Volume 28 (June & Dec. 2014) ISSN 0970-9169

GANIṆITA SANDESH

गणित संदेश

A Half Yearly International Research Journal
of

Rajasthan Gaṇita Parishad

Registered Head Office
Department of Mathematics
SPC Government College, AJMER - 305 001 (*INDIA*)

Website : www.rgp.co.in
E-mail : rgp@rgp.co.in

Issued Dec., 2016

E-mail : editor@rgp.co.in
E-mail : cmtr@rgp.co.in

GANṆITA SANDESH

गणित संदेश

Notes For Contributors

1. The editors will be pleased to receive contributions from all parts of India / abroad in any area of Mathematics (Research / Teaching etc.).
2. Manuscripts for publication should be sent through E-Mail directly to the editor or **rgp@rgp.co.in** along with hard copy in Triplicate duly computerised with double spacing preferably in Adobe Pagemaker 6.5, with text in Times New Roman font 11 pts. and Mathematical symbols (Math Type, Equation editor or Corel Equation).
3. Authors should provide abstract and identify 4 to 5 key words for subject classification.
4. Unduly long papers and papers with many diagrams/tables will not be normaly accepted. In general, length of the accepted paper should not exceed 10 printed pages.
5. The contributors are required to meet the partial cost of publication at the rate of ` 200/- or equivalent US $ per printed page size A4 (even number of pages) payable in advance on receipt of acceptance of their paper.

Membership Fees / Subcription Rates

	Period	In India (₹)	Outside India (US $)
❑ Admission / Enrolment Fee	First Time Only	100	100 (or equivalent)
❑ Life Membership		2,000	2,000 (or equivalent)
❑ Annual Membership fee for Teachers (Colleges / Universities), T.R.F. Registered Research Scholars	Financial year	250	250 (or equivalent)
❑ Educational / Research Institutions	Calendar year	500	500 (or equivalent)

❈ All payments must be made by Bank **D.D.** in favour of ***Rajasthan Ganita Parishad*** payable at **AJMER** or online **State Bank of India Account No. 10200807636, IFSC Code : SBIN0000603** under intimation to the **Treasurer, Rajasthan Ganita Parishad**, *Department of Mathematics,* SPC Government College, AJMER-305 001 (INDIA)

Ganita Sandesh Vol. 28 No. 1 and 2 (June and December, 2014) pp. 1 - 4 ISSN : 0970-9169

The 21st Century Definition of Mathematics

L. Radhakrishna

U.G.C. Center for Advanced Study, Dept of Mathematics

Bangalore University, BANGALORE 560001 (India)

Email : lrkwmr@gmail.com

Abstract

We review the three standard definitions of mathematics of the 20th century, along with their warring philosophies. The limitations of these standard philosophies are explained. To overcome these limitations, Davis and Hersh (1981) have designed a new philosophy called "Mathematical Humanism". This philosophy gives the latest definition of mathematics—the subject matter of this paper. Later, Hersh (1997, 2001) has made substantial study of the philosophy and the definition.

During the World Mathematics Year 2000 celebrations, the Fields Medalist David Mumford, in his address on Mathematics towards the Third Millennium announced: I love this definition of Hersh [Arnold et al.2000]. We call it as the 21st century definition of mathematics and give an updated exposition to benefit the researchers.

1. Introduction

Pythogoras (580-500 B.C.) invented the words *mathematics* (in Latin, it means that *which can be learnt*) and *philosophy (philo:* love, and *Sophia*: knowledge). In the 20th century, three philosophies of mathematics have been identified to correspond to three definitions (*vide* Section 2). The details of limitations of these philosophies — popularly referred as the standard philosophies— are cited in Sec.2. To overcome the difficulties Davis and Hersh (1981) exploited the Philosopher Karl Popper's classification of distinct realities.

Three major levels of distinct realities (Popper 1971) and the world of mathematics:

World 1: (Physical world) this is the world of mountains and valleys, of mass and energy, of bone and blood, and of stars and galaxies.

World 2: (Individual World) this is the world of emotions, thoughts, and awareness.

World 3: (Social world) this is the world of languages, traditions, social institutions. Mathematics belongs to World 3, an essential part of our non-material culture!

Mathematical Universe Hypothesis (MUH): In 2007, cosmologist Max Tegmark proposed MUH, which states that "our physical reality is a mathematical structure, and that our universe is not just described by mathematics—it is just mathematics." Hence Popper's and Tegmark's work shows how huge is the spell of mathematics. In the next section we will show how this huge spell will be restricted by the 20th century's narrow philosophies.

2. The Three Standard Philosophies and Definitions of Mathematics in the 20th Century

(A1) The Philosophy of Mathematical Platonism (Realism) states that the whole of mathematics exists externally and independently of human mind. Mathematicians discover what is already there in the universe. Most (65%) of the researchers are Platonists— applied mathematicians!

(A2) The definition of mathematics according to Platonism: *Mathematics is the exploration of the pre-existing world.* For example (i) The abstract structure 'smallest non-Abelian group' is discovered in the six symmetries of ammonia molecule (ii) straight line is discovered in the path of a light ray, (iii) Ellipse is discovered in the planetary orbit around the sun (iv) Fibonacci sequence is discovered in Phyllotaxis (Leaf arrangement in trees) via Schipp's formula, (v) Geometric progression is found in stimuli and discrimination of (rose) smell (vi) Matrix is found in the arrangement of influence of a politician [Radhakrishna 2013].

(B1) The Formalist Philosophy of Mathematics states that mathematics is the creation of the human mind and so it is invented. This is against the spirit of Platonism.

(B2) The Formalist Definition of Mathematics: Mathematics is the science of rigorous proof. Any logical proof must have a starting point. So one starts with undefined terms, and some unproved statements about these terms called axioms. Results obtained by valid logical deductions from these axioms are , called theorems i.e., formulas. In short, mathematics is the science of formal deductions from axioms to theorems. The theorems are neither true, nor false, as logic started with undefined terms. Venn diagrams, geometrical figures are not mathematical. There is no meaning and no intuition in mathematics. Logic rules the formalist mathematics.

(C1) The Philosophy of Constructivism or Intuitionism: The natural numbers

$$1, 2, 3, \ldots n, \ldots \text{ to } \infty$$

are given to us by intuition, which is the starting point for all mathematics. Only those mathematical objects are meaningful and exist, which can be constructed from the natural numbers.This is the philosophy of computational mathematics, and computer science. The number of keys on a key board of a computer is finite and so the emphasis on finiteness.

(C2) The Constructivist Definition of Mathematics: Starting from the natural numbers, all that can be constructed in a finite number of steps, is (genuine) mathematics.

Spurious mathematics: Real numbers cannot be constructed in a finite number of steps; for example the real number 'e', uses infinity "∞" in its construction

$$e = \lim_{n \to \infty} (1 + 1/n)^n$$

Thus constructivists avoid real numbers, consequently real analysis, complex and functional analysis. Also they do not allow indirect proofs of theorems, contrary to Formalists and Platonists.

3. The 21st Century Philosophy and Definition of Mathematics

The avoidance of real numbers by Constructivists, the lack of meaning to symbols by Formalists and the belief in the infallibility of mathematics by Platonists, have prompted Davis and Hersh to find a new philosophy of mathematics, that could overcome the limitations of the standard philosophies. They invented MATHEMATICAL HUMANISM as the new philosophy and the corresponding new definition of mathematics.

The 21st Century Philosophy of Mathematical Humanism: Mathematics is a purposeful activity connected to other fields of human activity. Mathematics is meaningful, fallible and rectifiable.

The 21st Century (Humanist) Definition of Mathematics:

Mathematics is the study of ***mental*** *objects with* ***reproducible*** *properties.* ...(3.1)

In algebra the mental objects are the elements of a set and the binary operation. In calculus functions and their derivatives are the mental objects.

An interesting observation on the 21st Century Definition:

(1) If we replace 'mental' in (3.1) with 'physical', we get the definition of SCIENCE.

(2) If we replace 'reproducible' in (3.1) by 'irreproducible', we get the definition of ARTS.

Statement (3.1) answers the query "What types of communications are called mathematics?"

4. Conclusion

Foundations of mathematics are not taught in Indian universities. So research scholars are not exposed to 'philosophy (Ph) of mathematics'. Obviously we produce Ph.D.s without Ph. In fact, philosophy gives direction to research. This situation can be redeemed with the help of recently published book [Radhakrishna 2013].

REFERENCES

[1] Arnold et al.(Editors) *Mathematics: Frontiers and Prospects,* American Mathematical Society, 2000

[2] Davis P.J., and Hersh R., *The Mathematical Experience,* Cambridge Birkhauser, 1981

[3] Hersh R., *What is Mathematics Really,* Oxford University Press, 1977
Later, 'Philosophical Perspectives in Mathematics Education' in Grinstein L. S. and Lipsey S. I. (Eds) *Encyclopaedia of Mathematics Education,* Rautledge Falmer 2001

[4] Max Tegmark, *Mathematical Universe Hypothesis,* in Clifford A. Pickover *The Math Book,* Sterling, 2009

[5] Radhakrishna L. *Write Mathematics Right: Principles of Professional Presentation, Exemplified with Humor and Thrills,* Narosa (India), Alpha Science International (Oxford U.K.) 2013

Ganita Sandesh Vol. 28 No. 1 and 2 (June and December, 2014) pp. 5 - 12 ISSN : 0970-9169

Integral Representations of Euler- Type for Kampe' de Fe'riet Functions of the Fourth Order

Anvar Hasanov [1], **Maged G. Bin Saad** [2*] **Rakhila B. Seilkhanova** [3]

[1] Institute of Mathematics, National University of Uzbekistan, 29 Durmon yuli street, Tashkent 700125, Uzbekistan
E-mail: anvarhasanov@yahoo.com

[2*] Department of Mathematics, Aden University, Aden, Kohrmakssar , P.O.Box 6014, Yemen
E-mail:mgbinsaad@yahoo.com, * Corresponding author

[3] S. Baishev University Aktobe, Department for Information Systems and Applied Mathematics, 030000 Aktobe, Zhubanov Str. 302, Kazakhstan.
E-mail:srahila@inbox.ru

Abstract

We obtain here certain integral representations for functions related to Kampe' de Fe'riet function of the fourth order ,which are the sufficiently general in nature and are capable of yielding a large number of simpler and useful results merely by specializing the parameters in them.

MSC. Primary 33C20, 33C65

Keywords. Hyper-geometric series, Kampe' de Fe'riet function of the fourth order, Eulerian integrals, Beta functions, Appell functions.

1. Introduction

Many special functions appear as solutions of ***differential equations*** or ***integrals*** of ***elementary functions.*** Therefore, tables of integrals usually include descriptions of special functions, and tables of special functions include most important integrals; at least, the integral representation of special functions. Because

Dedicated to Professor M.A.Pathan on his 75th birth anniversary symmetries of differential equations are essential to both physics and mathematics, the theory of integral representations is closely related to the theory of special functions as well as certain topics in ***mathematical physics.***

In this paper we consider Eulerian integral formulas of first kind and obtained a number of integral representations for functions related to Kampe' de Fe'riet function of the fourth order. For evaluations and extensions of results on Euler type integrals, we refer a paper [11]. A great interest in the theory of hypergeometric functions (that is, hyper-geometric functions of one , two and several variables) is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [10, p. 47-48]). Also, in this regard, it is noticed that the general sextic equation can be solved in terms of Kampe' de Fe'riet function (see [2] and [8]). Although the integrals involving and representing hypergeometric functions have numerous applications in pure and applied mathematics (see, for example, [4]-[7]), not all such integrals have been collected in tables or are readily available in the mathematical literature. It is noted that a few integrals involving functions related to Kampe' de Fe'riet function of two variables annexed those mathematical literature.

The Kampe' de Fe'riets hyper-geometric series of two variables $F_{l;m;n}^{p;q;k}$ (see [9] and [10]) is defined as follows:

$$F_{l;m;n}^{p;q;k}\left[\begin{matrix}(a_p);(b_q);(c_k);\\(\alpha_l);(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s\,r!s!}x^r y^s, \quad \ldots(1.1)$$

where for convergence

$$\left.\begin{matrix}p+q<l+m+1,\ p+k<l+n+1,\ |x|<\infty,\ |y|<\infty\\ \text{or}\\ p+q<l+m+1,\ p+k<l+n+1,\ |x|,<\infty,\ |y|<\infty,\text{and}\\ |x|^{\frac{1}{p-1}}+|y|^{\wedge\frac{1}{p-1}}<1,\text{if } p>l;\ \max\{|x|,|y|\}<1,\ if\ p<l\end{matrix}\right\} \quad \ldots(1.2)$$

where $\prod_{j=1}^{p}(a_j)_{r+s}=(a_1)_{r+s}(a_2)_{r+s}\ldots\ldots(a_p)_{r+s}$, with similar interpretations for $\prod_{j=1}^{l}(\alpha_j)_{r+s}$, , et cetera and $(a)_n$ denotes the Pochhammer symbol given by $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}, (a)_0=1$

and Γ being the well-known Gamma function. The Kampé de Fériet's function (1.1) being the most general hypergeometric function of two variables, this is because the Kampé de Fériet function reduces to the product of two generalized hypergeometric functions of one variable by choosing parameters suitably. It is often convenient to identify the various functions with integral representations. These integrals provide recursion formulas, asymptotic forms and analytic continuations of the special functions. In this paper we establish 17 Euler integral representations for Kampé de Fériet's functions of double series of the fourth order.

2. Integral Representations

First, we recall the definition of Euler integral of the first kind (the Beta function) [3]:

$$B\left(a,b\right)=\int_0^1 t^{a-1}\left(1-t\right)^{b-1}dt=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\qquad \operatorname{Re}(a)>0,\ \operatorname{Re}(b)>0. \qquad \dots (2.1)$$

By making a simple application of (2.1), we begin by presenting each of the following integral representations (2.2)–(2.19).

Theorem. Each of the following integral representations for Kampe' de Fe'riet functions holds true.

$$F_{1;2,2}^{2;2,2}\left[\begin{matrix}a,b;\ c,d;\ c_1,d_1;\\ e;\ f,g;\ f_1,g_1;\end{matrix}x,y\right] =\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\int_0^1 \xi^{a-1}\left(1-\xi\right)^{e-a-1}F_{0;2,2}^{1;2,2}\left[\begin{matrix}b;\ c,d;\ c_1,d_1;\\ -;\ f,g;\ f_1,g_1;\end{matrix}x\xi,y\xi\right]d\xi, \qquad \dots (2.2)$$

$$\operatorname{Re}(e)>\operatorname{Re}(a)>0,$$

$$F_{1;2,2}^{2;2,2}\left[\begin{matrix}a,b;\ c,d;\ c_1,d_1;\\ e+e_1;\ f,g;\ f_1,g_1;\end{matrix}x,y\right] =\frac{\Gamma(e+e_1)}{\Gamma(e)\Gamma(e_1)}\int_0^1 \xi^{e-1}\left(1-\xi\right)^{e_1-1}F_{0;3,3}^{2;2,2}\left[\begin{matrix}a,b;\ c,d;\ c_1,d_1;\\ -;e,f,g;e_1,f_1,g_1;\end{matrix}x,y\right]d\xi, \qquad \dots (2.3)$$

$$\operatorname{Re}(e)>0,\operatorname{Re}(e_1)>0.$$

$$F_{2;1,1}^{2;2,2}\left[\begin{matrix} a,b;c,d;c_1,d_1; \\ e+e_1,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e+e_1)}{\Gamma(e)\Gamma(e_1)}\int_0^1 \xi^{e-1}(1-\xi)^{e_1-1} F_{1;2,2}^{2;2,2}\left[\begin{matrix} a,b;c,d;c_1,d_1; \\ f;e,g;e_1,g_1; \end{matrix} x\xi, y(1-\xi)\right]d\xi, \quad \ldots (2.4)$$
$$\mathrm{Re}(e)>0, \mathrm{Re}(e_1)>0,$$

$$F_{2;1,1}^{2;2,2}\left[\begin{matrix} a,b;c,d;c_1,d_1; \\ e+e_1,f+f_1;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e+e_1)}{\Gamma(e)\Gamma(e_1)}\frac{\Gamma(f+f_1)}{\Gamma(f)\Gamma(f_1)}\int_0^1\int_0^1 \xi^{e-1}\eta^{f-1}(1-\xi)^{e_1-1}(1-\eta)^{f_1-1} \quad \ldots (2.5)$$
$$\times F_{0;3,3}^{2;2,2}\left[\begin{matrix} a,b;\quad c,d;\quad c_1,d_1; \\ -;e,f,g;e_1,f_1,g_1; \end{matrix} x\xi\eta, y(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\mathrm{Re}(e)>0, \mathrm{Re}(e_1)>0, \mathrm{Re}(f)>0, \mathrm{Re}(f_1)>0,$$

$$F_{2;1,1}^{2;2,2}\left[\begin{matrix} a,b;c,d;c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\int_0^1 \xi^{a-1}(1-\xi)^{e-a-1} F_{1;1,1}^{1;2,2}\left[\begin{matrix} b;c,d;c_1,d_1; \\ f;\quad g;\quad g_1; \end{matrix} x\xi, y\xi\right]d\xi, \quad \ldots (2.6)$$
$$\mathrm{Re}(e)>\mathrm{Re}(a)>0,$$

$$F_{2;1,1}^{2;2,2}\left[\begin{matrix} a,b;c,d;c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\frac{\Gamma(f)}{\Gamma(b)\Gamma(f-b)} \quad \ldots (2.7)$$
$$\times\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}(1-\xi)^{e-a-1}(1-\eta)^{f-b-1} F(c,d;g;x\xi\eta)F(c_1,d_1;g_1;y\xi\eta)d\xi d\eta,$$
$$\mathrm{Re}(e)>\mathrm{Re}(a), \mathrm{Re}(f)>\mathrm{Re}(b),$$

$$F_{2;1,1}^{2;2,2}\left[\begin{matrix} a,b;c,d;c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(c+c_1)}{\Gamma(c)\Gamma(c_1)}\int_0^1 \xi^{c-1}(1-\xi)^{c_1-1} F_{2;1,1}^{3;1,1}\left[\begin{matrix} a,b,c+c_1;d;d_1; \\ e,f;g;g_1; \end{matrix} x\xi, y(1-\xi)\right]d\xi \quad \ldots (2.8)$$
$$\mathrm{Re}(c)>0, \mathrm{Re}(c_1)>0,$$

$$F_{2;1,1}^{2;2,2}\left[\begin{matrix} a,b;c,d;c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(c+c_1)}{\Gamma(c)\Gamma(c_1)}\frac{\Gamma(d+d_1)}{\Gamma(d)\Gamma(d_1)}\int_0^1\int_0^1 \xi^{c-1}\eta^{d-1}(1-\xi)^{c_1-1}(1-\eta)^{d_1-1} \quad \text{... (2.9)}$$
$$\times F_{2;1,1}^{4;0,0}\left[\begin{matrix} a,b,c+c_1,d+d_1;-;-; \\ e,f;g;g_1; \end{matrix} x\xi\eta, y(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\mathrm{Re}(c)>0, \mathrm{Re}(c_1)>0, \mathrm{Re}(d)>0, \mathrm{Re}(d_1)>0,$$

$$F_{2;1,1}^{2;2,2}\left[\begin{matrix} a,b;c,d;c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(g_1)}{\Gamma(d_1)\Gamma(g_1-d_1)}\int_0^1 \xi^{d_1-1}(1-\xi)^{g_1-d_1-1} F_{2;1,0}^{2;2,1}\left[\begin{matrix} a,b;c,d;c_1; \\ e,f;\quad g;-; \end{matrix} x, y\xi\right]d\xi, \quad \text{... (2.10)}$$
$$\mathrm{Re}(g_1)>\mathrm{Re}(d_1)>0,$$

$$F_{2;1,1}^{2;2,2}\left[\begin{matrix} a,b;c,d;c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(g)}{\Gamma(d)\Gamma(g-d)}\frac{\Gamma(g_1)}{\Gamma(d_1)\Gamma(g_1-d_1)} \quad \text{... (2.11)}$$
$$\int_0^1\int_0^1 \xi^{d-1}\eta^{d_1-1}(1-\xi)^{g-d-1}(1-\eta)^{g_1-d_1-1} F_{2;0,0}^{2;1,1}\left[\begin{matrix} a,b;\ c;c_1; \\ e,f;-;-; \end{matrix} x\xi, y\eta\right]d\xi d\eta,$$
$$\mathrm{Re}(g)>\mathrm{Re}(d)>0, \mathrm{Re}(g_1)>\mathrm{Re}(d_1)>0,$$

$$F_{3;0,0}^{3;1,1}\left[\begin{matrix} a,b,c;\ d;d_1; \\ e,f,g;-;-; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\frac{\Gamma(f)}{\Gamma(b)\Gamma(f-b)}\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1} \quad \text{... (2.12)}$$
$$\times(1-\xi)^{e-a-1}(1-\eta)^{f-b-1} F_1(c;d,d_1;g;x\xi\eta, y\xi\eta)d\xi d\eta,$$
$$\mathrm{Re}(e)>\mathrm{Re}(a)>0,\ \mathrm{Re}(f)>\mathrm{Re}(b)>0,$$

$$F_{3;0,0}^{3;1,1}\left[\begin{matrix} a,b,c;\, d;d_1; \\ e,f,g;-;-; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(d+d_1)}{\Gamma(d)\Gamma(d_1)}\int_0^1 \xi^{d-1}(1-\xi)^{d_1-1}\,{}_4F_3\left(a,b,c,d+d_1;e,f,g;x\xi+y(1-\xi)\right)d\xi, \qquad \dots (2.13)$$

$$\operatorname{Re}(d)>0, \operatorname{Re}(d_1)>0,$$

$$F_{1;2,2}^{4;0,0}\left[\begin{matrix} a,b,c,d; & -; & -; \\ e; & f,g; & f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\int_0^1 \xi^{a-1}(1-\xi)^{e-a-1}\,F_{0;2,2}^{3;0,0}\left[\begin{matrix} b,c,d; & -; & -; \\ -; & f,g; & f_1,g_1; \end{matrix} x\xi, y\xi\right]d\xi, \qquad \dots (2.14)$$

$$\operatorname{Re}(e)>\operatorname{Re}(a)>0,$$

$$F_{1;2,2}^{4;0,0}\left[\begin{matrix} a,b,c,d; & -; & -; \\ e+e_1; & f,g; & f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(e+e_1)}{\Gamma(e)\Gamma(e_1)}\int_0^1 \xi^{e-1}(1-\xi)^{e_1-1}\,F_{0;3,3}^{4;0,0}\left[\begin{matrix} a,b,c,d; & -; & -; \\ -; & e,f,g; & e_1 f_1,g_1; \end{matrix} x\xi, y(1-\xi)\right]d\xi, \qquad \dots (2.15)$$

$$\mathrm{R}(\mathrm{e})>0, \mathrm{R}(\mathrm{e}_1)>0,$$

$$F_{2;1,1}^{4;0,0}\left[\begin{matrix} a,b,c,d;-;-; \\ e,f;\, g;g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\int_0^1 \xi^{a-1}(1-\xi)^{e-a-1}\,F_{1;1,1}^{3;0,0}\left[\begin{matrix} b,c,d;-;-; \\ f;\, g;g_1; \end{matrix} x\xi, y\xi\right]d\xi, \qquad \dots (2.16)$$

$$\operatorname{Re}(e)>\operatorname{Re}(a)>0,$$

$$F_{2;1,1}^{4;0,0}\left[\begin{matrix} a,b,c,d;-;-; \\ e,f;\, g;g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\frac{\Gamma(f)}{\Gamma(b)\Gamma(f-b)}\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1} \qquad \dots (2.17)$$

$$\times(1-\xi)^{e-a-1}(1-\eta)^{f-b-1}F_4\left(c,d;g,g_1;x\xi\eta, y\xi\eta\right)d\xi d\eta,$$

$$\operatorname{Re}(e)>\operatorname{Re}(a)>0, \operatorname{Re}(f)>\operatorname{Re}(b)>0,$$

$$F^{4;0,0}_{2;1,1}\left[\begin{matrix}a,b,c,d;-;-;\\e+e_1,f;g;g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(e+e_1)}{\Gamma(e)\Gamma(e_1)}\int_0^1 \xi^{e-1}(1-\xi)^{e_1-1}F^{4;0,0}_{1;2,2}\left[\begin{matrix}a,b,c,d; & -; & -;\\ & f;\,e,g;e_1,g_1;\end{matrix}x\xi,y(1-\xi)\right]d\xi, \quad \ldots(2.18)$$
$$\mathrm{Re}(e)>0,\mathrm{Re}(e_1)>0,$$

$$F^{4;0,0}_{2;1,1}\left[\begin{matrix}a,b,c,d;-;-;\\e+e_1,f+f_1;g;g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(e+e_1)}{\Gamma(e)\Gamma(e_1)}\frac{\Gamma(f+f_1)}{\Gamma(f)\Gamma(f_1)}\int_0^1\int_0^1 \xi^{e-1}\eta^{f-1} \quad \ldots(2.19)$$
$$\times(1-\xi)^{e_1-1}(1-\eta)^{f_1-1}F^{4;0,0}_{0;2,2}\left[\begin{matrix}a,b,c,d; & -; & -;\\ & -;\,e,g;e_1,g_1;\end{matrix}x\xi\eta,y(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\mathrm{Re}(e)>0,\mathrm{Re}(e_1)>0,\mathrm{Re}(f)>0,\mathrm{Re}(f_1)>0,$$

Proof. It is noted that each of the integral representations (2.2) to (2.19) can be proved directly by expressing the series definition of the involved special function in each integrand and changing the order of the integral sign and the summation, and finally using the Beta function $B(a,b)$ defined by (2.1).

We conclude this paper by remarking that by assigning suitable special values to the coefficients in (2.2) to (2.19), we can derive integral representations for Appell functions of two variables F_1, F_2, F_3 and F_4 (see [1] and [10]). The details involved in these derivations are fairly straightforward and are being left as an exercise for the interested reader. Also, the Euler ***integral*** of the first kind (2.1) can be applied to in order to establish other integral representations for more functions related to Kampe' de Fe'riet function of the fourth order.

REFERENCES

[1] Appell, P. and Kampe'de Fe'riet J., (1926)Fonctions hypergeometriques et hyperspheriques; polynomes d'Hermite, Gauthier - Villars, Paris.

[2] Coble A. B., (1911),The Reduction of the Sextic Equation to the Valentiner Form-- Problem , Math. Ann. 70, 337-350.

[3] Erde'lyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G., (1953), Higher Transcen- dental Functions, Vol. I, McGraw-Hill Book Company, New York, Toronto and London.

[4] Exton, H. (1978), Handbook of hypergeometric integrals, Mathematics and its Applications, Chichester: Ellis Horwood Ltd.

[5] Gradshteyn, I . S . and Ryzhik , I . M . ,(1994),Table of Integrals , Series and product, Academic Press , New York .

[6] Gradshteyn, I . S . and Ryzhik , I . M . ,(2000), Table of Integrals , Series and product, Academic Press , New York .

[7] Prudnikov, A. P. , Bryehkov, Y. A. and Mariehev, O. I.,(1986), Integrals and Series, vol. 2, Special functions, Gordon and Breach, New York.

[8] Sharma K., (2014), On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences , Vol 3 , 1-13.

[9] Srivastava H. M. and Panda R., (1976),An integrals representing for the product of two Jacobi polynomials , J. London Math. Soc. (2) 12, 419-425.

[10] Srivastava, H. M. and Karlsson P. W.,(1985), Multiple Gaussian hypergeometric series, Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York, Chichester, Brisbane, and Toronto.

[11] Subuhi Khan, Babita Agrawal, M.A.Pathan and Fahim Mohammad,(2007), Evaluation of certain Euler type integrals, Applied Mathematics and Computation,189,1993-2003

Ganita Sandesh Vol. 28 No. 1 and 2 (June and December, 2014) pp. 13 - 24 ISSN : 0970-9169

Applications of Mathematics in Real Life Phenomena: Chaos and Complexity in One Dimensional Systems

By

L. M. Saha[1] **and Til Prasad Sarma**[2]

[1]IIIMIT, Shiv Nadar University, GAUTAM BUDH NAGAR (U.P.) 201314

E-mail: lmsaha.msf@gmail.com

[2]Department of Education in Science and Mathematics, N C E R T, Sri Aurobindo Marg, NEW DELHI -110 016

E-mail: tpsarma@yahoo.com

Abstract

Real natural systems evolving around us are nonlinear in nature and their dynamics are not as simple as in cases of linear systems. Also, the ***Euclidean Geometry*** *invented by Euclid during 300 years B C is not enough to explain the shape and dimension of any real systems. Recent emergence of* ***Fractal Geometry****, suggested by Mandelbrot, is capable to describe a lot regarding the shape and dimension of real systems. Because of nonlinearity, real systems show complexities in behavior while evolving and chaos is one such complexity. Principles of nonlinear dynamics can only help to understand complex and chaotic behaviors observed in any nonlinear system.*

In this paper evolutionary phenomena of someone dimensional real systems have been discussed and complexities involved are discussed to understand the property nonlinearity within the system. Numerical simulations have been performed to explain complexities and chaotic motion in these systems. Specific models proposed are :simple models of spreading measles, cavity evolution of an external laser, blood cell evolution model of biology and *gross national product (GNP) model of economics. Measures of Lyapunov exponents (LCEs) and topological entropies are obtained as a measure of complexity and, are demonstrated through graphics and tables and interpreted completely. Results obtained are explained with complete interpretation.*

Keywords: Chaos, Lyapunov Exponent, Dynamic Lyapunov Indicator, Topological Entropy

AMS Subject classification: 92D40

1. Introduction

Almost all real systems that we observe around us are of evolutionary behavior and if we can write an appropriate mathematical model (Equation) for any such a system, we can describe its past and future behavior. As such systems are mostly nonlinear; one has to follow the Principles of Non-Linear Dynamics. For nonlinear system, we do not have any principle like *principle of superposition* that is true for linear systems. Nonlinearity in real system is a property of the system described by the parameters involved into it. As these parameters changes in values, one observes phenomena like bifurcation and chaos during evolution.

Chaos and complexities in real natural nonlinear systems are now very common phenomena. A system is chaotic if it shows unpredictability during evolution and becomes very sensitive to initial conditions. Two orbits originating nearby show divergence property after long term evolution. Chaos is measured by Lyapunov exponents denoted by, λ; actually this λ measures the evolutionary divergence between two orbits started with very small difference in initial conditions, (cf. [1] – [5]). A simple system evolves in simple ways but a complex or complicated system evolve in complicated ways and between simplicity and complexity there cannot be a common ground [6]. Chaos and irregular phenomena may not require very complicated equations.

Complexity in a dynamical system can be viewed as its systematic nonlinear properties. It is the order that results from the interaction among multiple agents within the system. A system is complex means its evolutionary behavior do not show regularity but chaotic or some other kind of irregularity [7 – 10]. Complexity and chaos observed in a system can well be understood by measuring elements like Lyapunov exponents (LCEs) and topological entropies etc. Topological entropy, a non-negative number, provides a perfect way to measure complexity of a dynamical system. For a system, more topological entropy means the system is more complex. Actually, it measures the exponential growth rate of the number of distinguishable orbits as time advances [11 -

15]. Though, positivity measure of Lyapunov exponents (LCEs) signifies presence of chaos, LCEs and topological entropies together provide measure of complexities in the system.

A complex system can be viewed as that of composed of many components which may interact with each other. A complex system exhibits some (and possibly all) of the following characteristics: (i) some degree of spontaneous order, (ii) robustness of the order and (iii) numerosity. Topological entropy provides the measure of complexity; more topological entropy implies the system is more complex. Actually, a topological entropy measures the exponential growth rate of the number of distinguishable orbits as time advances in the system. However positivity of its value does not justify the system be chaotic [16, 17]. Correlation Dimension provides the dimensionality of the system. It is a kind of fractal dimension and its numerical value is always non-integer [18 – 20].

The objective of the present article is to study complexity in one dimensional discrete nonlinear dynamical system and to obtain certain measure of such complexity like Lyapunov exponents and topological entropies.

2. Tools to Measure Complexity

(a) Lyapunov Exponents:

The Lyapunov exponent, (or Lyapunov characteristic exponent LCE), provides an average measure of exponential divergence of two orbits initiated with infinitesimal separation. The +velargest eigenvalue of a complex dynamical system is an indicator of chaos, [10].For a smooth map f on R^n and x_0 an initial point the Lyapunov exponent can be calculated as follows:

Two trajectories in phase space with initial separation δx_0 diverge (provided that the divergence can be treated within the linearized approximation) when

$$|\delta x(t)| \approx e^{\lambda t} |\delta x(0)| \qquad \text{...(2.1)}$$

where $\lambda > 0$ is the Lyapunov exponent.

The system described by the map f be *regular* as long as $\lambda \leq 0$ and *chaotic* when $\lambda > 0$.

(b) Topological Entropies:

Consider a finite partition of a state space X denoted by $P = \{ A_1, A_2, A_3, \ldots, A_N\}$. Then a measure μ on X with total measure $\mu(X) = 1$ defines the probability of a given reading as

$$p_i = \mu(A_i)\,,\ i = 1, 2, \ldots, N.$$

Then the entropy of the partition be given by

$$H(p) = -\sum_{i=0}^{N} p_i \log p_i \qquad \ldots (2.2)$$

3. Dynamic Models

(a) Logistic Model as Epidemic Spreading:

Logistic model, often used for spread of epidemic infections such as flue, measles etc., represented by map

$$x_{n+1} = x_n + r(1 - x_n)\,x_n \qquad \ldots (3.1)$$

where the x_n represents the number of infectious individuals after n time steps (e.g. days), r is the rate of spreading of the disease. The model (3.1) is derived from the actual population model, (cf.[22, 23]).

The system (1) has fixed points as $x_1^* = 0$ and $x_2^* = 1$. Using stability analysis one obtains that the fixed point $x_1^* = 0$ is stable when $-2 < r < 1$ and $x_2^* = 1$ is stable when $0 < r < 2$.Thus steady state of period one exists for $r < 2$ and at $r = 2$, system bifurcates and one observes period doubling bifurcations leading to chaos as r further increases as shown in bifurcation diagram, Fig. 1 (left figure). The right figure shows an iterative approach of steady state.

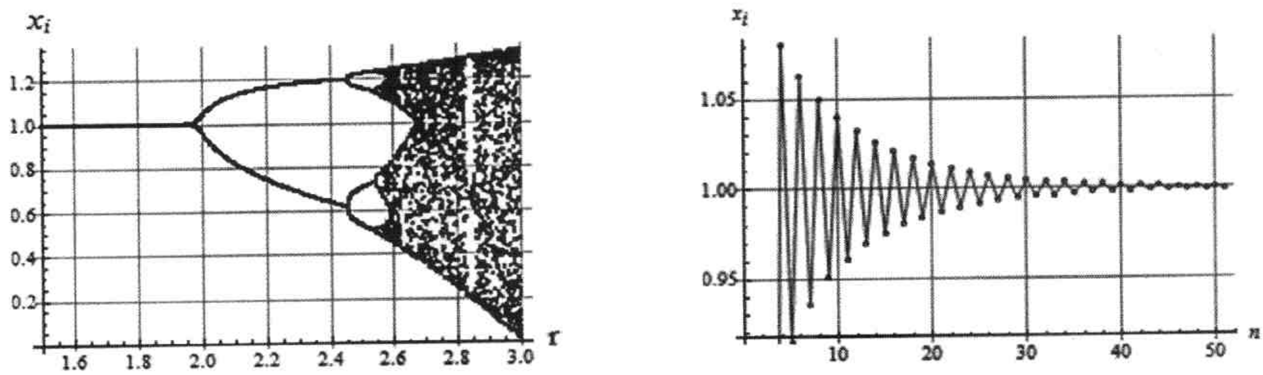

Fig. 1: Bifurcation diagram of system (3.1) for $1.5 \le r \le 3$.

The right hand figure shows an iterative approach of steady state.

Calculations have been performed to obtain the Lyapunov exponents, LCEs, and topological entropies and represented graphically, Fig. 2. LCEs are obtained for $1.5 \le r \le 3$, Fig. 2 (a) and topological entropies for $2.1 \le r \le 3$, Fig. 2 (b).Looking carefully these two figures one easily gets impression that complexity may happen where the system does not show chaos.

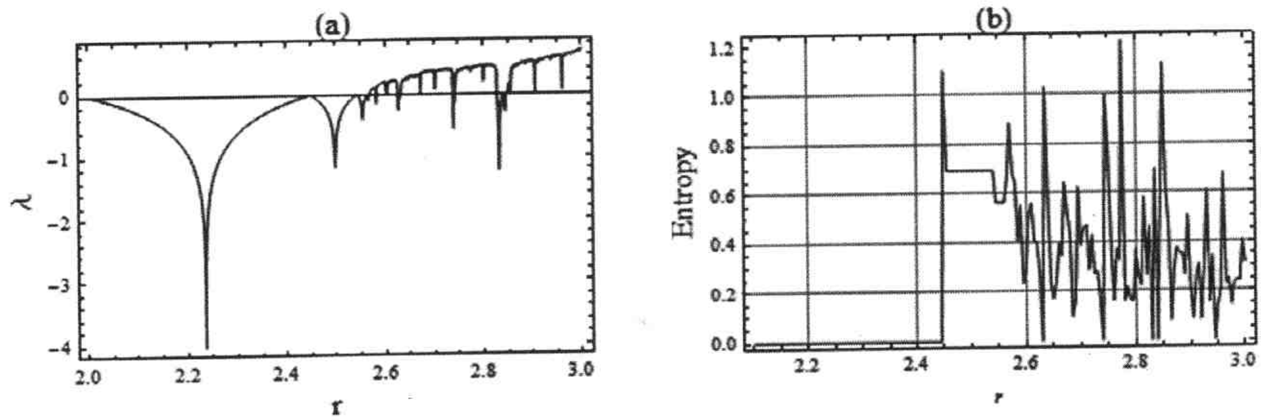

Fig. 2: Plot (a) is represents Lyapunov exponents for $1.5 \le r \le 3$ and plot (b) represents topological entropy for $2.1 \le r \le 3$.

(b) Cavity Evolution of an External Laser:

A highly simplified type discrete model for laser system, arising from Laser Physics [24–26], was described in some articles. The model describes evolution of certain Fabry-Perot cavity containing asaturable absorber and driven by an external laser and represented by

$$x_{n+1} = Q - \frac{A\, x_n}{1+x_n^2}, \ \forall \in \Re,\ n \in N \qquad ...(3.2)$$

Where Q is the normalized input field and A is a parameter depends on the specifics of the parameters and is always positive. As the model describes bistability character due to variation of values of A, it may be referred as bistability parameter also. The parameter Q, also describe similar criteria when varied which can be seen during numerical simulation. The variable x_n stands for the normalized field in the cavity at time t. Mathematical discussions provide in details regarding stability and bistability or multistability of the model in various ranges of parameter[25]. Recently, multistabilityin this model shows some interesting results in the processes of chaotic evolution [26].

The objective of study of this model is to see dynamical aspects of evolution of this map in parameter space (A, Q). Numerical investigations are performed to obtain bifurcation diagrams by varying A as well as Q. For chaotic case, for different range of A, plots of Lyapunov exponents and corresponding topological entropy have been obtained. The system evolve chaotically for higher values of A. Plots of time series and cobweb diagram are shown in Fig.3 for Q = 2.76 and A = 5.122.

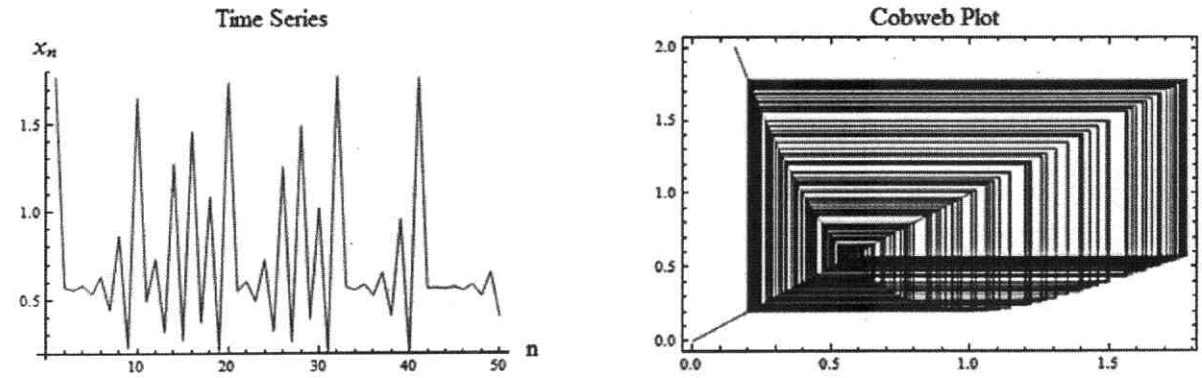

Fig.3: Time series and cobweb plots for chaotic map (1). Parameters are Q = 2.76 and A = 5.122.

We have drawn bifurcation diagrams for four ranges of values of A fixing Q = 2.76 as shown in Fig. 4. These gives clear picture of bistability, multistability, fold bifurcation together with chaos.

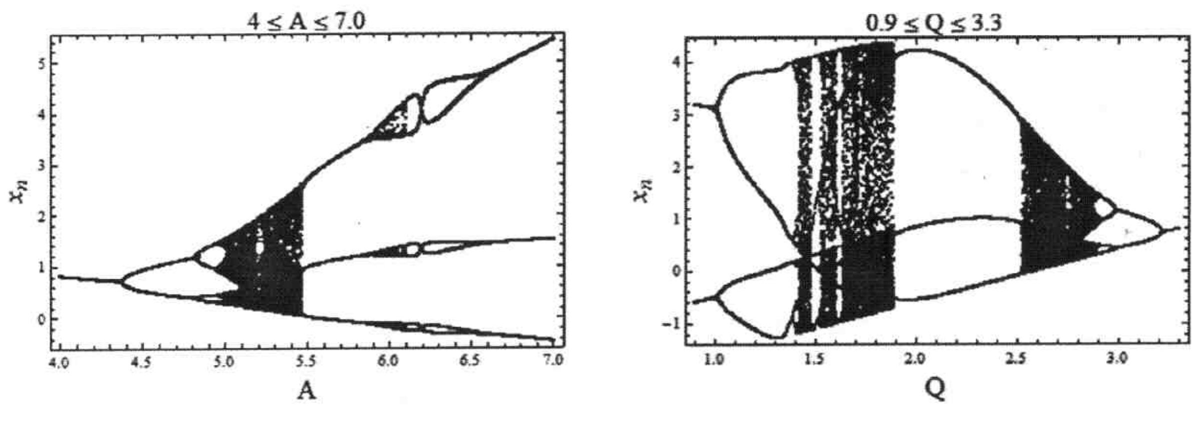

Fig. 4: Two bifurcation diagrams of laser map (3.2) by varying A and Q.

In the left figure Q is fixed,(Q = 2.76) and on the right figure A is fixed, (A = 5.4).

Plots of Lyapunov exponents for cases corresponding to Fig.4, first keeping Q fixed, (Q = 2.76), are given in Fig.5 below. These plots clearly describe that chaos appears for parameter range $5.1 \leq A \leq 5.4$ and again when $15.7 \leq A \leq 18$. The characteristic figure appear when A >> 15 signifies the phenomena of multistability.

Also, the plots of Lyapunov exponents corresponding to Fig.4, when we made Q to vary and A fixed, (A = 5.4), is shown below in Fig. 5(lower panel). Appearing of chaotic motion observed to be interesting. For both the cases, varying A as well as varying Q, one observes topological entropy appear to be positive. For example in the upper left plot near the region of A = 4.5, one finds Lyapunov exponent negative, (i.e. $\lambda < 0$), but on the right plot topological entropy is significantly positive. Thus, here the system, though regular, is highly complex. Similar situations appear in various regions in both, upper row plots and lower row plots, of Fig. 5. By varying both, A and Q, a 3 D plot of topological entropy is shown in Fig. 6.

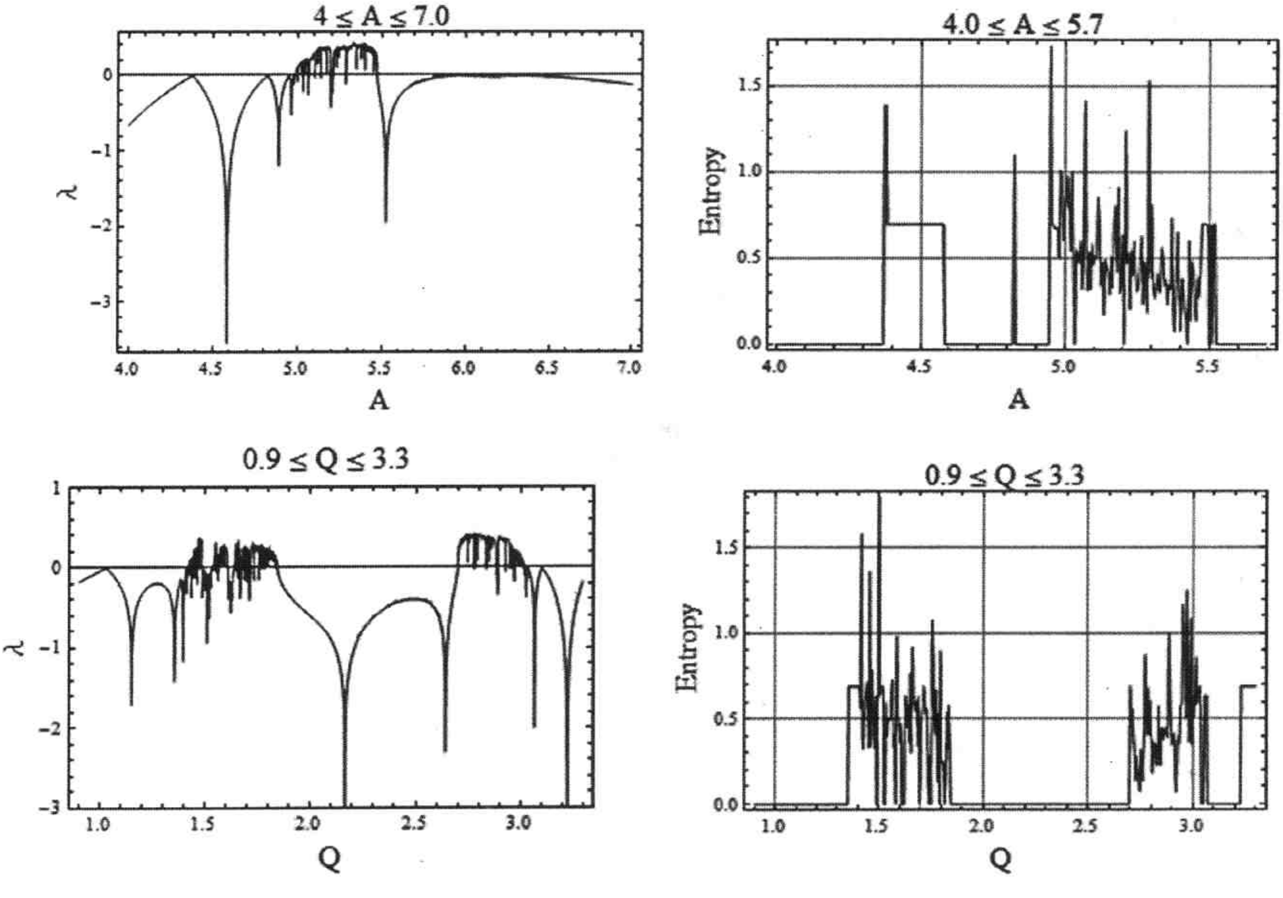

Fig. 5: Plots of LCEs (λ) and topological entropy. For upper row plots Q = 2.76 and for lower row plots A = 5.4.

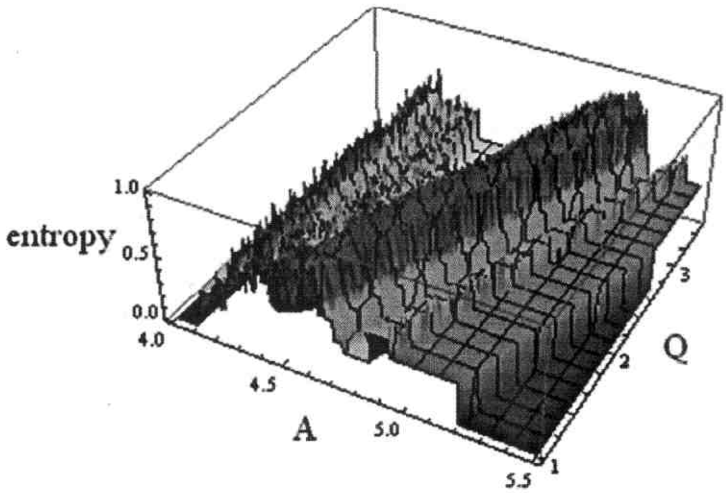

Fig. 6: Plots of topological entropies, (upper row), for two different rage of values of A keeping Q = 2.76 fixed.The lower one is a 3-D entropy plot varying A and Q; $4 \le A \le 5.5$ & $1 \le Q \le 3.5$.

(c) Blood Cell Model:

A simple blood cell model , [28–30], can be described by the following one dimensional discrete equation

$$C_{n+1} = (1-a)C_n + b\, C_n^r\, e^{-sC_n} \qquad \ldots(3.3)$$

Where C_n denotes the red cell count per unit volume in the n^{th}time interval, a is the rate of cell destroyed and parameters b, r, s are related with produced cell p_n per unit time given by

$$p_n = bC_n^r\, e^{-sC_n} \quad \text{with } 0 < a \le 1 \text{ and } r, s > 0.$$

Interesting bifurcation diagrams of blood cell model have been obtained and are shown in Fig.7.The left figure showing criteria of bistability and multistability. The right figure clearly showing period adding and chaos adding nature of bifurcation. This bifurcation figure provides the idea that this blood cell model is highly complex.

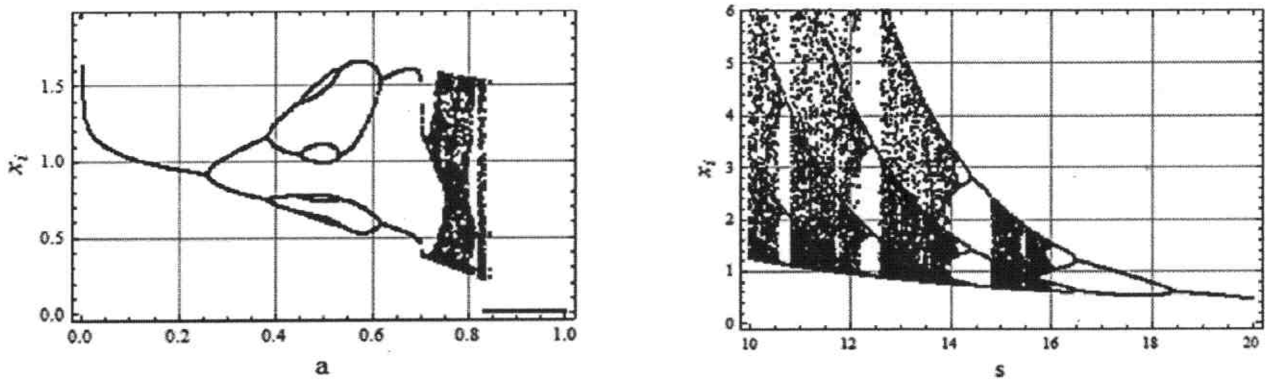

Fig.7 : Bifurcations of blood cell model by varying a and s for r = 8, b = 1.1 × 10^6 and (i) in left figure s = 16, (ii) in right figure a = 0.5. Here x_i stands for C_i .

Further numerical studies havebeen carried out to calculate LCEs and topological entropies for above blood cell model and shown in Fig. 8. Left figure is for LCEs and that on the right side is for topological entropies. Analyzing these two graphics, one definitely gets ideas that complexity and chaos are two different properties.

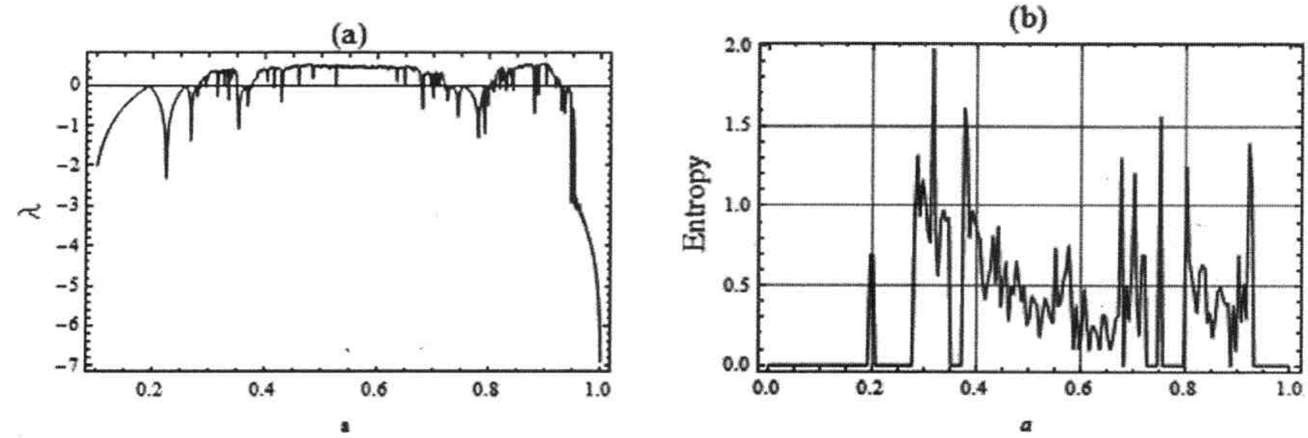

Fig.8: Left figure (a) is for LCEs and the right one is for topological entropies.

Parameter values are $r = 6$, $s = 16$, $b = 1.1\times10^6$ and $0 \le a \le 1$.

LCEs plot and topological entropy plot in Fig. 8, shows the complexity nature of blood cell model. We see positive topological entropy in many regions where LCEs are negative.

(d) Gross National Product (GNP) Model:

Gross national product (GNP) of a country measures the economic activity of the country. It is based on the labour and production output within the country. For GNP, we have considered the following model,[31],

$$x_{n+1} = \sigma\frac{B x_n^{\beta}(m - x_n)^{\gamma}}{(1+\lambda)} \qquad \ldots(3.4)$$

Here x_n is the capital-labor ratio at time n and per capita production function f is defined as

$$f(x) = \frac{Bx^{\beta}(m - x)}{(1+\lambda)}$$

where σ is the savings ratio, λ is the natural rate of population growth and parameters m, γ, β all are positive constant (> 0). This model is thought as a highly simplified model for GNP of a country.

For $\sigma = 0.5$, $\beta = 0.3$, $\gamma = 0.5$, $\lambda = 0.2$, m = 1and $0.5 \le B \le 3.5$, a bifurcation diagram diagram has been obtained, Fig. . Also, LCEs and topological entropies are calculated and shown in Fig. 9

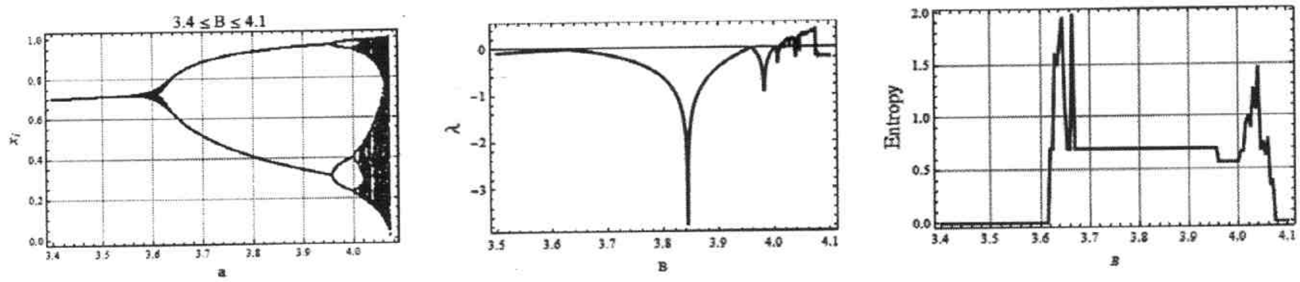

Fig.9: From the left; plots of bifurcation diagram, LCEs and topological entropies for $\sigma = 0.5, \beta = 0.3, \gamma = 0.5, \lambda = 0.2$, m = 1 and $3.4 \le B \le 4.1$.

In this case of GNP model, after observing figures shown in Fig.9, we see high positive value of topological entropy where the system is not chaotic i. e. regulsar.

4. Conclusion

The results obtained in this study show clearly that a complex system which is composed of many components interacting with each other and exhibits properties like spontaneous order, robustness of order and numerosity etc. Topological entropy, which provides the measure of complexity and be assumed different to chaos. A system may be regular but may exhibit complexity. Similarly, a chaotic system may not be complex. But, dealing with natural nonlinear real systems, for most of the systems one encounter complexity as well as chaos at different set of values of parameters of the system. To observe chaos, Lyapunov exponents are precise tools and for complexity it is the topological entropy.

The present study is based on feu one dimensional discrete nonlinear systems. However, similar results regarding chaos and complexity may also be obtained for higher dimensional systems, e.g. [32].

REFERENCES

[1] Birkhoff, George D. (1991): Dynamical Systems. American Mathematical Society Providence, Rhode Island

[2] Devaney, R. L. (2003): An Introduction to Chaotic Dynamical Systems (Studies in Nonlinearity), Westview Press; 2nd Edition.

[3] Lynch, Stephen (2007): Dynamical Systems with Applications using Mathematica. Birkhäuser, Boston.

[4] Nagashima, H. and Baba, Y. (2005): Introduction to Chaos: Physics and Mathematics of Chaotic Phenomena. IOP Publishing Ltd.

[5] Drazin, P. G. (1992): Nonlinear Systems. Cambridge University Press; 1st Edition.

[6] Ian Stewart. Does God Play Dice ?, Penguin Books 1989

[7] Walby, S. (2007). Complexity theory, systems theory, and multiple intersecting social inequalities. Philosophy of the Social Sciences, 37(4), 449-470

[8] Manson, S. M. (2001). Simplifying complexity: a review of complexity theory. Geoforum, 32(3), 405-414.

[9] Cilliers, P. (2000). What can we learn from a theory of complexity?. Emergence, 2(1), 23-33.

[10] Saha, L. M. and Budhraja, Mridula (2007) : The largest eigenvalue ! An indicator of chaos.IJAMM, 3, (2007), 61 – 71.

[11] Adler, R L Konheim, A G McAndrew, M H. Topological entropy, Trans. Amer. Math. Soc. 1965; 114: 309-319

[12] R. Bowen, R .Topological entropy for noncompact sets, Trans. Amer. Math. Soc. 1973: 184: 125-136

[13] Gottwald GA, Melbourne I. A new test for chaos in deterministic systems. Proc R SocLond A 2004;460:603–11.

[14] Benettin, G.; Galgani, L.; Giorgilli, A.; Strelcyn, J. M. (1980). "Lyapunov Characteristic Exponents for smooth dynamical systems and for hamiltonian systems; a method for computing all of them. Part 1 & II: Theory". *Meccanica* 15: 9 – 20 & 21 – 30.

[15] Bryant, P.; Brown, R.; Abarbanel, H. (1990). "Lyapunov exponents from observed time series". *Physical Review Letters* 65 (13): 1523 – 1526.

[16] Adler, R.L.; Konheim, Allan G.; McAndrew, M.H. (1965). "Topological entropy". *Transactions of the American Mathematical Society* 114 (2): 309–319.

[17] R. Bowen: Topological entropy for noncompact sets, Trans. Amer. Math. Soc. 184 (1973), 125-136

[18] P. Grassberger and ItamarProcaccia (1983)."Measuring the Strangeness of Strange Attractors". *Physica D: Nonlinear Phenomena* 9 (1–2): 189–208.

[19] Peter. Grassberger and ItamarProcaccia (1983)."Characterization of Strange Attractors". *Physical Review Letters* 50 (5): 346–349.

[20] Lozi, R. [1978] "Unattracteurétrange du type attracteur de Hénon, "Journal de Physique. Colloque C5, Supplément au n0 8, 39, 9 -10.

[21] M. Martelli, Introduction to Discrete Dynamical Systems and Chaos. John Wiley & Sons, Inc., New York, 1999.

[22] Robert May (1976): Simple mathematical models with very complicated dynamics. Nature 261: 459—67.

[23] S. Elaydi and D. Ribble (2010): Discrete Mathematical Models in Life Sciences. www.trinity.edu/org/hhm).

[24] R.Benefacio,L.A. Lugato(1982)Theory of optical bistability, In Dissipative Systems in Quantum Optics,Ed. R. Benefacio, Springer-Verlag, pp. 61 – 92.

[25] D.M.Hefferman(1985)Multistability,intermittencyandremergingFeigenbaumtreesin an externally pumped ring cavity laser system, *Phys. Lett.* **A 108**: 413 – 422.

[26] F.O'Cairbre,A.G. O'Farrell,A.O'Reilly(1995) Bistability, bifurcation and chaosina laser system, *Int. Jour. Bifurcation and Chaos***5**(4): 1021 – 1031.

[27] Vicente Aboites, Multistable chaotic dynamical systems and philosophy. . Int. J. Bifur. Chaos, **5** , No. 6, 2008), 1821 – 1824.

[28] A. Lasota (1977): Ergodic problems in biology, in *Dynamical Systems*, Vol. 2: Warsaw, Astérisque, Vol. 50, Société Mathématique de France, Paris, 1977, 239 – 250.

[29] Stephen Lynch (2005): Analysis of blood cell population model, Internet J. Bifurcation Chaos, 15: 669 – 677.

[30] Stephen Lynch (2007): Dynamical Systems with Applications using Mathematica, Birkhäuser, Boston.

[31] R. H. Day (1982): Irregular growth cycles, Amer. Econ. Rev., 72: 406 – 414.

[32] G. Litak, A. Syta, M. Budhraja, L.M. Saha (2009): Detection of the chaotic behaviour of a bouncing ball by the 0–1 test. Chaos, Solitons and Fractals, 42 (2009) 1511–1517.

Ganita Sandesh Vol. 28 No. 1 and 2 (June and December, 2014) pp. 25 - 32 ISSN : 0970-9169

Double Dirichlet Average of Generalized K-Wright Type Function Via Fractional Derivative

Chena Ram Choudhary and Anita

Department of Mathematics and Statistics,
Jai Narain Vyas University, JODHPUR, INDIA
Email : crc2007@rediffmail.com
anita.neetu89@yahoo.com

Abstract

The aim of the present paper is to establish the results of Dirichlet average of generalized K-Wright type function, using fractional derivative. In this paper the solution is obtained in compact form of double Dirichlet average of generalized K-Wright type function. Several special cases have also been obtained.

Keywords: Dirichlet average, generalized K-Wright Type function, fractional derivative, Mittag-Leffler function.

Subject Classifications: 26A33, 33C20, 33C65.

1. Introduction

The Dirichlet average of functions is introduced by Carlson [1-5], which represents certain type of integral average with respect to Dirichlet measure. Deora and Banerji [6] have found the double Dirichlet average of e^x by using fractional derivatives. The various types of Dirichlet average has been given by Carlson [8] Gupta and Agarwal [9] found the double Dirichlet average of functions by using fractional derivative. Recently Ram et al. [11, 12] also found the Dirichlet average of generalized k-Mittag-Leffler function and k-series. In this paper, the Dirichlet average of generalized K-Wright type function has been obtained.

2. Definitions

Some definitions which are needed in the preparation of this paper:

a. **Standard Simplex in R^n, n$\geq$1:**

The standard simplex in $R^n, n \geq 1$ is defined by Carlson [1, p.62].

$$E = E_n = \{S(u_1, u_2, ..., u_n) : u_1 \geq 0, ..., u_n \geq 0, u_1 + u_2 + ... + u_n \leq 1\} \quad ...(1)$$

b. **Dirichlet Measure:**

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The Dirichlet measure $d\mu_b$ is defined by

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1 - 1}, ..., u_{k-1}^{b_{k-1}-1} (1 - u_1 - ....u_{k-1}) b_k^{-1} \, du_1 ....du_{k-1} \quad ... (2)$$

where $B(b)$ is the multivariable Beta function which is defined as

$$B(b) = B(b_1, ..., b_k) = \frac{\Gamma(b_1).....\Gamma(b_k)}{\Gamma(b_1 + b_2 + ... + b_k)} \; ; \; \left(\text{Re}(b_j) > 0; j = 1, 2, .., k\right)$$

c. **Dirichlet Average [1, p.75]:**

The general Dirichlet average function is defined by Carlson [2] in the form

$$F(b, z) = \int_E f(u.z) \, d\mu_b(u) \quad ... (3)$$

where $d\mu_b$ is defined by (2) and

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - .... - u_{k-1}.$$

d. **Double Averages of Functions of One Variable [1,2] :**

Let z be a $k \times x$ matrix with complex elements z_{ij}, let $u = (u_1, u_2, ..., u_k)$ and $v = (v_1, v_2, ..., v_x)$ be an ordered k-tuple and x-tuple of real non-negative weights $\sum u_i = 1$ and $\sum v_j = 1$ respectively. Now, we define

$$u.z.v = \sum_{i=1}^{k} \sum_{j=1}^{x} u_i \, z_{ij} \, v_j$$

If z_{ij} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of $\left(z_{11}, \ldots, z_{kx}\right)$ denote by H(z).

Let $b = (b_1, b_2, \ldots, b_k)$ be an ordered k-tuple of complex numbers with positive real part $\mathrm{Re}(b) > 0$ and similarly for $\beta = (\beta_1, \beta_2, \ldots, \beta_x)$ then define $d\mu_b(u)$ and $d\mu_b(v)$.

Let f be the holomorphic on a domain D in the complex plane, if $\mathrm{Re}(b) > 0, \mathrm{Re}(\beta) > 0$ and $H(z) \subset D$ we define

$$F(b, z, \beta) = \int\int f(u.z.v) \, d\mu_b(u) \, d\mu_b(v)$$

Double average for function is defined by Gupta and Agrawal [9].

where $$u.z.v = \sum_{i=1}^{2} \sum_{j=1}^{2} (u_i z_{ij} v_j) = \sum_{i=1}^{2} \left[u_i (z_{i1} v_1 + z_{i2} v_2)\right]$$

$$= \left[u_1 z_{11} v_1 + u_1 z_{12} v_2 + u_2 z_{21} v_1 + u_2 z_{22} v_2\right]$$

Let $z_{11} = a, z_{12} = b, z_{21} = c, z_{22} = d$ and $\begin{cases} u_1 = u & u_2 = 1 - u \\ v_1 = v & v_2 = 1 - v \end{cases}$

thus $$z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Therefore $$u.z.v = uva + ub(1 - v) + (1 - u)cv + (1 - u)d(1 - v)$$

$$= uv(a - b - c + d) + u(b - d) + v(c - d) + d \qquad \ldots(4)$$

e. Fractional Derivative [7, p.181]:

The theory of fractional derivative with respect to an arbitrary function has been used by Erdélyi et al. [7]. The fractional derivative is obtained by proceeding via fractional integral. The Riemann-Liouville fractional integral of order α is defined by

$$D_z^{\alpha} F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1}\, dt \qquad \dots (5)$$

where $\mathrm{Re}(\alpha) < 0$ and F(x) is the form of $x^p f(x)$; $f(x)$ is an analytic at $x = 0$.

f. Generalized K-Wright Type Function:

Let $\alpha, \beta, \gamma \in C, \kappa \in R, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0$ and $q \in (0,1) \cup N$ then generalized K-Wright type function is defined by Ram et al. [10] in the form

$$W_{\kappa,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,\kappa}\, z^n}{\Gamma_\kappa(\alpha n + \beta)(n!)^2} \qquad \dots (6)$$

where $(\gamma)_{nq,\kappa}$ is the κ -Pochhammer symbol and $\Gamma_\kappa(x)$ is the κ - Gamma function.

3. Main Result and Proof

Theorem 1: The equivalence relation of double Dirichlet average of Generalized K-Wright type function $W_{\kappa,\alpha,\beta}^{\gamma,q}(u.z.v)$ with the fractional derivative for $(k = x = 2)$ is

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \frac{\Gamma(\rho_1 + \rho_2)}{\Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2} D_{x-y}^{-\rho_2} W_{\kappa,\alpha,\beta}^{\gamma,q}(x)(x-y)^{\rho_1 - 1} \qquad \dots (7)$$

Proof: Let us consider the double average for $(k = x = 2)$ of K-Wright type function $W_{\kappa,\alpha,\beta}^{\gamma,q}(u.z.v)$

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \int_0^1 \int_0^1 W_{\kappa,\alpha,\beta}^{\gamma,q}(u.z.v)\, dm_{\mu_1\mu_2}(u)\, dm_{\rho_1\rho_2}(v)$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,\kappa}}{\Gamma_{\kappa}(\alpha n+\beta)(n!)^2} \int_0^1 \int_0^1 [u.z.v]^n \, dm_{\mu_1\mu_2}(u)\, dm_{\rho_1\rho_2}(v) \quad \text{... (8)}$$

where $\mathrm{Re}(\mu_1)=0, \mathrm{Re}(\mu_2)=0, \mathrm{Re}(\rho_1)>0, \mathrm{Re}(\rho_2)>0$

and

$$dm_{\mu_1\mu_2}(u) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} u^{\mu_1-1}(1-u)^{\mu_2-1}\, du \quad \text{... (9)}$$

$$dm_{\rho_1\rho_2}(v) = \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} v^{\rho_1-1}(1-v)^{\rho_2-1}\, dv \quad \text{... (10)}$$

Using equation (4), (9) and (10) in (8), we have

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,\kappa}}{\Gamma_{\kappa}(\alpha n+\beta)(n!)^2}$$

$$\times \int_0^1 \int_0^1 [uv(a-b-c+d)+u(b-d)+v(c-d)+d]^n \, u^{\mu_1-1}(1-u)^{\mu_2-1} v^{\rho_1-1}(1-v)^{\rho_2-1} du\, dv \quad \text{... (11)}$$

To obtain the fractional derivative, we assume $a=c=x; b=d=y$ then

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,\kappa}}{\Gamma_{\kappa}(\alpha n+\beta)(n!)^2}$$

$$\int_0^1 \int_0^1 [v(x-y)+y]^n \, u^{\mu_1-1}(1-u)^{\mu_2-1} v^{\rho_1-1}(1-v)^{\rho_2-1} du\, dv$$

Now, using the definition of Beta and Gamma function and due to suitable adjustment, we arrive at

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,\kappa}}{\Gamma_{\kappa}(\alpha n+\beta)(n!)^2} \int_0^1 [v(x-y)+y]^n \, v^{\rho_1-1}(1-v)^{\rho_2-1} dv$$

On putting $v(x-y)=t$ in above equation, we get

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,\kappa}}{\Gamma_\kappa(\alpha n+\beta)(n!)^2}\int_0^{x-y}[t+y]^n\left(\frac{t}{x-y}\right)^{\rho_1-1}\left(1-\frac{t}{x-y}\right)^{\rho_2-1}\frac{1}{(x-y)}dt$$

$$=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,\kappa}}{\Gamma_\kappa(\alpha n+\beta)(n!)^2}(x-y)^{1-\rho_1-\rho_2}\int_0^1\left[(y+t)\right]^n t^{\rho_1-1}(x-y-t)^{\rho_2-1}\,dt$$

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}(x-y)^{1-\rho_1-\rho_2}\int_0^{x-y}W^{\gamma,q}_{\kappa,\alpha,\beta}(y+t)t^{\rho_1-1}(x-y-t)^{\rho_2-1}\,dt$$

by using the definition of fractional derivative from equation (5), we get

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}D^{-\rho_2}_{x-y}W^{\gamma,q}_{\kappa,\alpha,\beta}(x)(x-y)^{\rho_1-1}$$

This completes the proof of Theorem 1.

Corollary 1.1 If we take $\kappa=1$ in Theorem 1, we get the following form

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}D^{-\rho_2}_{x-y}W^{\gamma,q}_{\alpha,\beta}(x)(x-y)^{\rho_1-1}$$

Corollary 1.2 If we take $\kappa=1,q=1$ in Theorem 1, we get the following form

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}D^{-\rho_2}_{x-y}W^{\gamma}_{\alpha,\beta}(x)(x-y)^{\rho_1-1}$$

Corollary 1.3 Further, if we take $\kappa=q=\gamma=1$ and in Theorem 1, we get the following new result

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}D^{-\rho_2}_{x-y}W_{\alpha,\beta}(x)(x-y)^{\rho_1-1}\qquad\ldots(12)$$

KNOWN RESULT: Further, double Dirichlet average of Wright function reduces to single Dirichlet average of Wright function and suitable adjustment in the parameters in above equation (12) then we get the following known result earlier obtained by Sharma et al.[13] :

$$S(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)}(x-y)^{1-\beta-\beta'}D^{-\beta'}_{x-y}W_{\alpha,\beta}(x)(x-y)^{\beta-1}$$

Theorem 2. The equivalence relation of double Dirichlet average of Generalized K-Wright type function $W^{\gamma,q}_{\kappa,\alpha,\beta}(u.z.v)$ with the fractional derivative for $(k=x=2)$ is

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{(\mu_1)_n}{(\mu_1+\mu_2)_n}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}D_{x-y}^{-\rho_2}W_{\kappa,\alpha,\beta}^{\gamma,q}(x)(x-y)^{\rho_1-1}$$

Proof: Using equation (11)

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,\kappa}}{\Gamma_\kappa(\alpha n+\beta)(n!)^2}$$

$$\times\int_0^1\int_0^1[uv(a-b-c+d)+u(b-d)+v(c-d)+d]^n\,u^{\mu_1-1}(1-u)^{\mu_2-1}v^{\rho_1-1}(1-v)^{\rho_2-1}\,du\,dv$$

If we set $a=x, b=y, c=d=0$, then we have

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,\kappa}}{\Gamma_\kappa(\alpha n+\beta)(n!)^2}$$

$$\times\int_0^1\int_0^1[uv(x-y)+uy]^n\,u^{\mu_1-1}(1-u)^{\mu_2-1}v^{\rho_1-1}(1-v)^{\rho_2-1}\,du\,dv$$

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,\kappa}}{\Gamma_\kappa(\alpha n+\beta)(n!)^2}$$

$$\times\int_0^1\int_0^1[vx+y(1-v)]^n\,u^{\mu_1+n-1}(1-u)^{\mu_2-1}v^{\rho_1-1}(1-v)^{\rho_2-1}\,du\,dv$$

Now, by using the definition of Beta and Gamma function, we get

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\frac{\Gamma(\mu_1+n)\Gamma(\mu_2)}{\Gamma(\mu_1+\mu_2+n)}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,\kappa}}{\Gamma_\kappa(\alpha n+\beta)(n!)^2}$$

$$\times\int_0^1[vx+y(1-v)]^n\,v^{\rho_1-1}(1-v)^{\rho_2-1}\,dv$$

$$=\frac{(\mu_1)_n}{(\mu_1+\mu_2)_n}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,\kappa}}{\Gamma_\kappa(\alpha n+\beta)(n!)^2}\int_0^1[vx+y(1-v)]^n\,v^{\rho_1-1}(1-v)^{\rho_2-1}$$

On putting $v(x-y)=t$ in above equation, we get

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{(\mu_1)_n}{(\mu_1+\mu_2)_n}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{(\gamma)_{nq,\kappa}}{\Gamma_\kappa(\alpha n+\beta)(n!)^2}$$

$$\times \int_0^{x-y} [t+y]^n \left(\frac{t}{x-y}\right)^{\rho_1-1} \left(1-\frac{t}{x-y}\right)^{\rho_2-1} \frac{1}{(x-y)} dt$$

Now, using the definition of fractional derivative, we have

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{(\mu_1)_n}{(\mu_1+\mu_2)_n} \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2} D_{x-y}^{-\rho_2} W_{\kappa,\alpha,\beta}^{\gamma,q}(x)(x-y)^{\rho_1-1}$$

A number of several special cases of Theorem-2 can also be obtained but for the sake of brevity they are not presented here.

REFERENCES

[1] **Carlson, B.C.,** *Special Functions of Applied Mathematics,* Academics Press, New York (1977).

[2] **Carlson, B.C.,** Appell's function F_4 as a double average, *SIAM J. Math. Anal. 6 (1975), 960-965.*

[3] **Carlson, B.C.,** Hidden symmetries of special function, *SIAM Rev. 12 (1970), 332-345.*

[4] **Carlson, B.C.,** Dirichlet averages of x^t log x, *SIAM J. Math. Anal. 18(2) (1987), 550-565.*

[5] **Carlson, B.C.,** A connection between elementary and higher transcendental functions, *SIAM J. Appl. Math.17 (1969), 116-148.*

[6] **Deora, Y. and Banerji, P.K.,** Double Dirichlet average of e^x using fractional derivative, *J. Fractional Calculus 3 (1993), 81-86.*

[7] **Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G.,** *Tables of Integral Transforms*, Vol. 2 McGraw-Hill, New York (1954).

[8] **Gupta, S.C. and Agrawal, B.M.,** Dirichlet average and fractional derivatives, *J. Indian Acad. Math. 12(1) (1990), 103-115.*

[9] **Gupta, S.C. and Agrawal, B.M.,** Double Dirichlet average of e^x using fractional derivatives, *Ganita Sandesh 5(1) (1991), 47-52.*

[10] **Ram, C., Choudhary, P. and Gehlot K.S.,** The generalized k-Wright type function, *J. of Global Research in Mathematical Archives, 1(9) (2013), 9-14.*

[11] **Ram, C., Choudhary, P. and Gehlot K.S.,** Dirichlet averages of generalized k-Mittag-Leffler function, *Int. J. of Mathematical Archive, 4(7) (2013), 105-118.*

[12] **Ram, C., Choudhary, P. and Gehlot K.S.,** Representation of Dirichlet averages of k-series via fractional integrals and special functions, *Int. J. of Mathematics and its applications, 1(1) (2013), 1-11.*

[13] **Sharma, M., Ali, M.F. and Jain, R.,** Dirichlet average of Wright function and fractional derivative, *Int. J. of Innovative & Applied Research, 2(1) (2014), 71-74.*

Ganita Sandesh Vol. 28 No. 1 and 2 (June and December, 2014) pp. 33 - 42 ISSN : 0970-9169

Certain New Integral Formulas Involving the Generalized Multiple (Multiindex) Mittag-Leffler Function

N.U. Khan, T. Usman and M. Ghayasuddin

Department of Applied Mathematics,

Z.H. College of Engineering and Technology

Aligarh Muslim University, ALIGARH - 202 002, (India)

Email : nukhanmath@gmail.com;

talhausman.maths@gmail.com;

ghayas.math@gmail.com

Abstract

The object of the present paper is to establish two interesting unified integral formulas involving Multiple (multiindex) Mittag-Leffler function, which is expressed in terms of Wright hypergeometric function. Some deduction from these results are also considered as the special cases of our main result.

2000 AMS Subject Classification : 33C60, 33E12.

Keywords : *Multiple (multiindex) Mattag-Leffler Function, Wright Hypergeometric Function, Integrals.*

1. Introduction

A number of integral formulas involving a variety of special functions have been developed by many authors (see[2],[3],[4],[5], also see [7] and [10]) motivated by their works, we aim at presenting two unified integral formulas involving the Multiple (Multiindex) Mittage-Leffler Function, which are expressed in terms of Wright hypergeometric Function. Also some interesting cases of our main reslults are also considered.

The generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [1] and Wright ([12], [13], [14]) who studied the asymptotic expansion of the generalized Wright hypergeometric function defined by (see[15, p.21]).

Dedicated to Prof. M.A. Pathan on his 75th Birth Anniversary

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),(\alpha_2,A_2),....(\alpha_p,A_p); \\ (\beta_1,B_1),(\beta_2,B_2),....(\beta_q,B_q);\end{matrix}\; z\right]=\sum_{n=0}^{\infty}\frac{\Pi_{j=1}^{p}\Gamma\left(\alpha_j+A_jk\right)z^k}{\Pi_{j=1}^{q}\Gamma\left(\beta_j+b_jk\right)k!} \quad ...(1.1) $$

where the coefficients $A_1,....,A_p$ and $B_1,....,B_q$ are positive real numbers such that

$$ 1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\geq 0 $$

A special case of (1.1) is

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,1),(\alpha_2,1),....(\alpha_p,1); \\ (\beta_1,1),(\beta_2,1),....(\beta_q,1);\end{matrix}\; z\right]=\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{matrix}(\alpha_1),....,(\alpha_p); \\ (\beta_1,1),....,(\beta_q);\end{matrix}\; z\right] $$

where ${}_pF_q$ is the generalized hypergeometric series defined by (see [9, section 1.5])

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1),(\alpha_2),....(\alpha_p); \\ (\beta_1),(\beta_2),....(\beta_q);\end{matrix}\; z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}={}_pF_q\left(\alpha_1,....\alpha_p;\beta_1,....\beta_q;z\right) $$

where $(\lambda)_n$ is called the pochhammer's symbol (see [8], [9])

Kiryakova [11] defined the Multiple (multiindex) Mittag-Leffler function as follows :

Let $m>1$ be an integer, $\rho_1,....\rho_m>0$ and $\mu_1,....,\mu_m$ be arbitrary real numbers. By means of "Multiindices" $(\rho_i)(\mu_i)$ we introduce the so-called multiindex (m-ruple, multiple) Mittag-Leffler functions.

$$ E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\mu_1+\frac{k}{\rho_1}\right)....\Gamma\left(\mu_m+\frac{k}{\rho_m}\right)} \quad ...(1.2) $$

Also, we have the following interesting relation of Multiple (multiindex) Mittag-Leffler function to other special functions as follows :

(i) For $m = 2$, if we put $\frac{1}{\rho_1} = \alpha$, $\frac{1}{\rho_2} = 0$ and $\mu_1 = 1$, $\mu_2 = 1$ in (1,2), we have

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)} \qquad \text{...(1.3)}$$

(ii) For $m = 2$, if we put $\frac{1}{\rho_1} = \alpha$, $\frac{1}{\rho_2} = 0$ and $\mu_1 = \beta$, $\mu_2 = 1$ in (1.2), we have

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)} \qquad \text{...(1.4)}$$

(iii) For $m = 2$, if we put $\frac{1}{\rho_1} = 1$, $\frac{1}{\rho_2} = 1$ and $\mu_1 = v+1$, $\mu_2 = 1$ and replace z by $\frac{-z^2}{4}$, in (1.2), we have (see [11])

$$E_{(1,1),(1+v,1)}\left(\frac{-z^2}{4}\right) = \left(\frac{2}{z}\right)^v J_v(z) \qquad \text{...(1.5)}$$

where $J_v(z)$ is a Bessel function of first kind (see [8], [9]).

(iv) For $m = 2$, if we put $\frac{1}{\rho_1} = 1$, $\frac{1}{\rho_2} = 2$ and $\mu_1 = \frac{3-v+\mu}{2}$, $\mu_2 = \frac{3+v+\mu}{2}$ and replace z by $\frac{-z^2}{4}$, in (1.2), we have (see[11])

$$E_{(1,1),\left(\frac{3-v+\mu}{2},\frac{3+v+\mu}{2}\right)}\left(\frac{-z^2}{4}\right) = \frac{1}{z^{\mu+1}} 4S_{\mu,v}(z) \qquad \text{...(1.6)}$$

where $S_{\mu,v}(z)$ is a Struve function (see [8], [9])

(v) For $m = 2$, if we put $\frac{1}{\rho_1} = 1$, $\frac{1}{\rho_2} = 1$ and $\mu_1 = \frac{3}{2}$, $\mu_2 = \frac{3+2v}{2}$, and replace z by $\frac{-z^2}{4}$, in (1.2), we have (see [11])

$$E_{(1,1),\left(\frac{3}{2},\frac{3+2v}{2}\right)}\left(\frac{-z^2}{4}\right) = \frac{1}{z^{\mu+1}} 4H_v(z) \qquad \text{...(1.7)}$$

where $H_v(z)$ is a Lommel function (see [8], [9]).

For our present investigation, the following interesting and useful result due to Lavoie and Trottier [6] will be required :

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} dx = \left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \text{...(1.8)}$$

$\Re(\alpha) > 0$ and $\Re(\beta) > 0$

2. Main Result

We established two generalized integral formulas which are expressed in terms of generalized Wright hypergeometric functions, by inserting the Multiple (multiindex) Mittag-Leffler function with suitable arguments into the integrand of the integral.

First Integral

The following integral formula holds true : For $\Re(\alpha) > 0$, $\Re(\beta) > 0$,

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\,{}_2\Psi_{m+1}\left[\begin{matrix}(\beta,1), \quad (1,1); \\ \left(\mu_1,\frac{1}{\rho_1}\right),\ldots\ldots,\left(\mu_m,\frac{1}{\rho_m}\right),(\alpha+\beta,1);\end{matrix}\; y\right] \qquad \text{...(2.1)}$$

Second Integral

The following integral formula holds true : For $\Re(\alpha) > 0$, $\Re(\beta) > 0$,

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}\left[xy\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$$

$$= \left(\frac{2}{3}\right)^{2\alpha} {}_3\Psi_{m+1}\left[\begin{matrix} (\alpha,1), \quad (\beta,1), \quad (1,1); \\ \left(\mu_1,\frac{1}{\rho_1}\right),\ldots\ldots,\left(\mu_m,\frac{1}{\rho_m}\right),(\alpha+\beta,2); \end{matrix} y\right] \qquad \ldots(2.2)$$

Proof of (2.1) :

In order to derive (2.1), we denote the left-hand side of (2.1) by I, express $E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z)$ as a series with the help of (1.2) and then interchange the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, thus we get

$$I = \int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$$

$$= \sum_{k=0}^{\infty} \frac{(y)^k}{\Gamma\left(\mu_1+\frac{k}{\rho_1}\right)\ldots\left(\mu_m+\frac{k}{\rho_m}\right)} \int_0^1 x^{\alpha-1}(1-x)^{2\beta+2k-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta+k-1}$$

Evaluating the above integral with the help of (1.8), we get

$$I = \left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)\Gamma(1+k)}{\Gamma\left(\mu_1+\frac{k}{\rho_1}\right)\ldots\Gamma\left(\mu_m+\frac{k}{\rho_m}\right)\Gamma(\alpha+\beta+k)}(y)^k\frac{1}{k!}$$

Finally, summing up the above series with the help of definition (1.1), we arrive at the right hand side of (2.1). This complete the proof of our first result.

Proof of (2.2) :

Similarly, to derive (2.2), we denote the left-hand side of (2.2) by I', express $E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}(z)$ as a series with the help of (1.2) and then interchange the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, thus we get

$$I' = \int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_{\left(\frac{1}{\rho_i}\right),(\mu_i)}\left[xy\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$$

$$= \sum_{k=0}^{\infty} \frac{(y)^k}{\Gamma\left(\mu_1+\frac{k}{\rho_1}\right)\ldots\left(\mu_m+\frac{k}{\rho_m}\right)} \int_0^1 x^{\alpha+k-1}(1-x)^{2\beta+2k-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta+k-1}$$

Evaluating the above integral with the help of (1.8), we get

$$I' = \left(\frac{2}{3}\right)^{2\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)\Gamma(1+k)}{\Gamma\left(\mu_1+\frac{k}{\rho_1}\right)\ldots\Gamma\left(\mu_m+\frac{k}{\rho_m}\right)\Gamma(\alpha+\beta+2k)}(y)^k \frac{1}{k!}$$

Finally, summing up the above series with the help of definition (1.1), we arrive at the right hand side of (2.2). This complete the proof of our second result.

3. Special Cases

In this section, we define some special cases of our main results :

1. $\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_\alpha\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$

$$= \left(\frac{2}{3}\right)^{2\alpha} \Gamma(\alpha)\, {}_2\psi_2\left[\begin{matrix}(\beta,1), & (1,1); \\ (\alpha,1),\ldots\ldots,(\alpha+\beta,1);\end{matrix}\, y\right] \qquad \ldots(3.1)$$

where $\Re(\alpha) > 0,\ \Re(\beta) > 0$.

2. $\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_\alpha\left[xy\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$

$$= \left(\frac{2}{3}\right)^{2\alpha} {}_3\psi_2\left[\begin{matrix}(\alpha,1), & (\beta,1), & (1,1); \\ (\alpha,1),\ldots\ldots,(\alpha+\beta,2); & & \end{matrix}\, y\right] \qquad \ldots(3.2)$$

where $\Re(\alpha) > 0,\ \Re(\beta) > 0$.

The above results (3.1) and (3.2) can be established with the help of integral (2.1) and (2.2) by taking $m = 2$, $\frac{1}{\rho_1} = \alpha,\ \frac{1}{\rho_2} = 0,\ \mu_1 = 1,\ \mu_2 = 1$ and using equation (1.3).

3. $\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_{\alpha,\beta}\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$

$$= \left(\frac{2}{3}\right)^{2\alpha} \Gamma(\alpha)\, {}_2\psi_2\left[\begin{matrix}(\beta,1), & (1,1); \\ (\beta,\alpha),\ldots\ldots,(\alpha+\beta,1);\end{matrix}\, y\right] \qquad \ldots(3.3)$$

where $\Re(\alpha) > 0,\ \Re(\beta) > 0$

4. $\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} E_{\alpha,\beta}\left[xy\left(1-\frac{x}{4}\right)(1-x)^2\right]dx$

$$= \left(\frac{2}{3}\right)^{2\alpha} {}_3\Psi_2\left[\begin{matrix}(\alpha,1), & (\beta,1), & (1,1) \\ (\beta,\alpha),\ldots\ldots,(\alpha+\beta,2); & & \end{matrix} y\right], \qquad \ldots(3.4)$$

where $\Re(\alpha) > 0,\ \Re(\beta) > 0$.

The above results (3.3) and (3.4) can be established with the help of integral (2.1) and (2.2) by taking $m = 2$, $\frac{1}{\rho_1} = \alpha,\ \frac{1}{\rho_2} = 0$ and $\mu_1 = \beta,\ \mu_2 = 1$ and using equation (1.4).

5. $\int_0^1 x^{\alpha-1}(1-x)^{2\beta-v-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-\frac{v}{2}-1} J_v\left[2i\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)^{1/2}\right]dx$

$$= i^{-v}(y)^{\frac{-v}{2}}\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\,{}_1\Psi_2\left[\begin{matrix}(\beta,1) \\ (v+1,1),\ldots\ldots,(\alpha+\beta,1);\end{matrix} y\right], \qquad \ldots(3.5)$$

where $i^2 = -1,\ \Re(\alpha) > 0,\ \Re\left(\beta-\frac{v}{2}\right) > 0$.

6. $\int_0^1 x^{\alpha-\frac{v}{2}-1}(1-x)^{2\beta-v-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-\frac{v}{2}-1} J_v\left[2i\left(xy\left(1-\frac{x}{4}\right)(1-x)^2\right)^{1/2}\right]dx$

$$= i^{-v}(y)^{\frac{-v}{2}}\left(\frac{2}{3}\right)^{2\alpha} {}_2\Psi_2\left[\begin{matrix}(\alpha,1), & (\beta,1); \\ (v+1,1),(\alpha+\beta,2); & \end{matrix} y\right], \qquad \ldots(3.6)$$

where $i^2 = -1,\ \Re\left(\alpha-\frac{v}{2}\right) > 0,\ \Re\left(\beta-\frac{v}{2}\right) > 0$.

The above results (3.5) and (3.6) can be established with the help of integral (2.1) and (2.2) by taking $m = 2$, $\frac{1}{\rho_1} = 1,\ \frac{1}{\rho_2} = 1$ and $\mu_1 = v+1,\ \mu_2 = 1$ and replacing z by $\frac{-z^2}{4}$, and using equation (1.5) (see [11]).

7. $$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-2\mu-3}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-\mu-2} S_{\mu,v}\left[2i\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)^{1/2}\right]dx$$

$$= 4y^{-\mu-1}\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\,{}_2\Psi_3\left[\begin{matrix}(\beta,1) & (1,1) \\ \left(\frac{3-v-\mu}{2},1\right),\left(\frac{3+v+\mu}{2},1\right),(\alpha+\beta,1); & y\end{matrix}\right], \qquad \ldots(3.7)$$

where $i^2=-1,\ \Re(\alpha)>0,\ \Re(\beta-\mu)>1$

8. $$\int_0^1 x^{\alpha-\mu-1}(1-x)^{2\beta-2\mu-3}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-\mu-2} S_{\mu,v}\left[2i\left(xy\left(1-\frac{x}{4}\right)(1-x)^2\right)^{1/2}\right]dx$$

$$= 4y^{-\mu-1}\left(\frac{2}{3}\right)^{2\alpha}\,{}_3\Psi_3\left[\begin{matrix}(\alpha,1) & (\beta,1) & (1,1); \\ \left(\frac{3-v+\mu}{2},1\right),\left(\frac{3+v+\mu}{2},1\right),(\alpha+\beta,2); & y\end{matrix}\right], \qquad \ldots(3.8)$$

where $i^2=-1,\ \Re(\alpha-\mu)>1,\ \Re(\beta-\mu)>1.$

The above results (3.7) and (3.8) can be established with the help of integral (2.1) and (2.2) by taking $m=2$

$\frac{1}{\rho_1}=1,\ \frac{1}{\rho_2}=1$ and $\mu_1=\frac{3-v+\mu}{2},\ \mu_2=\frac{3+v+\mu}{2}$ and replacing z by $\frac{-z^2}{4}$, using equation (1.6) (see [11]).

9. $$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-2v-3}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-v-2} H_v\left[2i\left(y\left(1-\frac{x}{4}\right)\left(1-x^2\right)\right)^{1/2}\right]dx$$

$$= 4y^{-v-1}\left(\frac{2}{3}\right)^{2\alpha}\Gamma(\alpha)\,{}_2\Psi_3\left[\begin{matrix}(\beta,1) & (1,1); \\ \left(\frac{3}{2},1\right),\left(\frac{3+2v}{2},1\right),(\alpha+\beta,1); & y\end{matrix}\right], \qquad \ldots(3.9)$$

where $i^2=-1,\ \Re(\alpha)>0,\ \Re(\beta-\mu)>1.$

10. $$\int_0^1 x^{\alpha-\nu-2}(1-x)^{2\beta-2\nu-3}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-\nu-2} H_\nu\left[2i\left(xy\left(1-\frac{x}{4}\right)(1-x)^2\right)^{1/2}\right]dx$$

$$= 4y^{-\nu-1}\left(\frac{2}{3}\right)^{2\alpha} {}_3\Psi_3\left[\begin{matrix}(\alpha,1) & (\beta,1) & (1,1); \\ \left(\frac{3}{2},1\right),\left(\frac{3+2\nu}{2},1\right),(\alpha+\beta,2); \end{matrix} \; y\right], \quad \ldots(3.10)$$

where $i^2 = -1$, $\Re(\alpha-\nu) > 1$, $\Re(\beta-\mu) > 1$

The above results (3.9) and (3.10) can be established with the help of integral (2.1) and (2.2) by taking $m = 2$, $\frac{1}{\rho_1} = 1$, $\frac{1}{\rho_2} = 1$ and $\mu_1 = \frac{3}{2}$, $\mu_2 = \frac{3+2\nu}{2}$ and replacing z by $\frac{-z^2}{4}$, and using equation (1.7).

REFERENCES

[1] C.Fox., The asymptotic expansion of generalized hypergeometric functions, Proc. London Math. Soc. **27** (2) (1928), 389-400.

[2] J. Choi, P. Agarwal, S. Mathur, and S.D. Purohit, Certain new integral formulas involving the generalized Bessel functions, Bull. Korean Math. Soc. **51** (2014), 995-1003.

[3] J. Choi and P. Agarwal, Certain unified integrals involving a product of Bessel functions of first kind, Honam Mathematical J. **35** (4) (2013), 667-677.

[4] J. Choi and P. Agarwal, Certain Unified Integrals Associated with Bessel Functions, Boundary Value Problems, **2013** (95), 2013, 1-9.

[5] J. Choi, A. Hasnove, H.M. Srivastava, and M. Turaev, Integral representations for Srivastava's triple hypergeometric functions, Taiwanese J. Math. **15** (6), (2011), 2751-2762.

[6] J.L. Lavoie and G. Trottier, On the sum of certain Appell's series, Ganita, **20**(1) 1969), 31-32.

[7] P. Agarwal, S. Jain, S. Agarwal, and M. Nagpal, On a new class of integrals involving Bessel functions of the first kind, Communication in Numerical Analysis, **2014** (2014),1-7.

[8] E.D. Rainville, Special functions, The Macmillan Company, New York, (1971).

[9] H.M. Srivastava and H.L. Manocha, A treatise on generating functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and sons, New York, (1984).

[10] H.M. Srivastava and P.W. Karlsson, Multiple Gaussian Hypergeometric series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons. New York, Chichester, Brisbane, and Toronto, (1985).

[11] Virgina S. Kiryakova, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus, J. Comput. Appl. Math. **118** (2000), 241-259.

[12] E.M. Wright, The asymptotic expansion of the generalized hypergeometric functions. J. London Math. Soc. **10** (1935), 286-293.

[13] E.M. Wright, The asymptotic expansion of integral functions defined by Taylor series, Philos. Trans. Roy. Soc. London, A **238** (1940), 423-451.

[14] E.M. Wright, The asymptotic expansion of the generalized hypergeometric function II, Proc. Lond. Math. Soc. **46** (2) (1940), 389-408.

[15] Y.A. Brychkov, Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas, CRC Press, Taylor and Francis Group, Boca Raton, London, and New York, (2008).

Ganita Sandesh Vol. 28 No. 1 and 2 (June and December, 2014) pp. 43 - 54 ISSN : 0970-9169

Bianchi Type-V Cosmological Models With Perfect Fluid and Dark Energy

Vimal Chand Jain[1], Mahesh Kumar Yadav[2] and Nikhil Jain[3]

[1]Dept.of Mathematics, Govt. Engg. College, Barliya Chouraha, NH-8, AJMER, 305002
Email:vcjeca@gmail.com
[2]Department of Mathematics, Central University, SAGAR
Email: yadav1976mk@gmail.com
[3]Dept. of Mathematics, Govt. Women Engg. College, Nasirabad Road, Ajmer, 305002, India
Email: jain10480@gmail.com

Abstract

In this paper, we study Bianchi type-V cosmological model with perfect fluid and dark energy (phantom and quintessence). To get deterministic solution of the model we assume that shear (σ_i^j) *is proportional to expansion* (θ) *and potential* $V(\phi)=\frac{\lambda}{n}\phi^n$; *where* ϕ *is Higgs field. Some physical aspects including singularities of the model have also been discussed.*

Keywords: Bianchi type V. Perfect fluid. Dark energy

1. Introduction

Recent cosmological observations contradict the matter dominated universe with decelerating expansion indicating that our universe experiences accelerated expansion. The accelerating expansion of the universe is driven by mysterious energy with negative pressure known as Dark Energy (DE). The evidence of the existence of

DE comes from the Supernova observations [1, 2] and other observations such as cosmic microwave background (CMB) anisotropies measured with WMAP satellite [3] and large scale structure [4]. These observations suggest that nearly two-third of our universe consists of DE and the remaining consists of relativistic dark matter and baryons [5]. In spite of all the observational evidences, the nature of DE is still a challenging problem in theoretical physics. A variety of possible solutions such as cosmological constant [6], quintessence [7], phantom field [8], tachyon field [9], quintom [10], and the interacting DE models like Chaplygin gas [11], holographic models [12] and braneworld models [13] etc. have been proposed to interpret accelerating universe. However, none of these models can be regarded as being entirely convincing so far.

Recently, many authors have studied the Bianchi type I model in the presence of anisotropic DE. Rodrigues [14] constructed a Bianchi type I CDM cosmological model whose DE component preserves non-dynamical character but yields anisotropic vacuum pressure. Koivisto and Mota [15, 16] proposed a different approach to resolve the CMB anisotropy problem; the earlier isotropy of the universe could be distorted by the direction dependent acceleration of the later universe. Koivisto and Mota [16] investigated the Bianchi I cosmological model containing interacting DE fluid with non-dynamical anisotropic EoS and perfect fluid component. They suggested that if the EoS is anisotropic, the expansion rate of the universe becomes direction dependent at late times and cosmological models with anisotropic EoS can explain some of the observed anomalies in CMB. Mota et al. [17] explored the possibility of using the cosmological observationto probe and constrain an imperfect DE fluid. They concluded thata perfect fluid representation of DE might ultimately turn out to be a phenomenological sufficient description of all the observational consequences of DE. However, one cannot exclude the possibility of imperfectness in DE.Akarsu and Kilinc [18, 19] suggested that anisotropic fluid must not necessarilypromote anisotropy in the expansion whereas such fluid may also actto support isotropic behavior of the universe. It has been shown [18] thatanisotropic Bianchi I model in the presence of perfect fluid and minimallyinteracting DE shows isotropic behavior for the earlier times of the universe.

Primordial magnetic fields can have a significant impact on the CMBanisotropy depending on the direction of field lines [20, 21]. Many peoplehave investigated the influence of magnetic field on the dynamics of universeby analyzing anisotropic Bianchi models. Milaneschi and Fabbri [22]studied the anisotropy and polarization properties of CMB radiation in homogeneous Bianchi I cosmological model. Jacobs [23] explored the effects of a uniform, primordial magnetic field on Bianchi type I cosmological model.He concluded that the primordial magnetic field produced large expansionanisotropies during the radiation-dominated phase but it had negligible effectduring the dust-dominated phase. King and Coles [21] discussed thedynamics of magnetized axisymmetric Bianchi I universe with vacuum energy.He examined the behavior of scale factors perpendicular and parallelto the field lines. Roy et al. [24] investigated Bianchi type I cosmologicalmodels containing perfect fluid and magnetic field directed along x axis. Exactsolutions are obtained using the condition that expansion is proportionalto shear scalar.

The discovery of the accelerated mode of expansion of the universe stands as a major breakthrough of the observational cosmology. Survey of cosmological distant type Ia supernovae (SNeIa; Riess et al 1998; Perlmutter et al 1999) indicated the presence of a new unaccounted-for Dark energy (DE) that opposes the self-attractions of matter and causes the expansion of universe to accelerate. This acceleration is realized with negative pressure and positive energy density that violate the strong energy condition. This violation gives a reverse gravitational effect. Due to this effect, the universe gets a jerk and the transition from the earlier deceleration phase to the recent acceleration phase takes place (Caldwell et al. 2006). The cause of this sudden transition and the source of accelerated expansion are still unknown. The state of the art in cosmology has led to the following present distribution of the energy densities of the universe: 4% for baryonic matter, 23% for non-baryonic dark matter and 73% so-called DE (Spergel et al. 2007).

Inflationary universes play a significant role in solving number of outstanding problems in cosmology like homogeneity, the isotropy, the horizons, flatness and primordial magnetic monopole problem in grand unified field theory.

In this paper we have investigated Bianchi type-V dark energy perfect fluid cosmological model (phantom and quintessence). To get the deterministic solution in terms of cosmic time t, we have assumed that shear (σ) is proportional to the scalar expansion (θ) i.e. $\sigma \propto \theta$. The potential $V(\phi)=\frac{\lambda}{n}\phi^n$ is assumed to discuss the physical results where ϕ is the Higg's field. The vacuum energy is assumed as $\frac{1}{R^2}$ as considered by Chen and Wu[39] to discuss isotropization process where R is scale factor.

2. Metric and Field Equations

We consider Bianchi V metric in the form

$$ds^2 = -dt^2 + A^2dx^2 + B^2e^{2x}dy^2 + C^2e^{2x}dz^2 \qquad \ldots(1)$$

where $A, B \text{ and } C$ are functions of 't' alone.

The energy momentum tensor for perfect fluid is taken as

$${}_{pf}T_{ab} = \left(\rho_{pf} + p_{pf}\right)u_a u_b + p_{pf} g_{ab} \qquad \ldots(2)$$

where ρ_{pf} is the matter density and p_{pf} is the pressure.

For commoving observer

$$u^1 = 0 = u^2 = u^3 \text{ and } u^4 = 1 \qquad \ldots(3)$$

The energy momentum tensor for dark energy is described by scalar field ϕ as

$${}_{\phi}T_{ab} = \left(\frac{1}{2}\varepsilon\phi_4^2 + V(\phi)\right)u_a u_b + \left(\frac{1}{2}\varepsilon\phi_4^2 - V(\phi)\right)h_{ab} \qquad \ldots(4)$$

Where

$$h_{ab} = g_{ab} + u_a u_b \qquad \ldots(5)$$

The phantom field $(\varepsilon=-1)$ may be considered as perfect fluid for comoving observers and the density and isotropic pressure are given by Tsamparlis and Pariathanasis [38]

$$\rho_\phi = \left(-\frac{1}{2}\phi_4^2 + V(\phi)\right) \qquad \ldots(6)$$

$$p_\phi = \left(-\frac{1}{2}\phi_4^2 - V(\phi)\right) \qquad \ldots(7)$$

where as for quintessence field $(\varepsilon = 1)$ the density as pressure are

$$\rho_\phi = \left(\frac{1}{2}\phi_4^2 + V(\phi)\right) \quad \ldots (8)$$

$$p_\phi = \left(\frac{1}{2}\phi_4^2 - V(\phi)\right) \quad \ldots (9)$$

For phantom field, equation (6) with (7) and for quintessence field equations (8) with (9) leads to

$$\rho_\phi - p_\phi = 2V(\phi) \quad \ldots (10)$$

The Einstein's field equations are given by

$$R_i^j - \frac{1}{2}Rg_i^j = -8\pi G\left({}_{pf}T_i^j + {}_{\phi}T_i^j\right) + \Lambda g_i^j \quad \ldots (11)$$

The Klein-Gordon equation for dark energy (quintessence and phantom) with potential $V(\phi)$ is given by

$$\phi_{44} + \left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right)\phi_4 = -\varepsilon\frac{dV(\phi)}{d\phi} \quad \ldots (12)$$

where $\varepsilon = 1$ (quintessence) and $\varepsilon = -1$ (phantom).

For the line element (1) Einstein's field equations reduce to the following system of equations

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} - \frac{\alpha^2}{A^2} = -\left(p_{pf} + \varepsilon\frac{1}{2}\phi_4^2 - V(\phi)\right) + \Lambda \quad \ldots (13)$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4C_4}{AC} - \frac{\alpha^2}{A^2} = -\left(p_{pf} + \varepsilon\frac{1}{2}\phi_4^2 - V(\phi)\right) + \Lambda \quad \ldots (14)$$

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} - \frac{\alpha^2}{A^2} = -\left(p_{pf} + \varepsilon\frac{1}{2}\phi_4^2 - V(\phi)\right) + \Lambda \quad \ldots (15)$$

$$\frac{A_4B_4}{AB} + \frac{B_4C_4}{BC} + \frac{A_4C_4}{AC} - \frac{3\alpha^2}{A^2} = \left(p_{pf} + \varepsilon\frac{1}{2}\phi_4^2 - V(\phi)\right) + \Lambda \quad \ldots (16)$$

$$\frac{2A_4}{A} - \frac{B_4}{B} - \frac{C_4}{C} = 0 \quad \ldots (17)$$

where the sub-indices 4 denotes ordinary differentiation with respect to t.

3. Solution of the field equations

To get the deterministic solution of Einstein's field equations (13)-(17) in terms of cosmic time t, we assume that $\sigma \propto \theta$, where σ is shear and θ the expansion in the model.

This leads to

$$B = \beta C^n \qquad \dots (18)$$

Equation (17) leads to

$$A^2 = BC \qquad \dots (19)$$

Equations (14) and (15) leads to

$$\frac{A_{44}}{A} - \frac{B_{44}}{B} + \frac{C_4}{C}\left(\frac{A_4}{A} - \frac{B_4}{B}\right) = 0 \qquad \dots (20)$$

Using equations (18) and (19) in equation (20), we get

$$\frac{C_{44}}{C} + \frac{C_4^2}{C^2}(2n+1) = 0 \qquad \dots (21)$$

On solving equation (21) we get

$$C = (2n+2)(k_1 t + k_2)^{\frac{1}{2n+1}} \qquad \dots (22)$$

where k_1 and k_2 are constant of integration

using equation (22) in equations (18) and (19)

$$B = \beta(2n+2)^n (k_1 t + k_2)^{\frac{n}{2n+2}} \qquad \dots (23)$$

and

$$A = \beta^{\frac{1}{2}} (2n+2)^{\frac{n+1}{2}} (k_1 t + k_2)^{\frac{1}{4}} \qquad \dots (24)$$

Hence the metric (1) takes the form

$$ds^2 = -dt^2 + \beta(2n+2)^{n+1}(k_1t+k_2)^{\frac{1}{2}}dx^2 + \beta^2(2n+2)^{2n}(k_1t+k_2)^{\frac{2n}{2n+2}}dy^2 + (2n+2)^n(k_1t+k_2)^{\frac{2}{2n+2}}dz^2 \quad \ldots (25)$$

After suitable transforms it takes the form

$$ds^2 = -\frac{dT^2}{k_1^2} + (2n+2)^{n+1}T^{\frac{1}{2}}dX^2 + (2n+2)^{2n}T^{\frac{2n}{2n+2}}dY^2 + (2n+2)^n T^{\frac{2}{2n+2}}dZ^2 \quad \ldots (26)$$

4. Some Physical Properties

Scalar expansion (θ) is given as

$$\theta = \frac{k_1}{T} \quad \ldots (27)$$

Components of shear tensor (σ_i^j) is calculated as

$$\sigma_1^1 = \frac{k_1}{6T} \quad \ldots (28)$$

$$\sigma_2^2 = \frac{k_1(n-2)}{3(2n+2)T} \quad \ldots (29)$$

$$\sigma_3^3 = -\frac{nk_1}{3(n+1)T} \quad \ldots (30)$$

$$\sigma_4^4 = 0 \quad \ldots (31)$$

and shear (σ) is given as

$$\sigma^2 = \frac{1}{2}\left[\frac{1}{36} + \frac{(n-2)^2}{(2n+2)^2} + \frac{1}{9(2n+2)^2}\right]\frac{k_1^2}{T^2} \quad \ldots (32)$$

The scale factor is defined as

$$R^3 = ABC \quad \ldots (33)$$

In our model scale factor is given by

$$R = \beta^{\frac{1}{2}}\left[(2n+2)T^{\frac{n}{2n+2}}\right]^{\frac{n+1}{2}} \qquad \dots (34)$$

Case: I

For phantom field $(\varepsilon = -1)$ the equation (12) leads to

$$\phi_{44} + \left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right)\phi_4 = \frac{dV(\phi)}{d\phi} \qquad \dots (35)$$

To find the solution, we assume that

$$V(\phi) = \frac{\lambda}{n}\phi^n \qquad \dots (36)$$

Following Linde[37]

$$\phi_{44} << \frac{dV(\phi)}{d\phi} \qquad \dots (37)$$

Using equations (36) and (37) in equation (35)

$$\frac{k_1}{(k_1 t + k_2)}\phi_4 = \lambda\phi^{n-1} \qquad \dots (38)$$

which on integration leads to

$$\phi^{2-n} = \frac{\lambda(2-n)}{2k_1}\left(k_1 t^2 + 2k_2 t + 2k_3\right) \qquad \dots (39)$$

where k_3 is constant of integration

when $n = 4$ equation (39) reduces to

$$\phi^2 = -\frac{k_1}{\lambda\left(k_1 t^2 + 2k_2 t + 2k_3\right)} \qquad \dots (40)$$

Which matches the result with Linde[37]

The matter density (ρ_{pf}) and the pressure (p_{pf}) for the model (26) $(\varepsilon = -1)$ are given by

$$p_{pf} = \frac{k_1^2(2n+1)(n+3)}{8(n+1)^2 T^2} - \frac{k_1^2}{16T^2} + \frac{\alpha^2}{\beta(2n+2)^{1/n+1} T^{1/2}} + \frac{k_1 T}{2\lambda(k_1 t^2 + 2k_2 t + 2k_3)^2} - \frac{\lambda}{n}\left[\frac{k_1}{\lambda(k_1 t^2 + 2k_2 t + 2k_3)}\right]^{n/2} + \frac{1}{\beta\left[(2n+2)T^{\frac{n}{2n+2}}\right]^{n+1}} \quad \ldots (41)$$

$$\rho_{pf} = \frac{k_1^2(n^2+3n+1)}{4(n+1)^2 T^2} - \frac{3\alpha^2}{\beta(2n+2)^{n+1} T^{1/2}} + \frac{k_1 T}{2\lambda(k_1 t^2 + 2k_2 t + 2k_3)^2} - \frac{\lambda}{n}\left[\frac{k_1}{\lambda(k_1 t^2 + 2k_2 t + 2k_3)}\right]^{n/2} - \frac{1}{\beta\left[(2n+2)T^{\frac{n}{2n+2}}\right]^{n+1}} \quad \ldots (42)$$

Case: 2

For quintessence $(\varepsilon = 1)$ the equation (12) leads to

$$\phi_{44} + \left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right)\phi_4 = -\frac{dV(\phi)}{d\phi} \quad \ldots (42)$$

To find the solution, we assume that

$$V(\phi) = \frac{\lambda}{n}\phi^n \quad \ldots (43)$$

Following Linde[37]

$$\phi_{44} << \frac{dV(\phi)}{d\phi} \quad \ldots (44)$$

Using equations (36) and (37) in equation (35), we have

$$\frac{k_1}{(k_1 t + k_2)}\phi_4 = -\lambda\phi^{n-1} \quad \ldots (45)$$

which on integration leads to

$$\phi^{2-n} = \frac{\lambda(n-2)}{2k_1}\left(k_1 t^2 + 2k_2 t + 2k_3\right) \qquad \ldots (46)$$

where k_3 is constant of integration

when $n = 4$, equation (39) reduces to

$$\phi^2 = \frac{k_1}{\lambda\left(k_1 t^2 + 2k_2 t + 2k_3\right)} \qquad \ldots (47)$$

Which matches the result with Linde[37]

The matter density (ρ_{pf}) and the pressure (p_{pf}) for the model (26) $(\varepsilon = -1)$ are given by

$$p_{pf} = \frac{k_1^2(2n+1)(n+3)}{8(n+1)^2 T^2} - \frac{k_1^2}{16T^2} + \frac{\alpha^2}{\beta(2n+2)^{1/n+1} T^{1/2}} - \frac{k_1 T}{2\lambda\left(k_1 t^2 + 2k_2 t + 2k_3\right)^2} - \frac{\lambda}{n}\left[\frac{k_1}{\lambda\left(k_1 t^2 + 2k_2 t + 2k_3\right)}\right]^{n/2} + \frac{1}{\beta\left[(2n+2)T^{\frac{n}{2n+2}}\right]^{n+1}} \qquad \ldots (48)$$

$$\rho_{pf} = \frac{k_1^2\left(n^2+3n+1\right)}{4(n+1)^2 T^2} - \frac{3\alpha^2}{\beta(2n+2)^{n+1} T^{1/2}} - \frac{k_1 T}{2\lambda\left(k_1 t^2 + 2k_2 t + 2k_3\right)^2} - \frac{\lambda}{n}\left[\frac{k_1}{\lambda\left(k_1 t^2 + 2k_2 t + 2k_3\right)}\right]^{n/2} - \frac{1}{\beta\left[(2n+2)T^{\frac{n}{2n+2}}\right]^{n+1}} \qquad \ldots (49)$$

5. Conclusion

We find that spatial volume increases with time. Hence the model represents inflationary scenario. Since $\frac{\sigma}{\theta} \neq 0$, hence anisotropy is maintained throughout. However at n = 0, 2 the model (26)isotropizes along z axis and y axis respectively. The model has POINT TYPE singularity at T = 0 if $n > 0$. For $n < 0$ and $n \neq -1$, the model (26) has CIGAR TYPE singularity at T = 0. The model (26) starts with a big bang at T = 0 and the scalar expansion (θ) decreases with time. The Higg's field (ϕ) for phantom field is initially large but decreases due to lapse of time and for quintessence field the Higg's field (ϕ) also decreases with time. In phantom field $(\varepsilon = -1)$ as well as in quintessence field $(\varepsilon = 1)$ as $t \to \infty$ then $\phi^{(n-2)} \to 0$ and for n = 4, $\phi^2 \to 0$. The cosmological term (Λ), the expansion (θ) are initially large but decreases due to lapse of time.

REFERENCES

[1] Perlmutter, S. et al.: Astrophys. J. 483(1997)565; Perlmutter, S. et al.: Nature 391(1998)51; Perlmutter, S. et al.: Astrophys. J. 517(1999)565.

[2] Riess, A.G. et al.: Astron. J. 116(1998)1009.

[3] Bennett, C.L. et al.: Astrophys. J. Suppl. 148(2003)1; Spergel, D.N. et al.: Astrophys. J. Suppl. 148(2003)175.

[4] Verde, L. et al.: Mon. Not. R. Astron. Soc. 335(2002)432; Hawkins, E. et al.: Mon. Not. Roy. Astr. Soc. 346(2003)78; Abazajian, et al.: Phys. Rev. D69(2004)103501.

[5] Hinshaw, G. et al.: Astrophys. J. Suppl. 180(2009)225.

[6] Weinberg, S.: Rev. Mod. Phys. 61(1989)1; Peebles, P. J. E. and Ratra, B.: Rev. Mod. Phys. 75(2003)559.

[7] Sahni, V. and Starobinsky, A.A.: Int. J. Mod. Phys. D9(2000)373; Sahni, V.: Lect. Notes Phys. 653(2004)141; Padmanabhan, T.: Gen. Relativ.Grav.40(2008)529.

[8] Caldwell, R.R.: Phys. Lett. B23(2002)545; Nojiri, S. and Odintsov, S.D.: Phys. Lett. B147(2003)562; ibid. B1(2003)565.

[9] Sen, A.: JHEP 48(2002)204; Padmanabhan, T.: Phys. Rev. D66(2002)021301; Padmanabhan, T. and Choudhury, T.R.: Phys. Rev. D66(2002)081301.

[10] Feng, B.,Wamg, X.L. and Zhang, X.M.: Phys. Lett. B607(2005)35; Guo, Z.K. et al.: Phys. Lett. B608(2005)177.

[11] Kamenshchik, A., Moschella, U. and Pasquier, V.: Phys. Lett.B511(2001)265; Bento, M.C. Bertolami, O. and Sen, A.A.: Phys. Rev. D66(2002)043507.

[12] Wang, B., Gong, Y. G. and Abdalla, E.: Phys. Lett. B624(2005)141;Setare, M. R.: Phys. Lett. B642(2006)1; Hu, B. and Ling, Y.: Phys.Rev. D73(2006)123510; Kim, H., Lee, H. W. and Myung, Y. S.: Phys.Lett. B632(2006)605; Setare, M. R.: JCAP 0701(2007)023.

[13] Li, M.: Phys. Lett. B603(2004)1; Deffayet, C., Dvali, G.R. andGabadadze, G.: Phys. Rev. D65(2002)044023.

[14] Rodrigues, D.C.: Phys. Rev. D77(2008)023534.

[15] Koivisto, T. and Mota, D.F.: JCAP 806(2008)18.

[16] Koivisto, T. and Mota, D.F.: Astrophysical J. 679(2008)1.

[17] Mota, D. F. et al.: Mon. Not. Roy. Astron. Soc. 382(2007)793.

[18] Akarsu, O. and Kilinc, C.B.: Gen. Relativ. Grav.42(2010)1.

[19] Akarsu, O. and Kilinc, C.B.: Gen. Relativ. Grav.42(2010)763.

[20] Madsen, M.S.: Mon. Not. R. Astron. Soc. 237(1989)109.

[21] King, E.J. and Coles, P.: Class. Quantum Grav.24(2007)2061.

[22] Milaneschi, E. and Fabbri, R.: Astron. Astrophys. 151(1985)7.

[23] Jacobs, K.C.: Astrophys. J. 155(1969)379.

[24] Roy, S.R., Singh, J.P. and Narin, S.: Aust. J. Phys. 38(1985)239.

[25] Carroll, S.M. Hoffman, M. and Trodden, M.: Phys. Rev.D68(1992)023509.

[26] Collins, C.B. and Hawking, S.W.: Astrophys. J. 180(1973)317.

[27] Thorne, K.S.: Astrophys. J. 148(1967)51.

[28] Kantowski, R. and Sachs, R.K.: J. Math. Phys. 7(1966)433.

[29] Kristian, J. and Sachs, R.K.: Astrophys. J. 143(1966)379.

[30] Collins, C.B.: Phys. Lett. A60(1977)397.

[31] Roy, S.R. and Banerjee, S.K.: Class. Quantum Grav.11(1995)1943.

[32] Bali, R. and Kumawat, P.: Phys. Lett. B665(2008)332.

[33] Bali, R. Banerjee, R. and Banerjee, S.K.: Astrophys. Space Sci.317(2008)21.

[34] Singh, C.P. and Kumar, S.: Astrophys. Space Sci. 310(2007)31.

[35] Gron, Ø.: Am. J. Phys. 54(1986)1.

[36] Padmanabhan, T.: Structure Formation in the universe (CambridgeUniversity Press, 1993).

[37] Linde, A. D.: Class. Quantum Grav.18, 3275 (2001).

[38] Tsamparlis, M. and Paliathanasis, A.: Class Quantum Grav. 29, 015006(2012).

[39] Chen, W. and Wu, Y.S.:Phys. Rev. D41, 695 (1990).

Ganita Sandesh Vol. 28 No. 1 and 2 (June and December, 2014) pp. 55 - 60 ISSN : 0970-9169

Some Results Associated With Extended τ -Gauss Hypergeometric Functions

Rajeev Kumar Gupta, Bhupender Singh Shaktawat and Dinesh Kumar

Department of Mathematics and Statistics,

Jai Narain Vyas University, Jodhpur-342005, India.

Email : drrkbgupta@yahoo.co.in

b.s.shaktawatmath@gmail.com

dinesh_dino03@yahoo.com

Abstract

In this paper we define an extension of the τ - Gauss hypergeometric function ${}_3R_2^{\tau}(z)$, and investigate its various properties such as integral representation, derivative formula, Mellin transform and fractional calculus operators. Some interesting special cases of our main results are also pointed out. The established results provide extensions of the results given by Parmar [2].

2010 Mathematics Subject Classifications: 33B15, 33B20, 33C05, 33C20.

Key words: Generalized Gauss hypergeometric functions, τ -hypergeometric function, Mellin transform, Fractional calculus.

1. Introcuction and Preliminaris

In 2001, Virchenko et al. [7] have studied and investigate the following τ -Gauss hypergeometric function for $\tau > 0, |z| < 1; \mathrm{Re}(c) > \mathrm{Re}(b) > 0$ as

$$ {}_2R_1^{\tau}(z) = {}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}, \qquad \ldots(1.1) $$

They also gave the Euler type integral representation ([7], p.91, eq. (6)):

$$ {}_2R_1\left(a,b;c;\tau;z\right)=\frac{1}{B\left(b,c-b\right)}\int_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-zt^{\tau}\right)^{-a}dt. \qquad \dots (1.2)$$

The special case when $\tau=1$ in (1.1) and (1.2) yields the familiar representation of Gauss's hypergeometric function. Further, Saxena et al. [5] defined generalized hypergeometric function in the following form

$$ {}_3R_2^{\tau}\left(z\right)={}_3R_2\left(\lambda,a,b;c,d;\tau;z\right)=\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(\lambda)_n\Gamma(a+\tau n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)\Gamma(d+\tau n)}\frac{z^n}{n!}, \qquad \dots (1.3)$$

where $\tau>0,\left|z\right|<1$.

Recently, Parmar [2] defined the extended τ - Gauss hypergeometric function ${}_2R_1^{\tau}(z)$ as follows: For $a,b\in C$ and $c\in C\setminus Z_0^-$ is given by

$$ {}_2R_1^{\tau}\left(z\right)={}_2R_1^{\tau}\left(\left(a,p\right),b;c;z\right)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a;p)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}, \qquad \dots (1.4)$$

where, $p\geq 0,\tau>0,\left|z\right|<1,\mathrm{Re}(c)>\mathrm{Re}(b)>0$ when $p=0$.

Motivated mainly by investigations of the extended τ - Gauss hypergeometric function ${}_2R_1^{\tau}(z)$ given by Virchenko et al. ([7], p.90, eq. (5)), we introduced the extended τ - Gauss hypergeometric function ${}_3R_2^{\tau}(z)$ as follows:

For $\lambda,a,b\in C$ and $c,d\in C\setminus Z_0^-$, we have

$$ {}_3R_2^{\tau}\left(z\right)={}_3R_2^{\tau}\left(\left(\lambda,p\right),a,b;c,d;z\right)=\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(\lambda;p)_n\Gamma(a+\tau n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)\Gamma(d+\tau n)}\frac{z^n}{n!}, \qquad \dots (1.5)$$

where, $p\geq 0,\tau>0,\left|z\right|<1,\mathrm{Re}(d)>\mathrm{Re}(a)>0,\mathrm{Re}(c)>\mathrm{Re}(b)>0$ when $p=0$, and $(\lambda,p)_n$ is the generalized Pochhammer symbol ([6], p. 484, eq. (2)) defined as

$$ (\lambda,p)_n=\begin{cases}\dfrac{\Gamma_p(\lambda+n)}{\Gamma(\lambda)} & \left(\mathrm{Re}(p)>0;\lambda,n\in C\right),\\ (\lambda)_n & \left(p=0;\ \lambda,n\in C\right).\end{cases} \qquad \dots (1.6)$$

Remark 1. If we take $b=d$ then (1.5) reduces to extended τ - Gauss hypergeometric function ${}_2R_1^{\tau}(z)$ studied by Parmar ([2], p.422, eq.(2.1)); further if we set $\tau=1$ then (1.5) reduces to the extended Gauss hypergeometric function ([6], p.487, eq.(17)).

Remark 2. If we set $p=0$ in (1.5), then it reduces to the Gauss hypergeometric function ${}_3R_2(\lambda,a,b;c,d;\tau;z)$ studied by Saxena et al. [5].

Remark 3. If we set $\tau=1, p=0$ in (1.5), then it reduces to the classical Gauss's hypergeometric function ${}_3F_2$; further, if we take $b=d$ then it reduces to the classical Gauss's hypergeometric function ${}_2F_1$.

2. Integral Representation and Derivative Formula

Theorem 1. The following integral representation for ${}_3R_2^\tau(\mathrm{z})$ in (1.5) holds true:

$${}_3R_2^\tau((\lambda,p),a,b;c,d;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}{}_2R_1^\tau((\lambda,p),\mathrm{b},d;zt^\tau)dt \qquad ...(2.1)$$

where, $\mathrm{Re}(p)>0, \tau>0, \mathrm{Re}(d)>\mathrm{Re}(a)>0, \mathrm{Re}(c)>\mathrm{Re}(b)>0$ when $p=0$.

Proof. Considering the following elementary identity for the Beta function $B(m,n)$:

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}=\int_0^1 t^{m-1}(1-t)^{n-1}dt$$

in (1.5) and using the series representation of extended τ - Gauss hypergeometric function ${}_2R_1^\tau(z)$, then we get the desired integral representation (2.1).

If we take Remark 1-3 into account then we can obtain many special cases of (2.1).

Theorem 2. The following derivative formula for ${}_3R_2^\tau(\mathrm{z})$ holds true:

$$\left(\frac{d}{dz}\right)^n\left[z^{c-1}{}_3R_2^\tau((\lambda,p),a,b;c,d;\,\mathrm{w}z^\tau)\right]=\frac{z^{c-n-1}\Gamma(c)}{\Gamma(c-n)}{}_3R_2^\tau((\lambda,p),a,b;c-n,d;\,\mathrm{w}z^\tau). \qquad ...(2.2)$$

Proof. According to the uniform convergence of the series (1.5), differentiating term by term under the sign of summation; and finally using (1.5) then we obtain the R.H.S. of (2.2) after little simplifications.

Remark 4. If set b = d in (2.2), then we get known result due to Parmar ([2], p. 424, eq. (3.7)).

3. Mellin Transform

The Mellin transform of a suitable function $f(\tau)$ is defined, as usual, by

$$M\{f(\tau):\tau\to s\}=\int_0^\infty \tau^{s-1}f(\tau)\,d\tau, \qquad ...(3.1)$$

provided that the improper integral in (3.1) exists.

Theorem 3. The Mellin transform of the function ${}_3R_2^{\tau}(z)$ defined by (1.5) is given by

$$M\left\{{}_3R_2^{\tau}\left((\lambda,p),a,b;c,d;\ z\right):p\rightarrow s\right\}=\Gamma(s)(\lambda)_s\ {}_3R_2\left(\lambda+s,a,b;c,d;\tau;z\right), \quad \text{... (3.2)}$$

where, $\mathrm{Re}(s)>0$ and $\mathrm{Re}(\lambda+s)>0$ when $p=0$.

Proof. Using the definition (3.1) of the Mellin transform, we find from (1.5), that

$$M\left\{{}_3R_2^{\tau}\left((\lambda,p),a,b;c,d;z\right):p\rightarrow s\right\}$$
$$=\int_0^{\infty}p^{s-1}\left(\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(\lambda;p)_n\Gamma(a+\tau n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)\Gamma(d+\tau n)}\frac{z^n}{n!}\right)dp.$$

Interchanging the order of integration and summation, we have

$$=\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+\tau n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)\Gamma(d+\tau n)}\frac{z^n}{n!}\frac{1}{\Gamma(\lambda)}\int_0^{\infty}p^{s-1}\Gamma_p(\lambda+n)dp.$$

Now, using the result of Chaudhry and Zubair ([1], p.16, eqn.(1.110)) given by

$$\int_0^{\infty}p^{s-1}\Gamma_p(\lambda+n)dp=\Gamma(\lambda+s+n)\Gamma(s), \qquad (\mathrm{Re}(s)>0), \quad \text{... (3.3)}$$

then we get

$$M\left\{{}_3R_2^{\tau}\left((\lambda,p),a,b;c,d;\ z\right):p\rightarrow s\right\}$$
$$=\frac{\Gamma(s)\Gamma(c)\Gamma(d)}{\Gamma(\lambda)\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+s+n)\Gamma(a+\tau n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)\Gamma(d+\tau n)}\frac{z^n}{n!}$$
$$=\Gamma(s)\,(\lambda)_s\,\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(\lambda+s)_n\Gamma(a+\tau n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)\Gamma(d+\tau n)}\frac{z^n}{n!}.$$

By using, (1.3) into account then we get the result (3.2).This is the complete proof of Theorem 3.

Remark 5. If we put b = d in (3.2), then it reduces to given result by Parmar ([2], p. 424, eq. (4.2)).

4. Fractional Calculus Approach

In this section, we consider compositions of the Riemann-Liouville fractional integrals and derivatives $I_{\rho+}^{\alpha}$ and $D_{\rho+}^{\alpha}$ ([4], Section (2.3) and (2.4)):

$$\left(I_{\rho+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{\rho}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt, \qquad \left(\alpha\in C,\ \mathrm{Re}(\alpha)>0\right), \quad \text{... (4.1)}$$

and $$\left(D_{\rho+}^{\alpha} f\right)(x)=\left(\frac{d}{dx}\right)^{n}\left(I_{\rho+}^{n-\alpha} f\right)(x), \qquad \left(\alpha\in C,\ \mathrm{Re}(\alpha)>0;\ n=\left[\mathrm{Re}(\alpha)\right]+1\right) \qquad \ldots (4.2)$$

Theorem 4. Let $\rho\in R_{+}=[0,\infty)$, $\lambda,\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d},\omega\in C$ and $\mathrm{Re}(\alpha)>0, \mathrm{Re}(c)>0, \mathrm{Re}(\tau)>0$ then for $x>\rho$ the following relations holds true:

$$\left(I_{\rho+}^{\alpha}\left\{(t-\rho)^{c-1}\ {}_{3}R_{2}^{\tau}\left((\lambda,p),a,b;c,d;\ \omega(t-\rho)^{\tau}\right)\right\}\right)(x)$$
$$=\frac{(x-\rho)^{c+\alpha-1}\Gamma(c)}{\Gamma(c+\alpha)}\ {}_{3}R_{2}^{\tau}\left((\lambda,p),a,b;c+\alpha,d;\ \omega(x-\rho)^{\tau}\right), \qquad \ldots (4.3)$$

and

$$\left(D_{\rho+}^{\alpha}\left\{(t-\rho)^{c-1}\ {}_{3}R_{2}^{\tau}\left((\lambda,p),a,b;c,d;\ \omega(t-\rho)^{\tau}\right)\right\}\right)(x)$$
$$=\frac{(x-\rho)^{c-\alpha-1}\Gamma(c)}{\Gamma(c-\alpha)}\ {}_{3}R_{2}^{\tau}\left((\lambda,p),a,b;c-\alpha,d;\ \omega(x-\rho)^{\tau}\right), \qquad \ldots (4.4)$$

Proof. By virtue of the formula (4.1) and (1.5). the term by term fraction integration and the application of the relation (see [4], 2.44)

$$\left(I_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}, \qquad \left(\alpha,\beta\in C,\ \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0\right),$$

yields, for $x>\rho$

$$\left(I_{\rho+}^{\alpha}\left\{(t-\rho)^{c-1}\ {}_{3}R_{2}^{\tau}\left((\lambda,p),a,b;c,d;\ \omega(t-\rho)^{\tau}\right)\right\}\right)(x)$$
$$=\left(I_{\rho+}^{\alpha}\left[\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(\lambda;\mathrm{p})_{n}\Gamma(a+\tau n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)\Gamma(d+\tau n)}\frac{\omega^{n}}{n!}(t-\rho)^{c+\tau n-1}\right]\right)(x)$$
$$=\frac{(x-\rho)^{c+\alpha-1}\Gamma(c)}{\Gamma(c+\alpha)}\ {}_{3}R_{2}^{\tau}\left((\lambda,p),a,b;c+\alpha,d;\ \omega(x-\rho)^{\tau}\right).$$

Next, by using (4.2) and (1.5), we have

$$\left(D_{\rho+}^{\alpha}\left\{(t-\rho)^{c-1}\ {}_{3}R_{2}^{\tau}\left((\lambda,p),a,b;c,d;\ \omega(t-\rho)^{\tau}\right)\right\}\right)(x)$$
$$=\left(\frac{d}{dx}\right)^{n}\left(I_{\rho+}^{n-\alpha}\left[(t-\rho)^{c-1}\ {}_{3}R_{2}^{\tau}\left((\lambda,p),a,b;c,d;\ \omega(t-\rho)^{\tau}\right)\right]\right)(x).$$

Now, using (4.3) and (2.2), we obtain the R.H.S. of (4.4). This is the complete proof of Theorem 4.

Remark 6. If we take b = d in (4.3) and (4.4), then we can obtain known results given by Parmar ([2], p. 425, eq. (5.3) and (5.4)).

5. Concluding Remarks

In the present paper we investigated an extended τ - Gauss hypergeometric function ${}_3R_2^{\tau}(z)$, and obtained its several properties. The obtained results are extension of work done by Parmar [2].

Acknowledgments

The author (Dinesh Kumar) would like to express his deep thanks to NBHM (National Board of Higher Mathematics), India, for granting a Post-Doctoral Fellowship (sanction no.2/40(37)/2014/R&D-II/14131).

REFERENCES

[1]. M.A. Chaudhry and S.M. Zubair, On a class of incomplete Gamma functions with applications, CRC Press (Chapman and Hall), Boca Raton, FL, (2002).

[2]. R.K. Parmar, Extended τ -hypergeometric functions and associated properties, Computes rendus Mathematique, Vol. **353** (5) (2015), 421-426.

[3]. E.D. Rainville, Special functions, Macmillan Company, New York, 1960; reprinted by Chelsea Publishing Company Bronx, New York, (1971).

[4]. S.G. Samko, A.A. Kilbas, and O.I. Marichev, Fractional integrals and derivatives, theory and applications, Gordon and Breach, Yverdon et alibi, (1993).

[5]. R.K. Saxena, Chena Ram and Naresh, Some results associated with a generalized hypergeometric function, Bulletin of Pure and Applied Sciences, Vol. **24**(2) (2005), 305-316.

[6]. H.M. Srivastava, A. Cetinkaya and İ Onur Kıyamaz, A certain generalized Pochhammer symbol and its applications to hypergeometric functions, Appl. Math. Comput., Vol. **226** (2014), 484-491.

[7]. N. Virchenko, S.L. Kalla and A. Al-Zamel, Some results on a generalized hypergeometric function, Integral Transforms and Special Functions, Vol. **12**(1) (2001), 89-100.

Ganita Sandesh Vol. 28 No. 1 and 2 (June and December, 2014) pp. 61 - 70 ISSN : 0970-9169

Double Integral Involving The $\overline{\text{H}}$-Function and a General Class of Polynomials

Rajendra Pal Sharma

Govt. Engineering College, Ajmer-305025, Rajasthan, India

E-mail: drrpsharma@yahoo.com

Abstract

In the present paper, we first establish an interesting finite double integral involving a general class of polynomials and the $\overline{\text{H}}$ *-function. On account of the most general nature of the function occurring in this integral, our findings provide interesting unifications and extensions of a large number of new and known results. For the sake of illustration, we give here three special cases of our main integral which are also new and of interest by themselves. Certain known results given earlier by Ronghe, Soni, Anandani follow as simple special cases of our main findings.*

Keywords: Double integrals, a general class of polynomials, Fox H-function, Feynman integrals, $\overline{\text{H}}$ -function.

Mathematics Subject Classification (2010): 33C05, 33C99

1. Introduction

The $\overline{\text{H}}$ -function ([8],[9]) occurring in the paper will be defined and represented in the following manner [3]

$$\bar{H}_{P,Q}^{M,N}[z] = \bar{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \bar{\phi}(\xi)\, z^{\xi}\, d\xi, \quad ...(1.1)$$

where
$$\bar{\varphi}(\xi) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j\xi) \prod_{j=1}^{N} \left\{ \Gamma(1 - a_j + \alpha_j\xi) \right\}^{A_j}}{\prod_{j=M+1}^{Q} \left\{ \Gamma(1 - b_j + \beta_j\xi) \right\}^{B_j} \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j\xi)} \qquad \ldots(1.2)$$

which contains fractional powers of some of the gamma functions. Here, and throughout the paper $a_j\ (j = 1, \ldots, P)$ and $b_j\ (j = 1, \ldots, Q)$ are complex parameters, $\alpha_j \geq 0\ (j = 1, \ldots, P)$, $\beta_j \geq 0\ (j = 1, \ldots, Q)$ (not all zero simultaneously) and the exponents $A_j\ (j = 1, \ldots, N)$ and $B_j\ (j = M + 1, \ldots, Q)$ can take on non-integer values.

The contour L in (1.1) is imaginary axis $\mathrm{Re}(\xi) = 0$. It is suitably indented in order to avoid the singularities of the Gamma functions and to keep those singularities on appropriate sides.

Again, for $A_j\ (j = 1, \ldots, N)$ not an integer, the poles of the Gamma functions of the numerator in (1.2) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j - \beta_j\xi)\ (j = 1, \ldots, M)$ and $\Gamma(1 - a_j + \alpha_j\xi)\ (j = 1, \ldots, N)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

Evidently, when the exponents A_j and B_j are all positive integers, the $\overline{H}$-function reduces to the well-known Fox's H-function [4,17].

The basic properties and the following sufficient conditions for the absolute convergence of the defining integral for the $\overline{H}$-function have been given by Buschman and Srivastava [3].

$$\Omega \equiv \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} A_j\alpha_j - \sum_{j=M+1}^{Q} B_j\beta_j - \sum_{j=N+1}^{P} \alpha_j > 0 \qquad \ldots(1.3)$$

and

$$|\arg(z)| < \frac{1}{2}\pi\Omega, \qquad \ldots(1.4)$$

where Ω is given by (1.3).

The behaviour of the $\bar{H}$-function for small values of |z| follows easily from a result recently given by Rathie [11,p.306, eq. (6.9)]. We have

$$\bar{H}_{P,Q}^{M,N}[z] = O\left(|z|^{\alpha}\right),\ \alpha = \min_{1 \le j \le M}\left[\operatorname{Re}\left(b_j/\beta_j\right)\right],\ |z| \to 0, \qquad \ldots(1.5)$$

Investigations of the convergence conditions, all possible types of contours, type of critical points of the integrand of (1.1), etc. can be made by an interested reader by following analogous techniques given in the well known works of Braaksma [2]. Hai and Yakubovich [7]. We however omit the details.

Also $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava [16, p.1,eq. (1)]

$$S_n^m[x] = \sum_{r=0}^{[n/m]} \frac{(-n)_{mr}}{r!} A_{n,r} x^r,\ n = 0, 1, 2, \ldots, \qquad \ldots(1.6)$$

where m is an arbitrary positive integer and the coefficients $A_{n,r}(n, r \ge 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,r}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Lagurre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others [18,pp.158-161].

2. Main Integral

$$\int_0^1 \int_0^{\pi/2} e^{\omega(\alpha+\beta)\theta} (\sin\theta)^{\alpha-1} (\cos\theta)^{\beta-1}$$

$$x^{\rho-1}(1-x)^{\sigma-1}[ax+b(1-x)]^{-\rho-\sigma}\,{}_2F_1[c, d; \rho; ax/ax+b(1-x)]$$

$$S_n^m\left[y e^{\omega(u'+v')\theta} (\sin\theta)^{u'} (\cos\theta)^{v'} (ax)^{h'} [b(1-x)]^{k'} [ax+b(1-x)]^{-h'-k'}\right]$$

$$\overline{H}_{P,Q}^{M,N}\left[z e^{\omega(u+v)\theta}(\sin\theta)^{u}(\cos\theta)^{v}(ax)^{h}[b(1-x)]^{k}[ax+b(1-x)]^{-h-k}\right.$$

$$\left.\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]d\theta\,dx$$

$$= \frac{e^{\omega\pi\alpha/2}}{a^{\rho}b^{\sigma}}\sum_{r=0}^{[n/m]}\frac{(-n)_{mr}}{r!}A_{n,r}\,(y\,e^{\omega\pi u'/2})^{r}$$

$$\overline{H}_{P+5,Q+3}^{M,N+5}\left[z\,e^{\omega\pi u/2}\left|\begin{matrix}(1-\alpha-u'r,u;1),(1-\beta-v'r,v;1),\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\end{matrix}\right.\right.$$

$$\begin{matrix}(1-\rho-h'r,h;1),(1-\sigma-k'r,k;1),(1-\rho-\sigma-h'r-k'r+c+d,h+k;1),\\(1-\alpha-\beta-u'r-v'r,u+v;1),(1-\rho-\sigma-h'r-k'r+c,h+k;1),\end{matrix}$$

$$\left.\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(1-\rho-\sigma-h'r-k'r+d,h+k;1)\end{matrix}\right] \qquad \ldots(2.1)$$

where (i) $u'\geq 0,\ v'\geq 0\ ;\ u\geq 0,\ v\geq 0$ (not both zero simultaneously)

$h\geq 0,\ k\geq 0$ (not both zero simultaneously)

(ii) $\mathrm{Re}(\alpha)+u\min_{1\leq j\leq M}\left[\mathrm{Re}(b_j/\beta_j)\right]>0$

$\mathrm{Re}(\beta)+v\min_{1\leq j\leq M}\left[\mathrm{Re}(b_j/\beta_j)\right]>0,$

$\mathrm{Re}(\rho)+h\min_{1\leq j\leq M}\left[\mathrm{Re}(b_j/\beta_j)\right]>0$

$\mathrm{Re}(\sigma)+k\min_{1\leq j\leq M}\left[\mathrm{Re}(b_j/\beta_j)\right]>0$

(iii) The $\overline{\mathrm{H}}$-functions occurring in (2.1) satisfy conditions corresponding appropriately to those given by (1.3) and (1.4), and a and b are such that the expression $[ax+b(1-x)]$ is not equal to zero where $0\leq x\leq 1$.

Proof: To establish the double integral (2.1), we express the $\overline{\mathrm{H}}$-function occurring in its left-hand side in terms of Mellin-Barnes contour integral given by (1.1),the general class of polynomials occurring therein the series from given by (1.6) and then interchange the order of θ, x – and ξ -integrals (which is permissible under the conditions stated with (2.1))so that the left-hand side of the integral (2.1) (say Δ) assumes the following form after a little simplification:

$$\Delta = \sum_{r=0}^{[n/m]} \frac{(-n)_{mr}}{r!} A_{n,r} \left(y\, a^{h'} b^{k'} \right)^r \frac{1}{2\pi i} \int_L \overline{\varphi}(\xi) (z\, a^h b^k)^{\xi}$$

$$\left\{ \int_0^{\pi/2} e^{\omega(\alpha+\beta+u'r+v'r+u\xi+v\xi)\theta} (\sin\theta)^{\alpha+u'r+u\xi-1} (\cos\theta)^{\beta+v'r+v\xi-1} d\theta \right\}$$

$$\left\{ \int_0^1 x^{\rho+h'r+h\xi-1} (1-x)^{\sigma+k'r+k\xi-1} [a x + b(1-x)]^{-\rho-\sigma-h'r-k'r-h\xi-k\xi} \right.$$

$$\left. {}_2F_1 [c, d; \rho; a x / a x + b(1-x)]\, dx \right\} d\xi \qquad \ldots(2.2)$$

Now evaluating the inner-integrals occurring in (2.2), with the help of the known result [10] given below

$$\int_0^{\pi/2} e^{\omega(\alpha+\beta)\theta} (\sin\theta)^{\alpha-1} (\cos\theta)^{\beta-1} = e^{\omega\pi\alpha/2} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \ldots(2.3)$$

where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$

and a known integral [13]

$$\int_0^1 x^{\rho-1} (1-x)^{\sigma-1} [a x + b(1-x)]^{-\rho-\sigma} {}_2F_1 [c, d; \rho; a x / a x + b(1-x)]\, dx$$

$$= \frac{\Gamma(\rho)\ \Gamma(\sigma)\ \Gamma(\rho+\sigma-c-d)}{a^{\rho} b^{\sigma}\, \Gamma(\rho+\sigma-c)\ \Gamma(\rho+\sigma-d)} \qquad \ldots(2.4)$$

where $\text{Re}(\rho) > 0$, $\text{Re}(\sigma) > 0$, $\text{Re}(\rho + \sigma - c - d) > 0$, a and b are such that the expression $[ax + b(1-x)]$ is not equal to zero where $0 \le x \le 1$, we get after a little simplification

$$\Delta = \sum_{r=0}^{[n/m]} \frac{(-n)_{mr}}{r!} A_{n,r} (ye^{\omega\pi u'/2})^r \frac{1}{2\pi i} \frac{e^{\omega\pi\alpha/2}}{a^\rho b^\sigma} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha + u'r + u\xi)\,\Gamma(\beta + v'r + v\xi)}{\Gamma(\alpha + \beta + u'r + v'r + u\xi + v\xi)}$$

$$\frac{\Gamma(\rho + h'r + h\xi)\,\Gamma(\sigma + k'r + k\xi)\,\Gamma(\rho + \sigma + h'r + k'r + h\xi + k\xi - c - d)}{\Gamma(\rho + \sigma + h'r + k'r + h\xi + k\xi - c)\Gamma(\rho + \sigma + h'r + k'r + h\xi + k\xi - d)}$$

$$\overline{\varphi}(\xi)(ze^{\omega\pi u/2})^{\xi} d\xi \qquad \ldots(2.5)$$

On reinterpreting the Mellin-Barnes contour integral occurring in (2.5) in terms of the $\overline{\text{H}}$-function given by (1.1),we easily arrive at the desired result (2.1).

3. Special Cases

[i] If we take $d = \rho$, $c = 0$,in result (2.1), we arrive at the following integral which is also new and sufficiently general in nature

$$\int_0^1 \int_0^{\pi/2} e^{\omega(\alpha+\beta)\theta} (\sin\theta)^{\alpha-1} (\cos\theta)^{\beta-1}\, x^{\rho-1} (1-x)^{\sigma-1} [ax + b(1-x)]^{-\rho-\sigma}$$

$$S_n^m \left[ye^{\omega(u'+v')\theta} (\sin\theta)^{u'} (\cos\theta)^{v'} (ax)^{h'} [b(1-x)]^{k'} [ax + b(1-x)]^{-h'-k'} \right]$$

$$\overline{H}_{P,Q}^{M,N} \left[ze^{\omega(u+v)\theta} (\sin\theta)^{u} (\cos\theta)^{v} (ax)^{h} [b(1-x)]^{k} \right.$$

$$\left. [ax + b(1-x)]^{-h-k} \middle| \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right] d\theta\, dx$$

$$= \frac{e^{\omega\pi\alpha/2}}{a^{\rho}\,b^{\sigma}} \sum_{r=0}^{[n/m]} \frac{(-n)_{mr}}{r!} A_{n,r}\,(y\,e^{\omega\pi u'/2})^{r}$$

$$\overline{H}\,{}^{M,N+4}_{P+4,Q+2}\left[z\,e^{\omega\pi u/2}\left|\begin{matrix}(1-\alpha-u'r,u;1),(1-\beta-v'r,v;1),(1-\rho-h'r,h;1),\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\sigma-k'r,k;1)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P},\\ (1-\alpha-\beta-u'r-v'r,\,u+v;1),(1-\rho-\sigma-h'r-k'r\;,h+k;1)\end{matrix}\right] \qquad ...(3.1)$$

provided that the conditions easily obtainable from (2.1) are satisfied.

[ii] Now, we give an interesting special case of (2.1) involving ${}_P\overline{\Psi}_Q$ [5, p. 271, eq. (7)] which is also new and of interest by itself.

$$\int_0^1\int_0^{\pi/2} e^{\omega(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}$$

$$x^{\rho-1}(1-x)^{\sigma-1}[ax+b(1-x)]^{-\rho-\sigma}\,{}_2F_1[c,d;\rho;\,ax/ax+b(1-x)]$$

$$S_n^m\left[y\,e^{\omega(u'+v')\theta}(\sin\theta)^{u'}(\cos\theta)^{v'}(ax)^{h'}[b(1-x)]^{k'}[ax+b(1-x)]^{-h'-k'}\right]$$

$${}_P\overline{\Psi}_Q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,P};\\(b_j,\beta_j;B_j)_{1,Q};\end{matrix}\,z\,e^{\omega(u+v)\theta}(\sin\theta)^{u}(\cos\theta)^{v}(ax)^{h}[b(1-x)]^{k}[ax+b(1-x)]^{-h-k}\right]d\theta\,dx$$

$$= \frac{e^{\omega\pi\alpha/2}}{a^{\rho}\,b^{\sigma}} \sum_{r=0}^{[n/m]} \frac{(-n)_{mr}}{r!} A_{n,r}\,(y\,e^{\omega\pi u'/2})^{r}$$

$$\overline{H}\,{}^{1,P+5}_{P+5,Q+4}\left[-z\,e^{\omega\pi u/2}\left|\begin{matrix}(1-\alpha-u'r,u;1),(1-\beta-v'r,v;1),\\ (0,1),(1-b_j,\beta_j;B_j)_{1,Q},\end{matrix}\right.\right.$$

$$\begin{matrix}(1-\rho-h'r,h;1),(1-\sigma-k'r,k;1),(1-\rho-\sigma-h'r-k'r+c+d,h+k;1),\\ (1-\alpha-\beta-u'r-v'r,u+v;1),(1-\rho-\sigma-h'r-k'r+c,h+k;1),\end{matrix}$$

$$\left.\begin{matrix}(1-a_j,\alpha_j;A_j)_{1,P}\\ (1-\rho-\sigma-h'r-k'r+d,h+k;1)\end{matrix}\right] \quad \ldots(3.2)$$

provided that the conditions easily obtainable from (2.1) are satisfied.

[iii] Also, we give an interesting special case of (2.1) involving the g -function connected with a certain class of Feynman integrals [6, p. 98, eq. (1.3)]

$$\int_0^1\int_0^{\pi/2} e^{\omega(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}$$

$$x^{\rho-1}(1-x)^{\sigma-1}[ax+b(1-x)]^{-\rho-\sigma}\,{}_2F_1[c,d;\rho;ax/ax+b(1-x)]$$

$$S_n^m\left[ye^{\omega(u'+v')\theta}(\sin\theta)^{u'}(\cos\theta)^{v'}(ax)^{h'}[b(1-x)]^{k'}[ax+b(1-x)]^{-h'-k'}\right]$$

$$g\left[\gamma,\eta,\tau,p;ze^{\omega(u+v)\theta}(\sin\theta)^{u}(\cos\theta)^{v}(ax)^{h}[b(1-x)]^{k}[ax+b(1-x)]^{-h-k}\right]d\theta\,dx$$

$$=\frac{e^{\omega\pi\alpha/2}}{a^{\rho}b^{\sigma}2^{2+p}}\frac{K_{d-1}\Gamma(p+1)\Gamma(1/2+\tau/2)}{(-1)^{p}\pi^{1/2}\Gamma(\gamma)\Gamma(\gamma-\tau/2)}\sum_{r=0}^{[n/m]}\frac{(-n)_{mr}}{r!}A_{n,r}(ye^{\omega\pi u'/2})^{r}$$

$$\overline{H}^{1,8}_{8,6}\left[-ze^{\omega\pi u/2}\left|\begin{matrix}(1-\alpha-u'r,u;1),(1-\beta-v'r,v;1),(1-\rho-h'r,h;1),\\ (0,1),(-\tau/2,1;1),(-\eta,1;1+p),\end{matrix}\right.\right.$$

$$\begin{matrix}(1-\sigma-k'r,k;1)(1-\rho-\sigma-h'r-k'r+c+d,h+k;1),\\ (1-\alpha-\beta-u'r-v'r,u+v;1),(1-\rho-\sigma-h'r-k'r+c,h+k;1),\end{matrix}$$

$$\left.\begin{matrix}(1-\gamma,1;1),(1-\gamma+\tau/2,1;1),(1-\eta,1;1+p)\\ (1-\rho-\sigma-h'r-k'r+d,h+k;1)\end{matrix}\right] \quad \ldots(3.3)$$

where the g -function occurring in (3.2) is defined in the following manner [6, p.98, eq. (1.3)]

$$g(\gamma,\eta,\tau,p;z) = \frac{K_{d-1}\, p!\,\Gamma(1+\tau/2)\,B(1/2,1/2+\tau/2)}{(-1)^p 2^{2+p}\ \pi\,\Gamma(\gamma)\,\Gamma(\gamma-\tau/2)} \int_{-i\infty}^{i\infty} \frac{d\xi}{2\pi i} \frac{(-z)^{\xi}\,\Gamma(-\xi)\,\Gamma(\gamma+\xi)\,\Gamma(\gamma-\tau/2+\xi)}{(\eta+\xi)^{1+p}\,\Gamma(1+\tau/2+\xi)}$$

$$= \frac{K_{d-1}\Gamma(p+1)\Gamma(1/2+\tau/2)}{(-1)^p\, 2^{2+p}\pi^{1/2}\Gamma(\gamma)\Gamma(\gamma-\tau/2)} \overline{H}\,{}^{1,3}_{3,3}\left[-z \left| \begin{matrix} (1-\gamma,1;1),(1-\gamma+\tau/2,1;1),(1-\eta,1;1+p) \\ (0,1),(-\tau/2,1;1),(-\eta,1;1+p), \end{matrix} \right.\right] \quad \ldots(3.4)$$

$K_d \equiv 2^{1-d}\,\pi^{-d/2}/\Gamma(d/2)$ [9, p.4121, eq. (5)] and conditions easily obtainable from (2.1) are satisfied.

If we take $n=0$ (the polynomial $S_n^m[x]$ will reduce to $A_{0,0}$ and can be taken to be unity without loss of generality), the results (2.1),(3.1) and (3.2) reduce to the known results obtained by Sharma [14]. Also, on taking $n=0$ and $A_j\ (j=1,\ldots,N) = B_j\ (j=M+1,\ldots,Q) = 1$, in results (2.1) and (3.1), we get the corresponding results for the Fox H-function given by A.K. Ronghe [12] and Soni [15] respectively. Again, if we take $n=0$ and $A_j\ (j=1,\ldots,N) = B_j\ (j=M+1,\ldots,Q) = 1$ and $h=0$ in result (3.1), we get a known integral by Soni [15].If we further take $k=0$ in the result thus obtained we get another known result given by Anandani [1]. Specials cases of (3.1) can also be given but we omit the details.

REFERENCES

[1] Anandani P., Rend. Circ. Mat. Palermo Ser. **2,20**(1971a),70-82.

[2] Braaksma B.L.J., Compositio Math.,**15**(1963), 239-341.

[3] Buschman R.G. and Srivastava H.M., J. Phya. A; Math. Gen. **23** (1990), 4707-4710.

[4] Fox C., Trans. Amer. Math. Soc., **98** (1961), 395-429.

[5] Gupta K.C., Jain R. and Sharma A., J. Rajasthan Acad. Phy. Sci. **2(4)** (2003), 269-282.

[6] Gupta K.C. and Soni R.C., Kyungpook Math. J., Vol.**41(1)** (2001), 97-104.

[7] Hai N.T. and Yakubovich S.B., The double Mellin-Barnes type integrals and their applications to convolution theory, World Scientific Publishing Co. Pvt. Ltd., Singapore, New Jersey, London, Hong Kong (1992).

[8] Inayat-Hussian A.A., J.Phys. A: Math. Gen., **20** (1987), 4109-4117.

[9] Inayat-Hussian A.A., J. Phys. A: Math. Gen., **20** (1987), 4119-4128.

[10] MacRobert T. M., Math. Ann., **142**(1960-61), 450-452.

[11] Rathie A.K., Matematiche (Catania), **52** (1997), 297-310.

[12] Ronghe A.K., Vijnana Pari. Anu. Patrika, **28 (1)** (1985), 41-49.

[13] Sharma B.L., Quart. J. of Vikram University, (1965).

[14] Sharma R.P., Bull. Cal. Math. Soc., **96(4)** (2004), 285-290.

[15] Soni Santi S., Vijnana Pari. Anu. Patrika, **25**(1982), 183-187.

[16] Srivastava H.M., Indian J. Math., **14** (1972), 1-6.

[17] Srivastava H. M., Gupta K.C. and Goyal S.P., The H-functions of one and two variables with applications, South Asian Publishers, New Delhi, Madras (1982).

[18] Srivastava H.M. and Singh N.P., Rend. Circ. Mat. Palermo Ser.**2, 32**(1983),157-187.

Ganita Sandesh Vol. 28 No. 1 and 2 (June and December, 2014) pp. 71 - 80 ISSN : 0970-9169

Some Identities in Basic Hypergeometric Series

SANJAY GOYAL

Department of Mathematics

Government Polytechnic College, BANSWARA (Raj.) INDIA

email:sanjaygoyal443@gmail.com

Abstract

In this paper some interesting identities have been established connected with basic hypergeometric series of type $_{p+1}\phi_p$ which reduced to known and unknown results and has some application in physical sciences.

1. Introduction

The difference operator D in the theory of basic function replaces the ordinary differential operator $\frac{d}{dx}$.

This difference operator D is defined as

$$Df(x) = \frac{f(x) - f(qx)}{x}, \quad (0 < q \le 1)$$

This operator has much importance in the theory of basic hypergeometric function and has been used by many authors, Heine, E. [1], Rogers L.J. [2] Jackson, F.H. [3], Hahn,W. [4], Gasper, G [5].

This operator has been used to obtain some identities involving the function $_{p+1}\phi_p$.

When we take $p = 1$, then $_{p+1}\phi_p$ reduces to $_2\phi_1$.

2. Notations

Following notations have been used :

2.1 q-shifted factorial is defined as

$$(a)_n = \begin{cases} 1 & n=0 \\ (1-q^a)(1-q^{a+1})\dots(1-q^{a+k-1}), & \end{cases} \qquad |q| < 1$$

2.2. $Dx^a = \left(1 - q^a\right) x^{a-1}$

2.3. $(a)_{n+k} = (a)_n \, (a+n)_k$

2.4. $${}_{p+1}\phi_p\left(\begin{matrix} a_1,\dots a_{p+1} \\ b_1,\dots b_p \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{p+1})_k}{(q)_k (b_1)_k \cdots (bp)_k} x^k$$

3. Some Identities

The following new identities will be derived.

3.1. $$(a_1)_n \, x^{a_1-1} {}_{p+1}\phi_p\left(\begin{matrix} a_1+n, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix}; x\right) = D^n\left[x^{a_1+n-1} {}_{p+1}\phi_p\left(\begin{matrix} a_1, \dots, a_{p+1} \\ b_1, \dots b_p \end{matrix}; x\right)\right],$$

3.2. $$(b_1-n)_n \, x^{b_1-n-1} {}_{p+1}\phi_p\left(\begin{matrix} a_1, \dots, a_{p+1} \\ b_1 - n, b_2, \dots, b_p \end{matrix}; x\right) = D^n\left[x^{b_1-1} {}_{p+1}\phi_p\left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix}; x\right)\right],$$

3.3. $$(a_1)_n (a_2)_{n\dots} (a_{p+1})_n \, {}_{p+1}\phi_p\left(\begin{matrix} a_1+n, a_2+n, \dots, a_{p+1}+n \\ b_1+n, b_2+n, \dots, b_p+n \end{matrix}; x\right)$$

$$= (b_1)_n (b_2)_n \cdots (b_p)_n \, D^n\left[{}_{p+1}\phi_p\left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix}; x\right)\right],$$

3.4. $$(b_1-a_1) x^{b_1-a_1-1} \prod_{k=0}^{\infty} \frac{\left(1 - xq^{b_1-a_1-a_2,\dots,-a_{p+1}+n+k}\right)}{\left(1-xq^k\right)} \times_{p+1}\phi_p\left(\begin{matrix} a_1-n, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix}; xq^{b_1-a_1-a_2\dots-a_{p+1}+n}\right)$$

$$= D^n\left[x^{b_1-a_1+n+1} \prod_{k=0}^{\infty} \frac{\left(1 - xq^{b_1-a_1-a_2\dots-a_{p+1}+k}\right)}{\left(1-xq^k\right)}\right.$$

$$\left.\times_{p+1}\phi_p\left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix}; xq^{b_1-a_1-a_2\dots-a_{p+1}}\right)\right]$$

3.5. $$(b_1 - n)_n \, x^{b_1-1-n} \prod_{k=0}^{\infty} \left(\frac{\left(1 - xq^{b_1-a_1-a_2\ldots a_{p+1}+n+k}\right)}{\left(1 - xq^k\right)} \right)$$

$$\times_{p+1}\phi_p \left(\begin{matrix} a_1 - n, \ldots a_2 - n, \ldots, a_{p+1} - n \\ b_1 - n, b_2, \ldots, b_p \end{matrix} ; xq^{b_1-a_1-a_2\ldots-a_{p+1}+n} \right)$$

$$= D^n \left[x^{b_1-1} \prod_{k=0}^{\infty} \frac{\left(1 - xq^{b_1-a_1-a_2\ldots-a_{p+1}+k}\right)}{\left(1 - xq^k\right)} \right.$$

$$\left. \times_{p+1}\phi_p \left(\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{matrix} ; xq^{b_1-a_1-a_2\ldots-a_{p+1}} \right) \right],$$

3.6. $$(b_1 - a_1)_n (b_1 - a_2)_n \ldots (b_1 - a_{p+1})_n \prod_{k=0}^{\infty} \left(\frac{\left(1 - xq^{b_1-a_1-a_2,\ldots-a_{p+1}+n+k}\right)}{\left(1 - xq^k\right)} \right)$$

$$\times_{p+1}\phi_p \left(\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1 + n, b_2 \ldots b_p \end{matrix} ; xq^{b_1-a_1-a_2\ldots-a_{p+1}+n} \right)$$

$$= (b_1)_n (b_2)_n \ldots (b_p)_n \, D^n \left[\prod_{k=0}^{\infty} \frac{\left(1 - xq^{b_1-a_1-a_2\ldots-a_{p+1}+k}\right)}{\left(1 - xq^k\right)} \; {}_{p+1}\phi_p \left(\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} ; xq^{b_1-a_1\ldots-a_{p+1}} \right) \right],$$

4. Proof

First identity of 3.1

Expanding R.H.S. of identity 3.1 using 2.2, we get

4.1. $$D^n \left[x^{a_1+n-1} \, {}_{p+1}\phi_p \left(\begin{matrix} a_1, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} : x \right) \right]$$

$$= D^n\left[x^{a_1+n-1}\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_{p+1})_k}{(q)_k(b_1)_k\cdots(b_p)_k}.x^k\right] \qquad \text{[by 2.4]}$$

$$= D^n\left[\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_{p+1})_k}{(q)_k(b_1)_k\cdots(b_p)_k}.x^{a_1+n-1+k}\right]$$

Using 2.2 term by term n times, we get

$$= x^{a_1-1}\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_{p+1})_k}{(q)_k(b_1)_k\cdots(bp)_k}\left(1-q^{a_1+k}\right)\left(1-q^{a_1+k+1}\right)\ \ldots\left(1-q^{a_1+k+n-1}\right)x^k$$

Using 2.1 and 2.3, we get

$$= x^{a_1-1}\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_{p+1})_k}{(q)_k(b_1)_k\cdots(b_p)_k}(a_1+k)_n\,x^k$$

$$= (a)_n\,x^{a_1-1}\sum_{k=0}^{\infty}\frac{(a_1+n)_k(a_2)_k\cdots(a_{p+1})_k}{(q)_k(b_1)_k\cdots(b_p)_k}x^k$$

$$= (a)_n\,x^{a_1-1}{}_{p+1}\phi_p\left(\begin{matrix}a_1+n, a_2,\ldots,a_{p+1}\\ b_1,b_2,\ldots,b_p\end{matrix};x\right) = \text{L.H.S.}$$

4.2. R.H.S. of 3.2 $= D^n\left[x^{b_1-1}{}_{p+1}\phi_p\left(\begin{matrix}a_1,a_2,\ldots,a_{p+1}\\ b_1,\ldots,b_p\end{matrix}:x\right)\right]$

$$= D^n\left[x^{b_1-1}\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_{p+1})_k}{(q)_k(b_1)_k\cdots(b_p)_k}x^k\right] \qquad \text{[by 2.4]}$$

$$= D^n\left[\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_{p+1})_k}{(q)_k(b_1)_k\cdots(b_p)_k}x^{b_1-1+k}\right]$$

$$= D^{n-1}\left[\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1})_k}{(q)_k (b_1)_k \cdots (b_p)_k}\left(1 - q^{b_1 - 1 + k}\right) x^{b_1 + k - 2}\right]$$

Using difference operator defination n times,

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1}) k}{(q)_k (b_1)_k \cdots (b_p)_k}\left(1 - q^{b_1 + k - 1}\right) \cdots \left(1 - q^{b_1 + k - n}\right) x^{b_1 + k - n - 1}$$

Using q-shifted factorial, we get

$$(b_1 - n)_n \, x^{b_1 - n - 1} \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{p+1})_k}{(q)_k (b_1 - n)_k \cdots (b_p)_k} x^k$$

$$= (b_1 - n)_n \, x^{b_1 - n - 1} {}_{p+1}\phi_p\left(\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1 - n, \ldots, b_p \end{matrix}; x\right) = \text{LHS}$$

4.3. R.H.S. of 3.3 $= (b_1)_n (b_2)_n \cdots (b_p)_n D^n \left[{}_{p+1}\phi_p\left(\begin{matrix} a_1, \ldots, a_{b+1} \\ b_1, \ldots, b_p \end{matrix}; x\right)\right]$

$$= (b_1)_n (b_2)_n \cdots (b_p)_n D^n \left[\sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1})_k}{(q)_k (b_1)_k \cdots (b_p)_k} x^k\right]$$

Taking n-differences and using 2.2, we have

$$= (b_1)_n (b_2)_n \cdots (b_p)_n \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1})_k}{(q)_k (b_1)_k \cdots (b_p)_k}\left(1 - q^k\right)\left(1 - q^{k-1}\right) \cdots \left(1 - q^{k-n+1}\right) x^{k-n}$$

But when $k < n$, all differences are zero and we get terms from $k \geq n$.

Replacing k by $k + n$, we have

$$= (b_1)_n (b_2)_n \cdots (b_p)_n \sum_{k=0}^{\infty} \frac{(a_1)_{n+k} (a_2)_{n+k} \cdots (a_{p+1})_{n+k} \left(q^{k+1}\right)_n}{(q)_{n+k} (b_1)_{n+k} \cdots (b_p)_{n+k}} x^k$$

Using 2.3, we get $= (a_1)_n (a_2)_n \cdots (a_{p+1})_n \sum_{k=0}^{\infty} \frac{(a_1+n)_k (a_2+n)_k \cdots (a_{p+1}+n)_k}{(q)_k (b_1+n)_k \cdots (b_p+n)_k} x^k$

$$= (a_1)_n (a_2)_n \cdots (a_{p+1})_n \; {}_{p+1}\phi_p \left(\begin{matrix} a_1+n, \ldots, a_{p+1}+n \\ b_1+n, \ldots, b_p+n \end{matrix} ; x \right) = \text{L.H.S.}$$

4.4. L.H.S. $= (a_1)_n x^{a_1-1} {}_{p+1}\phi_p \left(\begin{matrix} a_1+n, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} ; x \right)$

Replacing a_1 by $(b_1 - a_1)$, we obtain

$$(b_1 - a_1)_n x^{b_1-a_1-1} {}_{p+1}\phi_p \left(\begin{matrix} b_1 - a_1 + n, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} ; x \right)$$

Using 3rd transformation of basic hyper geometric function

$$(b_1 - a_1)_n x^{b_1-a_1-1} \prod_{k=0}^{\infty} \frac{\left(1 - xq^{b_1-a_1+n+a_2+\ldots a_{p+1}-pb_1+k}\right)}{\left(1 - xq^k\right)}$$

$$\times {}_{p+1}\phi_p \left(\begin{matrix} a_1 - n, b_1 - a_2 \ldots, b_1 - a_{p+1} \\ b_1, b_2, \ldots, b_p \end{matrix} ; xq^{b_1-a_1+a_2\ldots a_{p+1}+n-pb_1} \right) \quad \ldots(4.4.1)$$

Changing upper parameter, we get

$$(b_1 - a_1)_n x^{b_1-a_1-1} \prod_{k=0}^{\infty} \frac{\left(1 - xq^{b_1-a_1,\ldots,a_{p+1}+n+k}\right)}{\left(1 - xq^k\right)} \times {}_{p+1}\phi_p \left(\begin{matrix} a_1 - n, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{matrix} ; xq^{b_1-a_1-a_2,\ldots,a_{p+1}+n} \right)$$

R.H.S. $= D^n \left[x^{a_1+n-1} {}_{p+1}\phi_p \left(\begin{matrix} a_1, a_2, \ldots a_{p+1} \\ b_1, b_2, \ldots b_p, \end{matrix} ; x \right) \right]$

Using (4.4.1)

$$D^n \left[x^{a_1+n-1} \prod_{k=0}^{\infty} \frac{\left(1 - xq^{a_1+a_2\ldots a_{p+1}-pb_1+k}\right)}{\left(1 - xq^k\right)} {}_{p+1}\phi_p \left(\begin{matrix} b_1 - a_1, b_1 - a_2, \ldots b_1 - a_{p+1} \\ b_1, b_2, \ldots, b_p \end{matrix} ; xq^{a_1+a_2+\ldots a_{p+1}-pb_1} \right) \right]$$

Changing upper parameters, we get

$$D^n\left[x^{b_1-a_1+n-1}\prod_{k=0}^{\infty}\frac{\left(1-xq^{b_1-a_1\ldots-a_{p+1}+k}\right)}{(1-xqk)}\cdot{}_{p+1}\phi_p\left(\begin{matrix}a_1,\ldots,a_{p+1}\\b_1,\ldots,b_p\end{matrix};xq^{b_1-a_1\ldots-a_{p+1}}\right)\right],$$

4.5. Taking LHS of identity 3.2, we have

$$(b_1-n)_n\,x^{b_1-1-n}\;{}_{p+1}\phi_p\left(\begin{matrix}a_1,a_2,\ldots,a_{p+1}\\b_1-n,b_2,\ldots,b_p\end{matrix};x\right)$$

using (4.4.1) and changing upper parameters we get the required result

$$(b_1-n)_n\,x^{b_1-1-n}\prod_{k=0}^{\infty}\frac{\left(1-xq^{a_1+a_2\ldots+a_{p+1}-pb_1+n+k}\right)}{\left(1-xq^k\right)}$$

$$\times{}_{p+1}\phi_p\left(\begin{matrix}b_1-n-a_1,b_1-n-a_2,\ldots,b_1-n-a_{p+1}\\b_1-n,b_2,\ldots,b_p\end{matrix};xq^{a_1+a_2+\ldots+a_{p+1}-bp_1+n}\right)$$

$$(b_1-n)_n\,x^{b_1-1-n}\prod_{k=0}^{\infty}\frac{\left(1-xq^{b_1-a_1-a_2\ldots-a_{p+1}+n+k}\right)}{\left(1-xq^k\right)}$$

$$\times{}_{p+1}\phi_p\left(\begin{matrix}a_1-n,a_2-n,\ldots,a_{p+1}-n\\b_1-n,b_2,\ldots,b_p\end{matrix};xq^{b_1-a_1-a_2\ldots-a_{p+1}+n}\right)$$

Now R.H.S. $= D^n\left[x^{b_1-1}{}_{p+1}\phi_p\left(\begin{matrix}a_1,a_2,\ldots,a_{p+1}\\b_1,b_2,\ldots,b_p\end{matrix};x\right)\right]$

Using (4.4.1) and changing upper parameters we get the required result

$$D^n\left[x^{b_1-1}\prod_{k=0}^{\infty}\frac{\left(1-xq^{b_1-a_1\ldots-a_{p+1}+k}\right)}{\left(1-xq^k\right)}\;{}_{p+1}\phi_p\left(\begin{matrix}a_1,a_2,\ldots,a_{p+1}\\b_1,b_2,\ldots,b_p\end{matrix};xq^{b_1-a_1-a_2\ldots-a_{p+1}}\right)\right]$$

4.6. Start with identity $(a_1)_n (a_2)_n \cdots (a_{p+1})_n \; {}_{p+1}\phi_p \begin{pmatrix} a_1+n, a_2+n, \ldots a_{p+1}+n \\ b_1+n, b_2, \ldots, b_p \end{pmatrix}$

$$= (b_1)_n (b_2)_n \cdots (b_p)_n D^n \left[{}_{p+1}\phi_p \begin{pmatrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{pmatrix} ; x \right]$$

$$\text{L.H.S.} = (a_1)_n \cdots (a_{p+1})_n \; {}_{p+1}\phi_p \begin{pmatrix} a_1+n, a_2+n, \ldots, a_{p+1}+n \\ b_1+n, b_2, \ldots, b_p \end{pmatrix}$$

Replacing upper parameters as follows :

a_1 by $b_1 - a$, a_2 by $b_1 - a_2, \ldots, a_{p+1}$ by $b_1 - a_{p+1}$, we have

$$= (b_1 - a_1)_n (b_1 - a_2)_n \cdots (b_1 - a_{p+1})_n \; {}_{p+1}\phi_p \begin{pmatrix} b_1 - a_1 + n, b_1 - a_2 + n, \ldots b_1 - a_{p+1} + n \\ b_1 + n, b_2, \ldots, b_p \end{pmatrix} ; x$$

Using transformation and changing upper parameters, we get

$$= (b_1 - a_1)_n (b_1 - a_2)_n \cdots (b_1 - a_{p+1})_n \prod_{k=0}^{\infty} \frac{\left(1 - xq^{b_1 - a_1 - a_2 \ldots - a_{p+1} + n + k}\right)}{\left(1 - xq^k\right)}$$

$$\times {}_{p+1}\phi_p \begin{pmatrix} a_1, a_2, \ldots, a_{p+1} \\ b_1 - n, b_2, \ldots, b_p \end{pmatrix} ; xq^{b_1 - a_1 - a_2 \ldots - a_{p+1} + n}$$

$$\text{R.H.S.} = (b_1)_n (b_2)_n \cdots (b_n)_n D^n \left[{}_{p+1}\phi_p \begin{pmatrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{pmatrix} ; x \right]$$

Using (4.4.1) and changing upper parameters, we obtain

$$= (b_1)_n (b_2)_n \cdots (b_p)_n D^n \left[\prod_{k=0}^{\infty} \frac{\left(1 - xq^{b_1 - a_1 - a_2, \ldots - a_{p+1} + k}\right)}{\left(1 - xq^k\right)} \times {}_{p+1}\phi_p \begin{pmatrix} a_1 \; a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_p \end{pmatrix} ; xq^{b_1 - a_1 - a_2 \ldots - a_{p+1}} \right]$$

5. Deductions

If we put $p = 1$, then the above results hold for ${}_2\phi_1$.

Taking $a_1 = a$, $a_2 = b$ and lower parameter $b_1 = c$, we obtain the following :

5.1 $$(a)_n\, x^{a-1}{}_2\phi_1(a+n,b;c;x) = D^n\left[x^{a+n-1}{}_2\phi_1(a,b;c:x)\right]$$

5.2 $$(c-n)_n\, x^{c-1-n}{}_2\phi_1(a,b;c-n;x) = D^n\left[x^{c-1}{}_2\phi_1(a,b;c;x)\right]$$

5.3 $$(a)_n(b)_n\, {}_2\phi_1\left({a+n,b+n \atop c+n};x\right) = (c)_n\, D^n\left[{}_2\phi_1(a,b;c;x)\right]$$

5.4 $$(c-a)_n\, x^{c-a-1}\prod_{k=0}^{\infty}\frac{\left(1-xq^{c-a-b+n+k}\right)}{\left(1-xq^k\right)}\,{}_2\phi_1\left({a-n,b \atop c};xq^{c-a-b+n}\right)$$

$$= D^n\left[x^{c-a+n-1}\prod_{k=0}^{\infty}\frac{\left(1-xq^{c-a-b+k}\right)}{\left(1-xq^k\right)}\,{}_2\phi_1\left({a\,b \atop c};xq^{c-a-b}\right)\right],$$

5.5 $$(c-n)_n\, x^{c-1-n}\prod_{k=0}^{\infty}\frac{\left(1-xq^{c-a-b+n+k}\right)}{\left(1-xq^k\right)}\,{}_2\phi_1\left({a-n,b-n \atop c-n};xq^{c-a-b+n}\right)$$

$$= D^n\left[x^{c-1}\prod_{k=0}^{\infty}\frac{\left(1-xq^{c-a-b+k}\right)}{\left(1-xq^k\right)}\,{}_2\phi_1\left({a\,b \atop c};xq^{c-a-b}\right)\right]$$

5.6 $$(c-a)_n(c-b)_n\prod_{k=0}^{\infty}\frac{\left(1-xq^{c-a-b+n+k}\right)}{\left(1-xq^k\right)}\,{}_2\phi_1\left(a,b;c+n;xq^{c-a-b+n}\right)$$

$$= (c)_n\, D^n\left[\prod_{k=0}^{\infty}\frac{1-xq^{c-a-b+k}}{\left(1-xq^k\right)}\,{}_2\phi_1\left(a,b;c;xq^{c-a-b}\right)\right]$$

The identity (5.5) is the basic analogue of well known result due to Jacobi for the ordinary hypergeometric function.

REFERENCES

[1] Heine, E., Theoric der kugel function, I (1878), pp 97-125

[2] Rogers, L.J, proc. Lond. Math. Soc. (1), 24 (1893), pp 337-352.

[3] Jackson, F.H., Quart, Journ math. (Oxford) (2), (1951), pp (1-16)

[4] Hahn, W, math, Nachrichten, 3(1950) PP (257-294).

[5] Gasper, G., Rahman, M. Basic Hypergeometric series.

Estd. 1987 Reg. No. 74 / Ajmer / 1988-89, Dated 23/12/1988

राजस्थान गणित परिषद्

RAJASTHAN GAṆITA PARISHAD

Executive Committee - 2016

Office Bearers

President
DR. K. C. SHARMA
M.S.J.Govt. College, **BHARATPUR**

Vice President
DR. RAM NARESH SARASWAT
Manipal Universuty, **JAIPUR**

General Secretary
DR. R. P. SHARMA
Govt. Engg. College, **AJMER**

Treasurer
DR. V. C. JAIN
Govt. Engg. College, **AJMER**

Editor
DR. ANIL GOKHROO
S.P.C.Govt. College, **AJMER**

Joint Secretary
SH. R. N. MEGHWAL
Govt. College, **SUJANGARH**

MEMBERS

(3 Years)
1. SH. NIKHIL JAIN, AJMER
2. SH. VIPIN KUMAR, PILANI
3. SH. HAMENT KUMAR, BHARATPUR

(2 Years)
1. DR. YASHMEEN, KOTA
2. DR. KAUR SINGH, SHRI GANGANAGAR
3. SH. LAJPAT SINGH, UDAIPUR

(1 Years)
1. PROF. C.B. GUPTA, PILANI
2. DR. A.K. MALIK, PILANI
3. DR. SUNIL JOSHI, JAIPUR

EX OFFICIO MEMBERS : 1. Past President, DR. V. C. JAIN, AJMER **2.** Past General Secretary, DR. P.K. MISHRA, SIKAR

CO-OPTED MEMBERS : 1. DR. D.C.GOKHROO, AJMER 2. DR. B.S. SINSINWAR, SHIVGANJ

Honorary Members

PROF. BANSAL, J. L., JAIPUR
DR. GOKHROO, D.C., AJMER
PROF. GUPTA, K.C., JAIPUR
DR. OM PRAKASH, JAIPUR
PROF. RAJBALI, JAIPUR,
PROF. SAXENA, R. K., JODHPUR
PROF. VERMA, G. R., KINGSTON, (USA)

Form IV (See Rule 8)

1. Place of Publication : AJMER (*Rajasthan*) INDIA
2. Periodicity of Publication : Half Yearly (*June and December*)
3. Printer's Name : MAHENDRA SINGH RAWAT
   Nationality : Indian
   Address : Bhawani Offset Printers, Topdara, AJMER (*Rajasthan*)
4. Publisher's Name : Dr. R. P. SHARMA
   Nationality : Indian
   Address : General Secretary, Rajasthan Ganita Parishad, Department of Mathematics, Govt. Engg.College, AJMER (*Rajasthan*)
5. Editor's Name : Dr. ANIL GOKHROO
   Nationality : Indian
   Address : Department of Mathematics, S. P. C. Government College, AJMER (*Rajasthan*)
6. Name and Addresses of individuals who own the newspaper and partners or shareholders holding more than 1% of the total capital. : Rajasthan Ganita Parishad, Regd. Head Office *Department of Mathematics,* SPC Government College, AJMER (*Rajasthan*) INDIA

I, Dr. R.P. Sharma, hereby declare that the particulars given above are true to the best of my knowledge and belief.

Dated : 30th December, 2016 ***Dr. R. P. Sharma***

Volume 28, No. 1 and 2 (*June and Dec.* 2014)

GAṆITA SANDESH

गणित संदेश

The Sequence

Published by General Secretary, Rajasthan Ganita Parishad

Courtesy : Partially by Prof. G.R.Verma, Kingston (U.S.A)