{matrix};q\left|\begin{matrix}(c;\gamma,\gamma')\\(a,\alpha);(a',\alpha')\\(d;\delta,\delta')\\(b,\beta);(b',\beta')\end{matrix}\right.\right]\right\}$$

$$= p^{-\lambda} H^{A+1,(M_1:N_1),(M_2:N_2)}_{C+1,D,(P_1:Q_1),(P_2:Q_2)} \left[\begin{matrix} (z_1 p^{-1})^{\rho} \\ (z_2 p^{-1})^{\sigma} \end{matrix} \; ;q \left| \begin{matrix} (1-\lambda;\rho,\sigma)(c;\gamma,\gamma') \\ (a,\alpha);(a',\alpha') \\ (d;\delta,\delta'), \\ (b,\beta);(b',\beta') \end{matrix} \right. \right] \quad \ldots(2.1)$$

where $\mathrm{Re}[s\log(z_1) - \log\sin\pi s] < 0$, $\mathrm{Re}[t\log(z_2) - \log\sin\pi t] < 0$ and for arbitrary λ.

Proof of theorem 1: We consider left hand side of equation (2.1) (say L) and make use of the definition (1.10) to obtain

$$L = L_q\left\{ x^{\lambda-1} \frac{1}{(2\pi i)^2} \int_{C_1^*}\int_{C_2^*} \frac{X_1(s,q)\,X_2(t,q)\,X_3(s,t,q)(z_1 x^{\rho})^s (z_2 x^{\sigma})^t}{G(q^{1-s})G(q^{1-t})\sin\pi s\,\sin\pi t} dsdt \right\}$$

On interchanging the orders of integration and Laplace operator which is justified under condition given with (1.10) we obtain

$$L = \frac{1}{(2\pi i)^2} \int_{C_1^*}\int_{C_2^*} \frac{X_1(s,q)\,X_2(t,q)\,X_3(s,t,q)(z_1)^s (z_2)^t}{G(q^{1-s})G(q^{1-t})\sin\pi s\,\sin\pi t} \; {}_qL_p\left\{ x^{\rho s+\sigma t+\lambda-1} \right\} \, ds\, dt\,.$$

On using the known result (1.6) the above expression reduces to

$$L = \frac{1}{(2\pi i)^2} \int_{C_1^*}\int_{C_2^*} \frac{X_1(s,q)\,X_2(t,q)\,X_3(s,t,q)(z_1)^s (z_2)^t}{G(q^{1-s})G(q^{1-t})\sin\pi s\,\sin\pi t} dsdt$$

$$\frac{\Gamma_q(\rho s+\sigma t+\lambda)(1-q)^{\rho s+\sigma t+\lambda-1}}{p^{\rho s+\sigma t+\lambda}},$$

which on using the definition (1.10), reduces to the right hand side of the theorem 1.

$$= p^{-\lambda} H^{A+1,\ (M_1:N_1),(M_2:N_2)}_{C+1,D+1,\ (P_1:Q_1),(P_2:Q_2)} \left[\begin{matrix} (z_1 p^{-1})^{\rho} \\ (z_2 p^{-1})^{\sigma} \end{matrix} \; ;q \left| \begin{matrix} (1-\lambda;\rho,\sigma)\ (c;\gamma,\gamma') \\ (a,\alpha);(a',\alpha') \\ (d;\delta,\delta') \\ (b,\beta);(b',\beta') \end{matrix} \right. \right]$$

Theorem-2: If $\mathrm{Re}(\mu) < 0$ and ρ being any positive integers, then for arbitrary λ, the following result holds:

$$ {}_qL_p\left\{x^{\lambda-1}H^{A,(M_1:N_1),(M_2:N_2)}_{C,D(P_1:Q_1),(P_2:Q_2)}\left[\begin{matrix} z_1x^{\rho} \\ z_2 \end{matrix};q\left|\begin{matrix}(c;\gamma,\gamma') \\ (a,\alpha);(a',\alpha') \\ (d;\delta,\delta') \\ (b,\beta);(b',\beta')\end{matrix}\right.\right]\right\} $$

$$ = p^{-\lambda}H^{A,(M_1:N_1+1),(M_2:N_2)}_{C,D(P_1+1:Q_1),(P_2:Q_2)}\left[\begin{matrix}(z_1p^{-1})^{\rho} \\ z_2\end{matrix};q\left|\begin{matrix}(c;\gamma,\gamma') \\ (1-\lambda;\rho),(a,\alpha);(a',\alpha') \\ (d;\delta,\delta'), \\ (b,\beta);(b',\beta')\end{matrix}\right.\right]. \qquad ...(2.2) $$

This is valid under conditions given with (1.10).

Theorem-3: If $\mathrm{Re}(\mu) < 0$ and σ being any positive integers, then for arbitrary λ, the following result holds:

$$ {}_qL_p\left\{x^{\lambda-1}H^{A,(M_1:N_1),(M_2:N_2)}_{C,D(P_1:Q_1),(P_2:Q_2)}\left[\begin{matrix} z_1 \\ z_2x^{\sigma} \end{matrix};q\left|\begin{matrix}(c;\gamma,\gamma') \\ (a,\alpha);(a',\alpha') \\ (d;\delta,\delta') \\ (b,\beta);(b',\beta')\end{matrix}\right.\right]\right\} $$

$$ = p^{-\lambda}H^{A,(M_1:N_1),(M_2:N_2+1)}_{C,D(P_1:Q_1),(P_2+1:Q_2)}\left[\begin{matrix} z_1 \\ (z_2p^{-1})^{\sigma}\end{matrix};q\left|\begin{matrix}(c;\gamma,\gamma') \\ (a,\alpha);(1-\lambda;\sigma),(a',\alpha') \\ (d;\delta,\delta'), \\ (b,\beta);(b',\beta')\end{matrix}\right.\right] \qquad ...(2.3) $$

This is valid under conditions given with (1.10).

The proof of theorems 2 and 3 follows similar as proof theorems 1.

3.Special cases

If we set $\alpha = \alpha' = \beta = \beta' = \gamma = \gamma' = \delta = \delta' = \sigma = \rho = 1$ in (2.1) and make use of the definition (1.14) we obtain the following results involving the fractional order derivatives of basic analogue of Meijer's G-function of two variables;

$$ {}_qL_p\left\{ x^{\lambda-1} G^{A,(M_1:N_1),(M_2:N_2)}_{C,D,(P_1:Q_1),(P_2:Q_2)}\left[\begin{matrix} z_1x \\ z_2x \end{matrix} ;q \left| \begin{matrix} c_1,\ldots,c_C \\ a_1,\ldots a_{P_1};a_1'\ldots,a_{P_2}' \\ d_1,\ldots,d_D \\ b_1,\ldots b_{Q_1};b_1'\ldots,b_{Q_2}' \end{matrix}\right.\right]\right\} $$

$$ = p^{-\lambda} G^{A+1,(M_1:N_1),(M_2:N_2)}_{C+1,D+1,(P_1:Q_1),(P_2:Q_2)}\left[\begin{matrix} z_1/p \\ z_2/p \end{matrix} ;q \left| \begin{matrix} 1-\lambda;c_1,\ldots,c_C \\ a_1,\ldots a_{P_1};a_1'\ldots,a_{P_2}' \\ d_1,\ldots,d_D; \\ b_1,\ldots b_{Q_1};b_1'\ldots,b_{Q_2}' \end{matrix}\right.\right], \qquad \ldots(3.1) $$

which are valid under the conditions given with (1.10) and for arbitrary λ.

if we set $A = C = D = 0$ in result (2.1) and making use of the known result the result (1.19), then we obtain the following q-Laplace of a product of two basic functions:

$$ {}_qL_p\left\{ x^{\lambda-1} H^{M_1,N_1}_{P_1,Q_1}\left[z_1x^{\rho};q \left| \begin{matrix} (a,\alpha) \\ (b,\beta) \end{matrix}\right.\right] H^{M_2,N_2}_{P_2,Q_2}\left[z_2x^{\sigma};q \left| \begin{matrix} (a',\alpha') \\ (b',\beta') \end{matrix}\right.\right]\right\} $$

$$ = p^{-\lambda} H^{1,(M_1:N_1),(M_2:N_2)}_{1,0,(P_1:Q_1),(P_2:Q_2)}\left[\begin{matrix} (z_1p^{-1})^{\rho} \\ (z_2p^{-1})^{\sigma} \end{matrix} ;q \left| \begin{matrix} (1-\lambda;\rho,\sigma) \\ (a,\alpha);(a',\alpha') \\ (b,\beta);(b',\beta') \end{matrix}\right.\right], \qquad \ldots(3.2) $$

where $\mathrm{Re}[s\log(z_1) - \log\sin\pi s] < 0$, $\mathrm{Re}[t\log(z_2) - \log\sin\pi t] < 0$ and for arbitrary λ.

We conclude with an observation that the method used here can be employed to yield a variety of interesting results involving the expansions and transformations for the generalized basic hypergeometric functions of two variables.

References

[1] Abdi, W.A., On q-Laplace transforms. Proc. Nat. Acad. Sc. (India), 29(A) (1960), 389-407.

[2] Gasper, G. and Rahman, M., Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[3] Hahn, W., Beiträge Zur Theorie der Heineschen Reihen, die 24 Integrale der hypegeometrischen q-Differenzengleichung, das q-Analogon der Laplace-Transformation, Math. Nachr. 2 (1949), 340-379.

[4] Purohit, S.D., Kalla, S. L., On q-laplace transforms ofthe q-bessel functions,Fractional calculus & Appl. Anal, vol 10(2)(2007), 189-197.

[5] Purohit, S.D., Yadav, R.K & Vyas, V.K., On application of q-Laplace transforms to basic analogue of the I-function. (In Press)

[6] Saxena, R.K. and Kumar, R., A basic analogue of the generalized H-function, Le Matematiche, Vol. L(1995), Fasc. II, 263-271.

[7] Saxena, R. K., Modi, G.C. and Kalla, S.L.: A basic analogue of Fox's H-function. Rev. Tec. Ing. Univ. Zulia, 6 (1983), 139-143.

[8] Saxena, R.K., Modi, G.C.and Kalla, S.L., A basic analogue of H-function of two variables. Rev. Tec. Ing. Univ. Zulia , 10(2) (1987), 35-38.

[9] Yadav, R.K. & Purohit, S.D., On q-Laplace transforms of certain generalized basic hypergeometric functions, Proc. 5th Int. Conf. SSFA, 5(2004), 74-81.

[10] Yadav, R.K. & Purohit, S.D., On q-Laplace transforms of certain multiple basic hypergeometric functions. The Math. Student 74 (1-4) (2005), 207-215.

[11] Yadav, R.K. & Purohit, S.D., On q-Laplace transforms of certain q-hypergeometric polynomials. Proc. Nat. Acad. Sci., India 76(A) III (2006), 235-242.

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Order Reduction Using Bacterial Foraging Optimization Technique

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Abstract

This brief considers the order reduction using the optimization method.Recently, the Bacterial Foraging optimization (BFO) technique has attracted considerable attention among various modern heuristic optimization techniques. The problem of reducing a high order system to its lower order system is considered important in analysis, synthesis and simulation of practical systems. In this paper, BFO optimization and dominant pole retention technique has been discussed such that the reduced order model retains important characteristics of original system and approximates its response subjected to unit step input. BFO method is based on the minimization of the integral square error (ISE) between the transient responses of original higher order and the reduced order models in order to calculate numerator coefficients of low order model, whereas the dominant pole retention technique is used to calculate the coefficients of denominator polynomial.The method is illustrated through numerical example available from the literature.

Keywords : Order reduction, BFO, Dominant pole, Integral Square Error.

1. Introduction

Many physical systems are translatedinto mathematical model via higher order differential equations. The exact study of high order systems (HOS) is both tedious and costly as HOS are often too complicated to be used in real problems. Hence, simplification procedures based on physical considerations or using mathematical approaches are generally employed to realize simple models for the original HOS. It is usually suggested to reduce the order of this model while keeping the dominant behavior of the original system. This will help to better understanding of the physical system and reduce computational complexity, reduce hardware complexity and simplify the controller design. This naturally then motivates the use of model order reduction to obtain reduced- order model, which should provide an adequate approximation of the original system response to particular inputs[1-2]. The problem of reducing a high order system to its lower order system is considered important in analysis, synthesis and simulation of practical systems. In this paper combined advantages of classical technique and error minimization by Bacterial foraging technique (BFO) is intended. In this algorithm, coefficients of denominator polynomial of reduced order transfer function model are obtained by dominant pole retention criterion[1] and the numerator coefficients are determined by minimizing the integral square error (ISE) between the transient responses of original high order system and reduced order model using BFO technique[1-2]. The proposed algorithm is applied to the hydro-electric power plant[2]. The method preserves the stability in the reduced order model if original HOS is stable. There are various techniques for order reduction available in the literature[5-13]. The suggested method is simple and computer oriented.

Bacterial Foraging Optimization- The bacterial foraging optimization (BFO) proposed by Passino in the year 2002 is based onnatural selection that tends to eliminate animals with poor foraging strategies. After many generations, poor foraging strategies are eliminated while only the individuals with good foraging strategy survive signifying survival of the fittest. BFO formulates the foraging behavior exhibited by E. coli bacteria as an optimization problem.

Strategy- The information processing strategy of the algorithm is to allow cells to stochastically and collectively swarm toward optima.

- Chemotaxis:This process simulates the movement of bacteriathrough two ways swimming and tumbling. Biologically an E.coli bacterium can move in two different ways. It can swim for a period of time in the same direction or it may tumble in different direction, and alternate between these two modes of operation for the entire lifetime [3-4].
- Swarming: An interesting group behavior has been observed for several motile species of bacteria including E.coli and S. typhimurium, where intricate and stable spatio-temporal patterns (swarms) are formed in semisolid nutrient medium. A group of E.coli cells arrange themselves in a traveling ring by moving up the nutrient gradient [3].
- Reproduction: The least healthy bacteria eventually die while each of the healthier bacteria (those having lower value of the objective function) split into two bacteria, which are then placed in the same location. This keeps the swarm size constant[3].
- Elimination and Dispersal: Sudden changes in the localenvironment where the population livesdue to various reasons e.g. a significant rise of temperature may kill a group of bacteria that are currently in aregion with high concentration of nutrient gradients.

Events can take place in such a manner that all the bacteria in aregion are killed or a group is dispersed into a new location[3-4].

Dominant Pole Retention technique- In the suggested technique, coefficients of denominator polynomial of reduced order transfer function model are obtained by dominant pole retention criterion. According to this method, the poles nearest to origin are retained. In this method, denominator coefficients of low order system's transfer function are chosen as dominant poles of original high order systems. Dominant poles are taken as they have more effect on system[1] [5].

2. Problem Statement

In this paper, denominator coefficients of low order systems transfer function are chosen as dominant poles of original high order systems while numerator coefficients are obtained by minimizing the ISE between the transient responses of HOS and LOS using BFO.

$$G_n(s)=\frac{a_0+a_1s+a_2s^2+\ldots+a_ns^{n-1}}{b_0+b_1s+b_2s^2+\ldots+b_ns^n} \qquad \ldots(1)$$

Consider an n^{th} order SISO LTI system with the following transfer function:

with the eigen values of the system to be:

$$-\lambda_1<-\lambda_2<\ldots.<-\lambda_n. \qquad \ldots(2)$$

Let the reduced order model be:

$$G_r(s)=\frac{\alpha_0+\alpha_1s+\alpha_2s^2+\ldots+\alpha_ns^{r-1}}{\beta_0+\beta_1s+\beta_2s^2+\ldots+\beta_rs^r} \qquad \ldots(3)$$

The coefficients of the denominator of $G_r(s)$ are chosen such that the eigen values of the low order system are the dominant roots of the full order system as follows:

$$-\lambda_1<-\lambda_2<\ldots.<-\lambda_r. \qquad \ldots(4)$$

while the coefficients of the numerator of transfer function are obtained by BFO algorithm.

3. Numerical Example

The transfer function based on numerical values of the 6^{th} order SISO model of hydro electric power system given by transfer function [2]:

$$G_6(s)=\frac{-0.1s^5-2.61s^4-7.343s^3-4.875s^2-0.0417s}{s^6+26.0999s^5+73.4331s^4+32.0833s^3+6.2497s^2+1.1666s+0.0417} \qquad \ldots(5)$$

Poles of the above system are:

$\lambda_1 = -0.0440, \lambda_2 = -0.0442 + j0.2048,\ \lambda_3 = -0.0442 - j0.2048, \lambda_4 = -0.3546,\ \lambda_5 = -2.6507,\ \lambda_6 = -22.9622.$

The coefficients of the denominator of $G_4(s)$ (reduced order model) are chosen such that the eigen values of the low order system are the dominant roots of the full order system. Taking dominant poles $\lambda_1 = -0.0440$, $\lambda_2 = -0.0442 + j0.2048,\ \lambda_3 = -0.0442 - j0.2048,\ \lambda_4 = -0.3546$.

$$s^4 + 0.4870s^3 + 0.094725s^2 + 0.018873s + 0.0006847$$

Therefore, the denominator polynomial will be:

$$s^4 + 0.4870s^3 + 0.094725s^2 + 0.018873s + 0.0006847 \qquad \ldots(6)$$

while numerator polynomial of transfer function is obtained by BFO. So reduced order transfer function will be:

$$G_4(s) = \frac{-0.3037s^3 - 0.08336s^2 - 0.0004s - 2.89e^{-005}}{s^4 + 0.4870s^3 + 0.094725s^2 + 0.018873s + 0.0006847} \qquad \ldots(7)$$

4. Results

In this technique,order of this model is reduced while keeping the dominant behavior of the original system as closely as possible for step input.The ISE between HOS (6^{th} order) and LOS (4^{th} order) is1.099.Step responses of original HOS and reduced LOS are compared as shown in Fig.1.

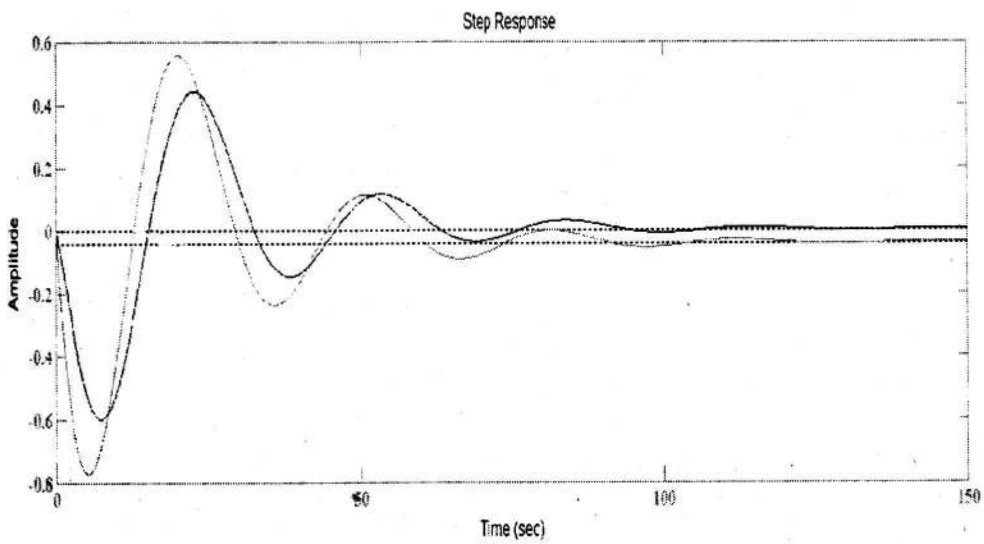

Fig.1: Step response of High order & Low order systems

From the Fig.1, it can be seen that the step responses of HOS and LOS are matching with each other.

5. Conclusions

In this paper, it was shown that Bacterial Foraging Optimization with Dominant pole retention method is a useful tool for the solution of model reduction problems. The reduced order modeling is doneto obtain the lower order approximate (4th order model) of higher order system(6th order model) through the Bacterial foraging optimization technique and Dominant pole retention method. The step responses of HOS and LOS are compared, which matches with each other. The ISE between HOS and LOS has also been calculated. Computational time is about 30 seconds.

References

[1] G.Parmar, R.Prasad and S.Mukherjee, "Reduction of linear dynamic systems using dominant pole retention method and modified cauer continued fraction",*Journal of Computer Science,*Vol.1, No.6, pp.547-554, May-June, 2006.

[2] S. Y. Kumar, P.K. Ghosh and R. Ghosh,"Power System Modeling using Bacterial Foraging andParticle Swarm Optimization Techniques",*Advances in Electronic and Electric Engineering,*Vol. 4, No. 1, pp. 49-56, 2014.

[3] S. Y. Kumar, P. K.Ghosh, and S. Mukherjee, "Model Order Reduction using Bio-inspired PSO and BFO Soft -Computing for Comparative Study",*Int. J. of Information Systems and Communications,*Vol. 1, No. 1, June 2011.

[4] R. M. Thomas, "Survey of Bacterial Foraging Optimization Algorithm", *International Journal of Science and Modern Engineering* (IJISME), Vol. 1, March 2013.

[5] S.Mukherjee, and R.Nishra,"Order reduction of linear systems using an error minimization technique", *Journal of Franklin Inst.*, Vol. 323, No. 1, pp. 23-32, 1987.

[6] C.B. Vishwakarma and R. Prasad,"MIMO System Reduction Using Modified Pole Clustering and Genetic Algorithm", *Modeling and Simulation in Engineering*, pp. 1-5, 2009.

[7] K. M. Passino, "Biomimicry of bacterial foraging for distributed optimization and control," *IEEE Control Systems Magazine*, Vol. 22, No. 3, pp. 52–67, 2002.

[8] N.N. Puri, and D.P. Lan, "Stable model reduction by impulse response error minimization using Mihailov criterion andPade's approximation", *Trans. ASME, J. Dyn. Syst. Meas. Control,* Vol. 110, pp. 389-394, 1988.

[9] S. Panda, J.S. Yadav, N.P. Padidar, and C. Ardil, "Evolutionary Techniques for Model Order Reduction of Large Scale Linear Systems", *Int. J. Applied Sci. Eng. Technol.5*,pp.22-28, 2009.

[10] R Prasad., A. K.Mittal and S. P. Sharma, "A mixed method for the reduction of multi-variable systems", *Journal of Institution of Engineers, India, IE(I) Journal- EL*, Vol. 85, pp. 177-181, March 2005.

[11] S. Mukherjee,"Order reduction of linear system using eigen spectrumanalysis", *Journal of Institution of Engineers, India, IE (I) Journal-EL*,Vol.77, pp.76-79, 1996.

[12] S. Mukherjee, M. Satakshi, and R.C. Mittal, "Model order reduction using response matching technique", *J. Franklin Inst. 342*, pp. 503–519, 2005.

[13] T. Reis and T. Stykel,"Balanced truncation model reduction of second-order systems", *Mathematical and Computer Modeling of Dynamical Systems*, pp. 391-406, 2008.

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Balanced Quaternary Logic Gates in Reversible Computing

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Abstract

Reversible Logic Technology phenomena has emerged as potential logic design style for implementation in Quantum Computing, VLSI Design and Dissipation less Computing. Our design methodology is based on first analyzing the Binary and Ternary circuits using their approach and then replacing them with their equivalent realization using Balancing in Quaternary Logic. Based on this technology, we proposed three balanced quaternary reversible logic gates and these are- NOT Gate, Feynman Gate and Toffoli Gate. By using these balanced quaternary reversible gates Half-Adder, Full-Adder, Multiplier etc. circuits can be designed easily with their best performance.

Keywords—reversible computing; M-S gate; Feynman gate; Toffoli gate.

I. Introduction

Multiple-valued logic synthesis, specially, binary logic synthesis is the most/traditional basis for quantum computation (QC) [1]. Several advantages are offered from multiple-valued logic over binary logic, such as more powerful quantum information processing, better security for quantum cryptography. Among them GF(4) [2] based synthesis of quaternary reversible logic plays a very important role in reversible computing. Several different type gates like NOT gate, Feynman gate, Toffoli gate etc. plays important role in reversible computing. The advantages of this approach is that quaternary logic functions having many input variables can be expressed as quaternary Galois field sum of products (QGFSOP) expression and can be realized easily with quaternary Feynman and Toffoli gate [3].

After analyzing the synthesis of ternary reversible logic it finds that a considerable number of good works have been done, but it has some limitations that conventional binary logic functions cannot be easily represented using the ternary base and the developed methods are applicable only for logic functions expressed in ternary base. Using the quaternary logic, not only quaternary logic functions but also binary logic functions can be represented by 2-bit together grouping into quaternary values. For a Hilbert space with N dimension, a binary system requires $n_2 = \log_2 N$ qubits (quantum bits), whereas a quaternary system requires $n_4 = \log_4 N = \log_2 N/2$ qudits (quantum digits). Muthukrishnan and troud [4] proposed a family of 2-qudit multiple-valued (d>=2) gates, which are realized in liquid ion-trap technology.

This paper introduces the balancing rules and their implementation for quaternary reversible logic gates. In quaternary logic 4-digits 0,1,2 and 3 are realized and these states are called quaternary standard states and these four standard states are replaced by -2,-1.+1 and +2, respectively. For more precise representation, some conventions are also assumed for balanced quaternary logic states. -2 and -1 states are assumed Low (L) and High (H), respectively. All integers can be represented in balanced quaternary. Any un-balanced (standard number) quaternary can be converted into balanced quaternary notation by subtracting +2 and +1 from 0,1 and 2,3, respectively. For example, $(2301)_{\text{un-bal.4}} = (12LH)_{\text{bal.4}} = (177)_{10}$.

This paper is summarized in different sections are as: Reversible logic in section II, The Basics of Quaternary Algebra in section III, Related work in section IV, Proposed balanced quaternary reversible logic gates in section V and Conclusion with future scope in section VI.

II. Reversible Logic

A. Reversible Logic Conditions

- Number of output variables must be equal to the number of input variables [5][6][7][8].
- "Bijective condition" must be satisfied. It means one to one unique correspondence between input and output variables [5][6][7][8].
- To make a function reversible 0 or 1 bit must be added for input garbage for output. 0 and 1 bit lines are called ancilla lines [5][6][7][8].
- The number of constant inputs and outputs must be minimum as should as possible [5][6][7][8].
- Fan-out is not allowed in reversibility [5][6][7][8].

  NOT gate is only conventional gate which shows all the reversible conditions and come into the category of reversible logic gates [5].

B. Reversible logic Gates

The basic reversible logic gates are NOT gate (NG), Feynman gate (FG) [9], Fredkin gate (FR) [10], Toffoli gate (TG) [11].

C. Unitary Matrix

A square matrix is called unitary matrix which has complex entries and which inverse is equal to its conjugate transpose U^*. It means that $U^*U = UU^* = I$, where U^* is the conjugate transpose of U and I is the identity matrix. A unitary matrix represents the input-output combinational relation of gate/circuit. The truth table of feynman gate is shown in table I with its unitary matrix in table II.

TABLE I. Truth table of feynman gate

Input		Output	
x	y	x'	y'
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

TABLE II. Unitary matrix of feynman gate

		Output			
		00	01	10	11
Input	00	1	0	0	0
	01	0	1	0	0
	10	0	0	0	1
	11	0	0	1	0

III. The Basics of Quaternary Algebra

The set Q_{gf} has the elements (0, 1, 2 and 3) exhibits an algebraic structure of quaternary Galois Field (GF (4)). The two binary operations addition and multiplication are defined in TableIII (a) and Table1 (b)

TABLE III (A) GF(4) ADDITION; (B) GF(4) MULTIPLICATION

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

(a)

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

(b)

(A1) a + (b + c) = (a + b) + c (associative law for addition)

(A2) a + b = b + a (commutative law for addition)

(A3) There is an element 0 such that $a + 0 = a$ for all a

(A4) For any a, there is an element (–a) as $a + (-a) = 0$

(M1) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associative law for multiplication)

(M2) $a \cdot b = b \cdot a$ (commutative law for multiplication)

(M3) There is an element 1 (not equal to 0) such that $a \cdot 1 = a$ for all a

(M4) For any $a \neq 0$, there is an element a^{-1} as $a \cdot a^{-1} = 1$

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ (distributive law)

The axioms rules are also defined from above described tables.

IV. Related Work

By survey of different reversible logic gates/circuits we find that reversible gates NOT, Feynman, Fredkin, Toffoli are made by the researchers with the help of different approaches but some weaknessesare in the gate/circuit design.

Realization of quaternary reversible circuits is more complex to reversible binary circuits. Recently, researchers have addressed a very few but promising research articles on realization and implementation of quaternary reversible circuits. In 2006, Mozammel H. A. Khan [12] proposed a successful implementation of quaternary Feynman and Toffoli gate. The realization of quaternary Feynman and Toffoli gate is shown using M-S primitive gate. Md. Mahmud Muntakim Khan et al. [13] addressed an optimized realization of quaternary Toffoli gate in 2007. In 2008, Mozzammel H. A. Khan [12] proposed a improved (from previous work) realization of quaternary Toffoli gate using quaternary control shift gates. Finally in 2013, Bikromadittya et al. [14] proposed the balanced ternary reversible logic NOT, Feynman and Toffoli gate. With the help of literature we find that all balanced reversible gates are not standard gates, so some approaches are necessary to build the gates/circuits standard, are discussed in this work.

V. Proposed balanced Quaternary Reversible Logic Gates

For the design of quaternary balanced reversible gates four states – 2, –1, +1 and +2 are used for corresponds to 0, 1, 2 and 3. The unique column vector representations of these states are represented by Figure 1.

$$-2 = L = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad -1 = H = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad +1 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad +2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Fig. 1. Vector Representation of Balanced Quaternary States

A. BALANCED QUATERNARY NOT GATE

It inverts the input to the corresponding output of the input, e.g. -2 inverts into +2 and -1 inverts into +1. Fig. 8 shows the symbol of Balanced quaternary reversible NOT gate with its Truth Table.

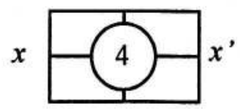

Fig. 2. Symbol of Balanced Quaternary NOT Gate

TABLE IV. TRUTH TABLE OF BALANCED QUATERNARY NOT GATE

Input	Output
A	A'
+2	−2
+1	−1
−1	+1
−2	+2

The unitary matrix of the balanced quaternary reversible NOT gate is as-

$$\begin{array}{c|c|c|c|c|} & -2 & -1 & +1 & +2 \\ \hline -2 & 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 1 & 0 \\ \hline +1 & 0 & 1 & 0 & 0 \\ \hline +2 & 1 & 0 & 0 & 0 \\ \hline \end{array} \Rightarrow \begin{pmatrix}0001\\0010\\0100\\1000\end{pmatrix}$$

Fig. 3. Unitary Matrix of Quaternary NOT Gate

The Figure 4 shows the balanced quaternary NOT operation e.g. input -2 produces +2.

$$\begin{pmatrix} 0001 \\ 0010 \\ 0100 \\ 1000 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Fig. 4. Balanced Quaternary NOT Operation

B. BALANCED QUATERNARY FEYNMAN GATE

It is a 2*2 reversible logic gate. The 1st input is unchanged for the 1st output and the 2nd input is changed by the balanced quaternary NOT gate. It is not dependent on the 1st input. Table V and VI shows the operation and Truth Table of balanced quaternary Feynman gate.

TABLE V. BALANCED QUATERNARY FEYNMAN OPERATION.

Control Input	Target Output
− 2	NOT of Input
− 1	
+ 1	
+ 2	

TABLE VI. TRUTH TABLE OF BALANCED QUATERNARY FEYNMAN GATE

Input		Output	
A (Control)	B	A' = A	[B'=F(A,B)]
− 2	− 2	− 2	+ 2
− 2	− 1	− 2	+ 1
− 2	+ 1	− 2	− 1
− 2	+ 2	− 2	− 2
− 1	− 2	− 1	+ 2
− 1	− 1	− 1	+ 1
− 1	+ 1	− 1	− 1
− 1	+ 2	− 1	− 2
+ 1	− 2	+ 1	+ 2
+ 1	− 1	+ 1	+ 1
+ 1	+ 1	+ 1	− 1
+ 1	+ 2	+ 1	− 2
+ 2	− 2	+ 2	+ 2
+ 2	− 1	+ 2	+ 1
+ 2	+ 1	+ 2	− 1
+ 2	+ 2	+ 2	− 2

Figures 5, 6 and 7 show the symbol, unitary matrix and column vector representation of balanced quaternary Feynman gate.

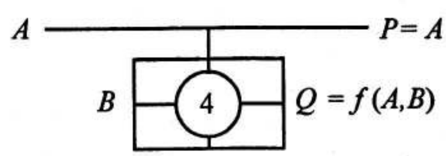

Fig. 5. Symbol of Balanced Quaternary Feynman Gate

$$\begin{pmatrix} 0001000000000000 \\ 0010000000000000 \\ 0100000000000000 \\ 1000000000000000 \\ 0000000100000000 \\ 0000001000000000 \\ 0000010000000000 \\ 0000100000000000 \\ 0000000000010000 \\ 0000000000100000 \\ 0000000001000000 \\ 0000000010000000 \\ 0000000000000001 \\ 0000000000000010 \\ 0000000000000100 \\ 0000000000001000 \end{pmatrix} \begin{pmatrix} 0001000000000000 \\ 0010000000000000 \\ 0100000000000000 \\ 1000000000000000 \\ 0000000100000000 \\ 0000001000000000 \\ 0000010000000000 \\ 0000100000000000 \\ 0000000000010000 \\ 0000000000100000 \\ 0000000001000000 \\ 0000000010000000 \\ 0000000000000001 \\ 0000000000000010 \\ 0000000000000100 \\ 0000000000001000 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Fig. 6. Unitary Matrix of Balanced Quaternary Feynman Gate

Fig. 8. Balanced Quaternary Feynman Operation

$$\begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0 \end{pmatrix}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{pmatrix}$$

Fig. 7. Column Vector representation of

-2-2,-2-1,-2+1,-2+2,-1-2,-1-1,-1+1,-1+2,+1-2,+1-1,+1+1,+1+2,+2-2,+2-1,+2+1,+2+2.

Figure 8 shows the Feynman operation when the Input -2-2 converted into -2+2 Output.

C. BALANCED QUATERNARY TOFFOLI GATE

It is a 3*3 reversible logic gate. The 1st and 2nd inputs are unchanged, and the 3rd input changed by the balanced quaternary NOT operation on the 3rd input to convert 3rd output. It is also not dependant on the 1st and 2nd input which can have any quaternary balanced states. The balanced quaternary Toffoli operations are shown by the table VII.

Figure 9 shows the symbol of balanced quaternary Toffoli gate, which is newly proposed, with Truth Table (table must be too large, so here we are not showing here its truth table).

TABLE VII. BALANCED QUATERNARY TOFFOLI GATE OPERATION

Control Input		Target Output
A	B	
Any	Any	NOT of Input

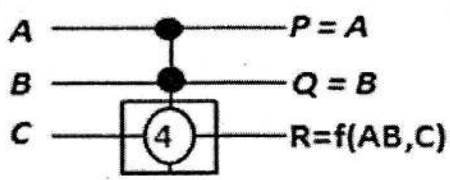

Fig. 9. Symbol of Balanced Quaternary Toffoli Gate

VI. Conclusion and Future Scope

Due to efficient realization of balanced quaternary logic there would be a very high prospect to generate quaternary reversible logic gates. In this article, a significant realization of balanced quaternary reversible gates is proposed which will promote a standard balancing in reversible computing. We have proposed a methodology and balancing principles for the realization of balanced quaternary reversible gates. It significantly optimizes the hardware complexity of our proposed design of balanced quaternary reversible NOT, Feynman and Toffoli gates.

With the help these proposed approaches for balanced quaternary reversible gates half-adder, full-adder and multiplier circuits can also proposed.

REFERENCES

[1] M. Nielsen, I. Chuang, Quantum Computation and Quantum Information, versity Press, 2000.

[2] M.H.A. Khan, M.A. Perkowski, "GF(4) Based Synthesis of Quaternary Reversible / Quantum Logic Circuits", Proc. of Int. Symp. Multiple- Valued Logic (ISMVL 2007), Oslo, Norway, May 2007.

[3] M. H. A. Khan, M. A. Perkowski, and P. Kerntopf, "Multi-output Galois field sum of products synthesis with new quantum cascades", *Proc. 33rd IEEE Int. Symp. On Multiple-Valued Logic (ISMVL 2003)*, Tokyo, Japan, 16-19 May 2003, pp. 146-153 . R.

[4] A. Muthukrishnan, C. R. Stroud Jr., "Multivalued logic gates for quantum computation", *Physical Review A*, 62, 2000, 052309/1-8.

[5] W. David, Mahesh Nalasani." Reversible Logic", IEEE Pottentials 2005 pp. 38-41. Information Processing Letters, Vol. 70, 199, pp 157-163, 2005.

[6] K. Chakrabarty, J.P.Hayes "Balanced Boolean Functions", IEEE Proceeding Computers-and Digital Techniques, Vol. 145. Pp. 52-62, 1998.

[7] A. Mishchenko, M. Perkowski, "Logic Synthesis of Reversible Wave Cascades", International Workshop on Logic Synthesis, 2002.

[8] A.G.Rao, A.Dwivedi, "Design of Multifunctional DR Gate and its Application in ALU Design",ICIT, 2014. International Conference on Information Technology on, vol., no., pp.339,345, 10-11 Sept., 2014.

[9] Feynman. "Quantum Mechanical Computers," Optic News, pp.11-20, 1985.

[10] E. Fredkin, and T. Toffoli, "Conservative Logic," International Journal of Theoretical Physics, 21:219-253, 1982.

[11] T. Toffoli, "Reverible computing," Tech memo MIT/LCS/TM- 151, MIT Lab for Computer Science, 1980.

[12] Khan, M.H.A., "Quantum Realization of Quaternary Feynman and Toffoli Gates," *Electrical and Computer Engineering, 2006. ICECE '06. International Conference on*, vol., no., pp.157,160, 19-21 Dec. 2006.

[13] Khan, M.M.; Biswas, A.K.; Chowdhury, S.; Tanzid, M.; Mohsin, K.M.; Hasan, M.; Khan, A.I., "Quantum realization of some quaternary circuits," *TENCON 2008 - 2008 IEEE Region 10 Conference*, vol., no., pp.1,5, 19-21 Nov. 2008.

[14] Mondal, B.; Sarkar, P.; Saha, P.K.; Chakraborty, S., "Synthesis of Balanced Ternary Reversible Logic Circuit," *Multiple-Valued Logic (ISMVL), 2013 IEEE 43rd International Symposium on*, vol., no., pp.334,339, 22-24 May 2013.

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Ternary Implication Gate realization using Memristor based Logic Synthesis

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Abstract

With continuous growth in demand for high speed computations, future sustainability of current technology come under question amongst the alternative technologies. Memristor based design has become interesting option. Memristor is a low cost element that can implement an imply gate. We designed a new synthesis method for ternary circuit with single output using implication gate in memristor. We realized only two working memristor. By this algorithm we minimize the number of pulses and delay time in sum of product. In this work, finally we find the result with high quality and fast.

Keywords— Memristor; number of pulses; Material Implication (IMPLY) gate; sequential realization of combinational logic; Logic Synthesis; Ternary Logic

I. Introduction

In the electrical engineering some basic elements like resistor, capacitor, and inductor are used. In 1971 Leon Chua found fourth element memristor. Memristor consists from memory and resistor [3]. Memristor is a two terminal device and programmable. Basically, Memristor used for Memory and used for store information. The memristor synonyms first time by Hewlett-Packard Corporation in 2008 and emerged many applications [6]. Memristor have been proposed for standard binary combinational logic [1, 6], multiple valued and fuzzy logic [7, 5] memory design logic in-memory design [5], and analog circuits [4]. Our work is motivated for low cost calculation.

Memristor is used for input, output, latch in different stages of the computing process and computational logic. The memristor is used for advance computer architecture, store information inside the memory. In the memristor the imply gate behavior as a switch. The Imply gate and memristor are combine the boolean operator and memory [7].

II. Background of Memristor

Memristor behavior similar to resistor because its resistor also depends on past state. A memristor has voltage based memory and voltage depends on its past state so memristor behavior like resistor and also its unit is ohm.

Memristor is used in different types of computations Memristor is non-linear, analog, and memory properties with passive electrical elements. It was shown an imply gate in ternary form used in working memristor like T={0, 1, 2} here 0 is unknown 1 is true and 2 is false state of a memristor. The memristor is used for memristance.

III. Realization of Basic imply gate with ternary logic

The Imply gate is a basic gate and we represent this gate by P?Q equal to ?P+Q. This gate also called Implication gate and we can realization the basic gates NOT, AND, OR, NAND [24] using implication gates in logic synthesis. In the truth table if $\neg P = 0$ and $Q = 2$ then result be 0, $\neg P = 2$ and $Q = 0$ then result be 0 if $\neg P = 1$ and $Q = 0$ then result be 1.

TABLE II. TRUTH TABLE OF IMPLY GATE IN TERNARY

P	Q	$P \to Q$
0	0	0
0	1	1
0	2	0
1	0	0
1	1	1
1	2	2
2	0	1
2	1	1
2	2	1

IV. Imply Sequence diegram wih woking memristor

We have design a circuit using imply sequence diagram notation. This notation is used to design application of synthesizing and analyzing the memristor circuit. In this notation we used this 8 symbol represents a pulse applied to it. In this notation, we used horizontal lines represents physical memristor. The top most side input always be negative. In this memristor the left side value be negative and right side valve is the value of the memristor. Here 0 is also indicated additional input

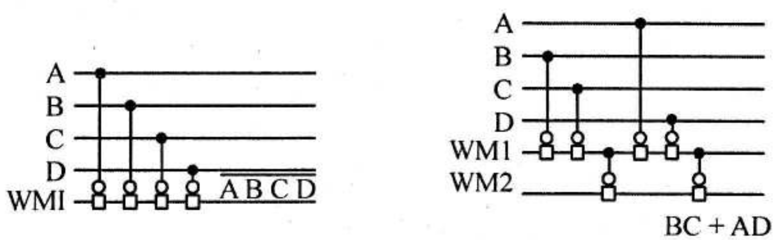


Fig. 1. Imply sequence Diagram (ISD).

(a) Synthesis with one working memristor (b) Synthesis with two Working Memristor [2].

V. Proposed work

In our algorithm we used only two working memristors. It is denoted by WM1 and WM2.This is most advantage that its uses only two memristor. The present method used for the minimize the number of pulses with two working memristor so in this method we can reduce the physical size and cost of the circuit. This algorithm provides single output functions.

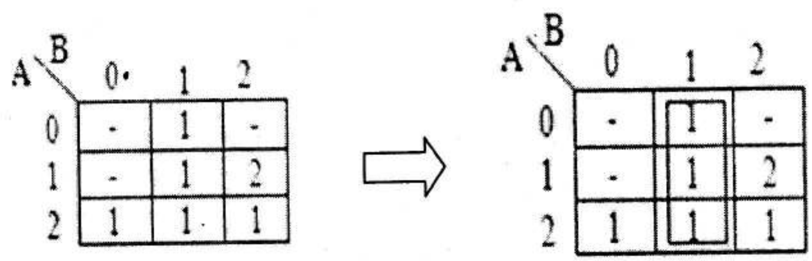
In this paper we want to simplification of the our explanation so we used standard logic and kmap but there requirements only truth table internally and output need the graphical representation of ISD notation. So here we make the some groups of the 1's.

In this notation we use the imply gate and this method realizes the simple logic gates like NAND gate. We select the largest group of 1's. These literals should be all positive. The product implicant is uses the variable with the product of literals. In this paper we used ternary logic. First we make a function in ternary with k-map function. We select largest group of 1's. In this ternary logic we use the value as 0 is denoted by dases (-), 1 is denoted by 1 and 2 is denoted by 2 in ternary function f(A,B). So we select the largest group of 1's in this function. Now select group the 1's are replaced by dases (don'tcares) and add the original don't cares with remaining 1's in group form according ternary logic. The maximum don't cares appear map then it reduce the number of pulses and imply gates. If there are no more find the largest group then we can add the inverter insert to the circuit and k-map be negated. When we use the realization logic gates then we use inverters and NAND gates.

In the Figure 2 shows the K-map function f(A,B) which need realized with logic synthesis algorithm the function f(A,B) is a trivial function.

In the Fig shows the k-map with the ternary logic function f(A,B). The k-map is applied according to ternary logic, which need realized with the logic synthesis algorithm. The function f(A,B) is a trivial with ternary logic. This function we use and explanation here.

We used here Ternary logic based function f(A,B) and we designed this function according to Ternary Logic properties T = {0, 1, 2}. In figure 2 shown the need of Working Memristors are 2 WMs. The function is used based on Implication Gate circuit.



Function f(A,B) largest grouping

A \ B	0	1	2
0	-	-	-
1	-	-	2
2	1	-	1

Remainder function r1

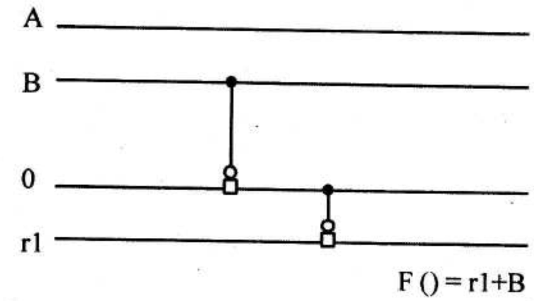


Fig. 2. The Step 1st of function f shown with K-map

In the first step we find the largest group of 1's and all the literals are positives and select this group from this example. In first step we find B at last. It is most positive prime and no include other positive prime covering minterm AB.

B is also realized NAND gate and first we take the input B and passing WM1 then we find B and again passing another working memristor WM2 then the end of the step f(A,B) = r1+B.

We find the remainder function r1, which is shown in figure 2.

In this function we have seen after grouping there is no possible group. In this type of grouping we take always MINTERM form and provide us positive primes so there grouping is possible but we select group according positive primes and select the MINTERM. If we find negative primes then we take inversion of the remainder function r1. So r1 replace by r2 with NOT gate.

The end of the step we find the function f(A,B) = ¬ r2+B. This is inversion of r1 to r2.

In the inverter we used one imply gate it shown in figure 3.

A \ B	0	1	2
0	-	-	-
1	-	-	1
2	2	-	2

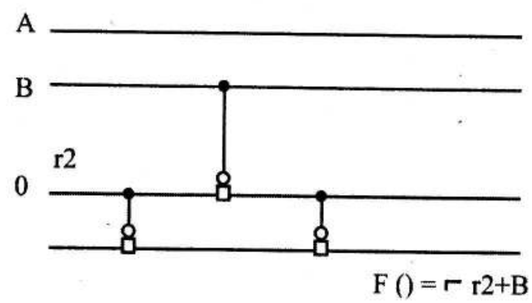


Fig. 3 Inversion of r1 to r2

Now we take the again the largest grouping number of 1's if there are not any group possible then combine don't care with 1's and make new group because we need the positive primes so we make group again and select 2nd row in figure 4 In this figure 4 shown there uses working memristor are 2 Working Memristors based circuits design.

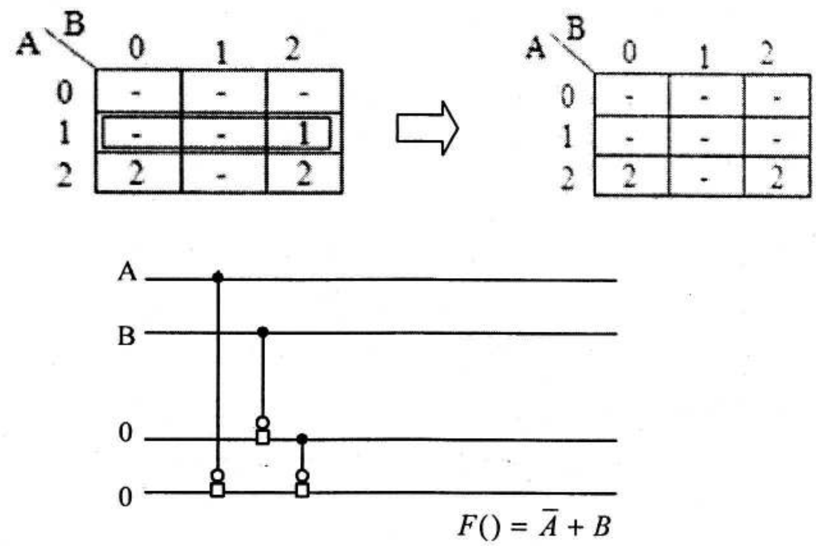


Fig. 4 Final step of the synthesis design

We don't have more 1's in the K-map and also no remainder function, so we can say this is final step of synthesis. Now the final function is

F(A,B) = ¬A+B.

The figure 4 shows two layers with positive primes A and B. So we can say this method is provided better result. Other method use takes all positive primes in a layer. So this is most important method to find the positive primes. In the final step we count the pulses and initialize of the memristor its count automatically one pulse which denoted by 0. If Memristor reused then it count one pulse.

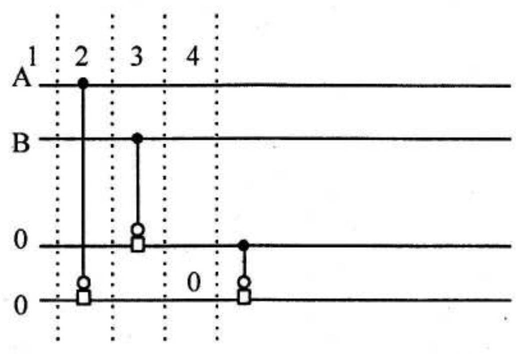


Fig. 5. Pluses counts in ISD notation

In this Ternary logic we have apply imply sequence diagram notation (ISD) and count the pulses. So we have 4 pulses in this synthesis design with ternary logic. So in the Figure 5 have counts the pluses.

VI. Conclusion

In this paper we have used Imply Sequence Diagram (ISD) notation with two Working Memristor. This presented method used for MEMRMIN 2WMs. The create approach an algorithm for minimize the number of pulses for memristor based Ternary logic circuits. These circuits are low cost, size and required power also low. It count the minimize pulses. It is provide better results than ESOP SOP for functions. Our method provided the improvement solution in the Ternary Logic circuits. This method allow don't cares. If the method have higher don't cares then result is better results. This method is better than other methods from [2,8,9,23] . This method also reduce the number of working memristor but ternary logic circuit design become complex. At last final result is the number of pulsed be reduce in the ternary logic and the realization Imply Gate with memristor in ternary logic design. This is first approach in Ternary Logic Synthesis Design.

According to our current research based future work include as following (1) This method can be used for multiple-valued logic. (2) The realization of the POS in Ternary logic Synthesis with working memristor. (3) Reduce the number of pulses and Working Memristors in the bi-decomposition algorithm 8. Ternary logic Imply gate based circuits in the Fuzzy Logic Design.

REFERENCES

[1] J. Borghetti, G. S. Snider, P. J. Kuekes, J. J. Yang, D. R. Stewart and S. R.Williams, "Memristive switches enable 'stateful' logic operations via material implication," Nature, vol. 464, pp. 873-876, 2010.

[2] Raghuvanshi, A.; Perkowski, M., "Logic synthesis and a generalized notation for memristor-realized material implication gates," Computer-Aided Design (ICCAD), 2014 IEEE/ACM International Conference on , vol., no., pp.470,477, 2-6 Nov. 2014.

[3] L.O. Chua, "Memristor-missing circuit element," IEEE Trans. Circuit Theory, Vol. 18, pp. 507-519, 1971.

[4] M. Laiho, and E. Lehtonen, "Arithmetic Operations within Memristor-Based Analog Memory," 12th International Workshop on Cellular Nanoscale Networks and their Applications (CNNA) 2010, Nanotechnology 23 (2012) 305205 (6pp).

[5] M. Klimo, O. ?uch, "Memristors can implement fuzzy logic," Oct. 2011.

[6] D.B. Strukov, G.S. Snider, D.R. Steward, and R.S.Williams, "The missing memristor found," Nature, Vol. 453, pp.80-83, 2008.

[7] S. Kvatinsky, E. G. Friedman, A. Kolodny, and U.C. Weiser, "Memristor-based Material Implication (IMPLY) Logic: design Principles and Methodologies," IEEE Trans. Comput. pp. 1-15, 2013.

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