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Certain Classes of Generating Functions Associated with the Aleph-Function of Several Variables I

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Abstract

Jaimini B.B. et al [2] *have established some generating relationship for Fox's H-function and multivariable H-function. In this paper, we present four generating functions involving multivariable Aleph-function, the I-function of several variables and the Aleph-function of two variables. The mains results of our document are quite general in nature and capable of yielding a very large number of generating function involving polynomials and various special functions occuring in the problem of mathematical analysis and mathematical physics and mechanics.*

Key words :- Generalized multivariable Aleph-function, Aleph-function of two variables, generating functions, multivariable I-function.

2010 Mathematics Subject Classification :- 33C99, 33C60, 44A20.

1. Introduction and Preliminaries

The Aleph-function of several variables is an extension of the multivariable I-function defined by C.K. Sharma and Ahmad [4], itself is an a generalization of G and H-functions of multiple variables. The multiple Mellin-Barnes integra l occuring in this paper will be referred to as the multivariable Aleph-function throughout our present study and will be defined and represented as follows.

We have $\aleph(z_1,\ldots,z_r) = \aleph^{0,n:m_1,n_1,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}$

$$\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \begin{matrix} \left[\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n}\right],\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right]: \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right]: \end{matrix}$$

$$\left[\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1}\right],\left[\tau_{i^{(1)}},\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right];\ldots;$$

$$\left[\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,m_1}\right],\left[\tau_{i^{(1)}},\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right];\ldots;$$

$$\left[\left(c_j^{(r)}\right),\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right]$$

$$\left[\left(d_j^{(r)}\right),\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\int_{L_r}\psi(s_1,\ldots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}ds_1\ldots ds_r \qquad \ldots(1.1)$$

with $\omega=\sqrt{-1}$

For more details, see Ayant [1].

The real number τ_i are positives for $i=1,\ldots,R,\tau_{i^{(k)}}$ are positives for $i^{(k)}=1,\ldots,R^{(k)}$

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|\arg z_k|<\frac{1}{2}A_i^{(k)}\pi,$

where $$A_i^{(k)}=\sum_{j=1}^{n}\alpha_j^{(k)}-\tau_i\sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\text{ with } k=1,\ldots,r;\, i=1,\ldots,R,\, i^{(k)}=1,\ldots,R^{(k)} \qquad \ldots(1.2)$$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\ldots,z_r) = 0\left(|z_1|^{\alpha_1},\ldots,|z_r|^{\alpha_r}\right),\ \max\left(|z_1|,\ldots,|z_r|\right) \to 0$$

$$\aleph(z_1,\ldots,z_r) = 0\left(|z_1|^{\beta_1},\ldots,|z_r|^{\beta_r}\right),\ \min\left(|z_1|,\ldots,|z_r|\right) \to \infty\ ,$$

where $k = 1,\ldots,r : \alpha_k = \min\left[\mathrm{Re}\left(d_j^{(k)} / \delta_j^{(k)}\right)\right], j = 1,\ldots,m_k$

and $\quad B_k = \max\left[\mathrm{Re}\left(\left(c_j^{(k)} - 1\right) / \gamma_j^{(k)}\right)\right], j = 1,\ldots,n_k$

we will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R; V = m_1, n_1; \ldots; m_r, n_r \qquad \ldots(1.3)$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \ldots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \qquad \ldots(1.4)$$

$$A = \left\{\left(a_j; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}\right)_{1,n}\right\}, \left\{\tau_i\left(a_{ji}; \alpha_{ji}^{(1)}, \ldots, \alpha_{ji}^{(r)}\right)_{n+1,p_i}\right\} \qquad \ldots(1.5)$$

$$B = \left\{\tau_i\left(b_{ji}; \beta_{ji}^{(1)}, \ldots, \beta_{ji}^{(r)}\right)_{m+1,q_i}\right\} \qquad \ldots(1.6)$$

$$C = \left\{\left(c_j^{(1)}; \gamma_j^{(1)}\right)_{1,n_1}\right\}, \tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}}\Bigg\}, \ldots, \left\{\left(c_j^{(r)}; \gamma_j^{(r)}\right)_{1,n_r}\right\}, \tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_{i^{(r)}}}\Bigg\}$$

$$\ldots(1.7)$$

$$D = \left\{\left(d_j^{(1)}; \delta_j^{(1)}\right)_{1,m_1}\right\}, \tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}}\Bigg\}, \ldots, \left\{\left(d_j^{(r)}; \delta_j^{(r)}\right)_{1,m_r}\right\}, \tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_{i^{(r)}}}\Bigg\}$$

$$\ldots(1.8)$$

The multivariable Aleph-function write :

$$\aleph(z_1,\ldots,z_r) = \aleph^{0,n:V}_{U:W}\left(\begin{array}{c|c} z_1 & A:C \\ \vdots & \cdots\cdots \\ z_r & B:D \end{array}\right) \qquad \ldots(1.9)$$

2. Generating Relationships Involving Multivariable Aleph-Function

We will note $U_{10} = p_i+1, q_i, \tau_i : R$ and $U_{11} = p_i+1, q_i+1, \tau_i; R$

Theorem 1. Let

$$C^{\mu,\alpha}_{n,m}(x) = \sum_{k=0}^{[n/m]} \binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\binom{n}{mk}\Omega_k x^k \qquad \ldots(2.1)$$

Suppose

$$A^{\lambda,\mu,\alpha}_{n}[z_1,\ldots,z_r,t] = \sum_{l=0}^{\infty} \frac{(\mu-\alpha)_l\, t^l}{(\mu+n)_l\, l!}\aleph^{0,n+1:V}_{U_{10}:W}\left(\begin{array}{c|c} z_1 & (1-\lambda-n-l;\epsilon_1,\ldots,\epsilon_r),A:C \\ \vdots & \cdots\cdots \\ z_r & B:D \end{array}\right) \qquad \ldots(2.2)$$

also let $$\theta_1[x,z_1,\ldots,z_r] = \sum_{k=0}^{\infty} \frac{\Omega_k x^k}{(mk)!}\aleph^{0,n+1:V}_{U_{10}:W}\left(\begin{array}{c|c} z_1 & (1-\lambda-mk;\epsilon_1,\ldots,\epsilon_r),A:C \\ \vdots & \cdots\cdots \\ z_r & B:D \end{array}\right) \qquad \ldots(2.3)$$

then

$$\sum_{n=0}^{\infty} C^{\mu,\alpha}_{n,m}(x)A^{\lambda,\mu,\alpha}_{n}[z_1,\ldots,z_r,t]\frac{t^n}{n!} = (1-t)^{-\lambda}\theta_1\left[x\left(\frac{t}{1-t}\right)^m;\frac{z_1}{(1-t)^{\epsilon_1}},\ldots,\frac{z_r}{(1-t)^{\epsilon_r}}\right] \qquad \ldots(2.4)$$

Theorem 2. Suppose

$$B^{\lambda,\mu,\alpha}_{n,m,k}[z_1,\ldots,z_r,t] = \sum_{l=0}^{\infty} \frac{(\mu-\alpha)_l\, t^l}{(\mu-n)_l\, l!}\aleph^{0,n+1:V}_{U_{11}:W}\left(\begin{array}{c|c} z_1 & (-\lambda-n-l;\epsilon_1,\ldots,\epsilon_r),A:C \\ \vdots & \cdots\cdots \\ z_r & (-\lambda-mk;\epsilon_1,\ldots,\epsilon_r),B:D \end{array}\right) \qquad \ldots(2.5)$$

and

$$C_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right]=\sum_{k=0}^{[n/m]}\binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\frac{(-)^{mk}\,\Omega_k x^k}{(n-mk)!}B_{n,m,k}^{\lambda,\mu,\alpha}\left[z_1,\ldots,z_r,t\right] \quad \ldots(2.6)$$

also let
$$\theta_2\left[x,z_1,\ldots,z_r\right]=\sum_{k=0}^{\infty}\Omega_k x^k \aleph_{U:W}^{0,n:V}\left(\begin{array}{c|c} z_1 & A:C \\ \vdots & \ldots. \\ z_r & B:D\end{array}\right) \quad \ldots(2.7)$$

then

$$\sum_{n=0}^{\infty}C_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right]t^n=(1+\eta)^{\alpha+1}\theta_2\left[x(-\eta)^m,z_1(1+\eta)^{\epsilon_1},\ldots,z_r(1+\eta)^{\epsilon_r}\right], \quad \ldots(2.8)$$

where $\eta=\dfrac{t}{1-t}$ and $\eta(0)=0$ .....(2.9)

Theorem 3. Suppose

$$C_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,\ldots,z_r,t\right]=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l\,t^l}{(\mu+n)_l\,l!}\aleph_{U_{11}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1 & (-\lambda-\omega k-n-l;\epsilon_1,\ldots,\epsilon_r),A:C \\ \vdots & \ldots\ldots \\ z_r & (-\lambda-\omega k-mk;\epsilon_1,\ldots,\epsilon_r),B:D\end{array}\right) \quad \ldots(2.10)$$

and

$$D_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right]=\sum_{k=0}^{[n/m]}\binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\frac{(-)^{mk}\,\Omega_k x^k}{(n-mk)!}C_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,\ldots,z_r,t\right] \quad \ldots(2.11)$$

then

$$\sum_{n=0}^{\infty}D_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right]t^n=(1+\eta)^{\alpha+1}\theta_2\left[x(-\eta)^m(1+\eta)^{\omega},z_1(1+\eta)^{\epsilon_1},\ldots,z_r(1+\eta)^{\epsilon_r}\right] \quad \ldots(2.12)$$

where η and θ_2 are defined by (2.9) and (2.7) respectively.

Theorem 4. Suppose

$$E_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,\ldots,z_r,t\right]=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l\,t^l}{(\mu+n)_l\,l!}\aleph_{U_{11}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1 & (1-\lambda-\omega k-n-l;\epsilon_1,\ldots,\epsilon_r),A:C \\ \vdots & \ldots\ldots \\ z_r & (-\lambda-\omega k-mk;\epsilon_1,\ldots,\epsilon_r),B:D\end{array}\right) \quad \ldots(2.13)$$

and

$$F_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right]=\sum_{k=0}^{[n/m]}\binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\frac{(-1)^{mk}\,\Omega_k x^k}{(n-mk)!}E_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,\ldots,z_r,t\right]\ldots(2.14)$$

also let

$$\theta_3\left[x,z_1,\ldots,z_r\right]=\sum_{k=0}^{\infty}\Omega_k x^k \aleph_{U_{11}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1 & \left(1-\lambda-\omega k-mk;\epsilon_1,\ldots,\epsilon_r\right),A:C \\ \vdots & \cdots\cdots \\ z_r & \left(-\lambda-\omega k-mk;\epsilon_1,\ldots,\epsilon_r\right),B:D \end{array}\right)\qquad\ldots(2.15)$$

then

$$\sum_{n=0}^{\infty}F_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right]t^n=(1+\eta)^{\lambda}\,\theta_3\left[x(-\eta)^m(1+\eta)^{\omega},z_1(1+\eta)^{\epsilon_1},\ldots,z_r(1+\eta)^{\epsilon_r}\right],\ldots(2.16)$$

where η is defined by (2.9).

Proof of theorem 1.

Let
$$M\{\}=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\ldots,s_r)\prod_{k=1}^{r}\theta_k(s_k)\{\}\qquad\ldots(2.17)$$

We denote the left hand side of the assertion (2.4) of the theorem 1 by

$$P(x,z_1,\ldots,z_r,t)=\sum_{n=0}^{\infty}C_{n,m}^{\mu,\alpha}(x)\,A_n^{\lambda,\mu,\alpha}\left[z_1,\ldots,z_r,t\right]\frac{t^n}{n!}\qquad\ldots(2.18)$$

on making substitution for $C_{n,m}^{\mu,\alpha}(x)$ from (2.1) and $A_n^{\lambda,\mu,\alpha}\left[z_1,\ldots,z_r,t\right]$ for (2.2), we have

$$P(x,z_1,\ldots,z_r,t)=\sum_{n,l=0}^{\infty}\sum_{k=0}^{[n/m]}\binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\frac{\Omega_k x^k}{(mk)!(n-mk)!}\frac{(\mu-\alpha)_l\,t^l}{(\mu+n)_l\,l!}$$

$$\aleph_{U_{10}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1 & \left(1-\lambda-n-l;\epsilon_1,\ldots,\epsilon_r\right),A:C \\ \vdots & \cdots\cdots \\ z_r & B:D \end{array}\right)t^n\qquad\ldots(2.19)$$

Now on using the definition of multivariable Aleph-function from (1.1) and changing the order of summation and integration and then on making series re-arrangement therein, it takes the following form :

$$P\left(x, z_1, \ldots, z_r, t\right) = M\left[\sum_{n,l,k=0}^{\infty} \binom{\mu+n+mk-1}{n}^{-1}\binom{\alpha+n+mk-1}{n}\right.$$

$$\left.\frac{(\mu-\alpha)_l\,\Omega_k x^k t^{l+n+mk}}{(\mu+n+mk)_l\, l!(mk)!n!}\Gamma\left(\lambda+n+mk+l+\sum_{i=1}^{r}\epsilon_1\, s_i\right)\right]ds_1\ldots ds_2 \quad \ldots(2.20)$$

Now in view of the following relation : $\dfrac{\Gamma(\rho+n+l)}{n!} = (\rho+n)_l\binom{\rho+n-1}{n}\Gamma(\rho)$ ...(2.21)

and interpreting the linner serie into Gauss' hypergeometric function ${}_2F_1$, we have :

$$P\left(x, z_1, \ldots, z_r, t\right) = M\left[\sum_{k,n=0}^{\infty} \binom{\mu+n+mk-1}{n}^{-1}\binom{\lambda+n+mk+\sum_{i=1}^{r}\epsilon_i\, s_i-1}{n}\right.$$

$$\binom{\alpha+n+mk-1}{n}{}_2F_1\left[\lambda+n+mk+\sum_{i=1}^{r}\epsilon_i\, s_i, \mu-\alpha;\mu+n+mk;t\right]t^n$$

$$\left.\frac{\Omega_k x^k t^{mk}}{(mk)!}\Gamma\left(\lambda+mk+\sum_{i=1}^{r}\epsilon_1\, s_i\right)\right]ds_1\ldots ds_r \quad \ldots(2.22)$$

Using the following combinatorial identity, see Raina [3].

$$\sum_{k=0}^{\infty}\binom{\mu+k-1}{k}^{-1}\binom{\lambda+k-1}{k}\binom{\alpha+k-1}{k}{}_2F_1\left[\lambda+k,\mu-\alpha;\mu+k;z\right]z^k = (1-z)^{-\lambda} \quad \ldots(2.23)$$

with $|z| < 1$, , Finally interpreting the result thus obtained with the Mellin-barnes contour integral (1.1), we arrive at the desired result (2.4).

3. Multivariable I-Function

If $\tau_i, \tau_{j(1)}, \ldots, \tau_{j(r)} \to 1$, the Aleph-function of several variables reduces to the I-function of several variables. The four generating relationships involving multivariable I-function defined by Sharma et al [4] have been derived in this section.

We will note $U_{10} = p_i + 1,\ \ q_i; R$ and $U_{11} = p_i + 1,\ \ q_i + 1; R$

Corollary 1. Let

$$C_{n,m}^{\mu,\alpha}(x) = \sum_{k=0}^{[n/m]} \binom{\mu+n-1}{n-mk}^{-1} \binom{\alpha+n-1}{n-mk} \binom{n}{mk} \Omega_k x^k \qquad \ldots(3.1)$$

Suppose

$$A_n^{\lambda,\mu,\alpha}[z_1,\ldots,z_r,t] = \sum_{l=0}^{\infty} \frac{(\mu-\alpha)_l\, t^l}{(\mu+n)_l\, l!} I_{U_{10}:W}^{0,n+1:V} \left(\begin{array}{c|c} z_1 & (1-\lambda-n-l;\epsilon_1,\ldots,\epsilon_r), A : C \\ \vdots & \cdots\cdots \\ z_r & B : D \end{array} \right) \qquad \ldots(3.2)$$

also let $$\theta_1[x,z_1,\ldots,z_r] = \sum_{k=0}^{\infty} \frac{\Omega_k x^k}{(mk)!} I_{U_{10}:W}^{0,n+1:V} \left(\begin{array}{c|c} z_1 & (1-\lambda-mk;\epsilon_1,\ldots,\epsilon_r), A : C \\ \vdots & \cdots\cdots \\ z_r & B : D \end{array} \right) \qquad \ldots(3.3)$$

then

$$\sum_{n=0}^{\infty} C_{n,m}^{\mu,\alpha}(x)\, A_n^{\lambda,\mu,\alpha}[z_1,\ldots,z_r,t] \frac{t^n}{n!} = (1-t)^{-\lambda}\, \theta_1\left[x\left(\frac{t}{1-t}\right)^m ; \frac{z_1}{(1-t)^{\epsilon_1}}, \ldots, \frac{z_r}{(1-t)^{\epsilon_r}} \right] \qquad \ldots(3.4)$$

Corollary 2. Suppose

$$B_{n,m,k}^{\lambda,\mu,\alpha}[z_1,\ldots,z_r,t] = \sum_{l=0}^{\infty} \frac{(\mu-\alpha)_l\, t^l}{(\mu+n)_l\, l!} I_{U_{11}:W}^{0,n+1:V} \left(\begin{array}{c|c} z_1 & (-\lambda-n-l;\epsilon_1,\ldots,\epsilon_r), A : C \\ \vdots & \cdots\cdots \\ z_r & (-\lambda-mk;\epsilon_1,\ldots,\epsilon_r), B : D \end{array} \right) \qquad \ldots(3.5)$$

and

$$C_n^{\lambda,\mu,\alpha}[x,z_1,\ldots,z_r,t] = \sum_{k=0}^{[n/m]} \binom{\mu+n-1}{n-mk}^{-1} \binom{\alpha+n-1}{n-mk} \frac{(-)^{mk}\,\Omega_k x^k}{(n-mk)!} B_{n,m,k}^{\lambda,\mu,\alpha}[z_1,\ldots,z_r,t] \qquad \ldots(3.6)$$

also let $$\theta_2[x,z_1,\ldots,z_r] = \sum_{k=0}^{\infty} \Omega_k x^k I_{U:W}^{0,n:V} \left(\begin{array}{c|c} z_1 & A : C \\ \vdots & \cdots\cdots \\ z_r & B : D \end{array} \right) \qquad \ldots(3.7)$$

then

$$\sum_{n=0}^{\infty} C_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right]t^n = (1+\eta)^{\alpha+1}\,\theta_2\left[x(-\eta)^m, z_1(1+\eta)^{\epsilon_1},\ldots, z_r(1+\eta)^{\epsilon_r}\right], \quad \ldots(3.8)$$

where $\eta = \dfrac{t}{1-t}$ and $\eta(0)=0$ ...(3.9)

Corollary 3. Suppose

$$C_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,\ldots,z_r,t\right] = \sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l\,t^l}{(\mu+n)_l\,l!}\,I_{U_{11}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1 & (-\lambda-\omega k-n-l;\epsilon_1,\ldots,\epsilon_r),A:C \\ \vdots & \cdots\cdots \\ z_r & (-\lambda-\omega k-mk;\epsilon_1,\ldots,\epsilon_r),B:D \end{array}\right) \ldots(3.10)$$

and

$$D_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right] = \sum_{k=0}^{[n/m]}\binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\frac{(-)^{mk}\,\Omega_k x^k}{(n-mk)!}\,C_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,\ldots,z_r,t\right] \ldots(3.11)$$

then

$$\sum_{n=0}^{\infty} D_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right]t^n = (1+\eta)^{\alpha+1}\,\theta_2\left[x(-\eta)^m, z_1(1+\eta)^{\omega}, z_1(1+n)^{\epsilon_1},\ldots, z_r(1+\eta)^{\epsilon_r}\right],$$

...(3.12)

where η and θ_2 are defined by (2.9) and (2.7) respectively.

Corollary 4. Suppose

$$E_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,\ldots,z_r,t\right] = \sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l\,t^l}{(\mu+n)_l\,l!}\,I_{U_{11}:W}^{\,0,n+1:V}\left(\begin{array}{c|c} z_1 & (1-\lambda-\omega k-n-l;\epsilon_1,\ldots,\epsilon_r),A:C \\ \vdots & \cdots\cdots \\ z_r & (-\lambda-\omega k-mk;\epsilon_1,\ldots,\epsilon_r),B:D \end{array}\right) \ldots(3.13)$$

and

$$F_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right] = \sum_{k=0}^{[n/m]}\binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\frac{(-)^{mk}\,\Omega_k x^k}{(n-mk)!}\,E_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,\ldots,z_r,t\right] \ldots(3.14)$$

also let $$\theta_3\left[x,z_1,\ldots,z_r\right]=\sum_{k=0}^{\infty}\Omega_k x^k I_{U_{11}:W}^{0,n+1:V}\left(\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\middle|\begin{matrix}\left(1-\lambda-\omega k-mk;\epsilon_1,\ldots,\epsilon_r\right),A:C\\ \cdots\cdots\\ \left(-\lambda-\omega k-mk;\epsilon_1,\ldots,\epsilon_r\right),B:D\end{matrix}\right) \quad \ldots(3.15)$$

then $$\sum_{n=0}^{\infty}F_n^{\lambda,\mu,\alpha}\left[x,z_1,\ldots,z_r,t\right]t^n=(1+\eta)^{\lambda}\,\theta_3\left[x(-\eta)^m(1+n)^w,z_1(1+\eta)^{\epsilon_1},\ldots,z_r(1+\eta)^{\epsilon_r}\right] \quad \ldots(3.16)$$

Remark. We obtain the same result with the multivariables H-function defined by Srivastava et al [6]. These results is the extensions of the formulas due to Srivastava et al [7].

4. Aleph-Function of Two Variables

If $r=2$, we obtain the Aleph-function of two variables defined by K.Sharma [5], and we have the following formulas.

Corollary 5.

Let $$C_{n,m}^{\mu,\alpha}(x)=\sum_{k=0}^{[n/m]}\binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\binom{n}{mk}\Omega_k x^k \quad \ldots(4.1)$$

Suppose

$$A_n^{\lambda,\mu,\alpha}\left[z_1,z_2,t\right]=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l\,t^l}{(\mu+n)_l\,l!}\aleph_{U_{10}:W}^{0,n+1:V}\left(\begin{matrix}z_1\\ \vdots\\ z_2\end{matrix}\middle|\begin{matrix}\left(1-\lambda-n-l;\epsilon_1,\epsilon_2\right),A:C\\ \cdots\cdots\\ B:D\end{matrix}\right) \quad \ldots(4.2)$$

also let $$\theta_1\left[x,z_1,z_2\right]=\sum_{k=0}^{\infty}\frac{\Omega_k x^k}{(mk)!}\aleph_{U_{10}:W}^{0,n+1:V}\left(\begin{matrix}z_1\\ \vdots\\ z_2\end{matrix}\middle|\begin{matrix}\left(1-\lambda-mk;\epsilon_1,\epsilon_2\right),A:C\\ \cdots\cdots\\ B:D\end{matrix}\right) \quad \ldots(4.3)$$

then $$\sum_{n=0}^{\infty}C_{n,m}^{\mu,\alpha}(x)\,A_n^{\lambda,\mu,\alpha}\left[z_1,z_2,t\right]\frac{t^n}{n!}=(1-t)^{-\lambda}\,\theta_1\left[x\left(\frac{t}{1-t}\right)^m;\frac{z_1}{(1-t)^{\epsilon_1}},\frac{z_2}{(1-t)^{\epsilon_2}}\right] \quad \ldots(4.4)$$

Corollary 6. Suppose

$$B_{n,m,k}^{\lambda,\mu,\alpha}\left[z_1,\ldots,z_r,t\right]=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l\,t^l}{(\mu+n)_l\,l!}\aleph_{U_{11}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1 & (-\lambda-n-l;\epsilon_1,\epsilon_2),A:C \\ \vdots & \cdots\cdots \\ z_2 & (-\lambda-mk;\epsilon_1,\epsilon_2),B:D \end{array}\right) \quad \ldots(4.5)$$

and

$$C_n^{\lambda,\mu,\alpha}(x,z_1,z_2,t)=\sum_{k=0}^{[n/m]}\binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\frac{(-)^{mk}\,\Omega_k x^k}{(n-mk)!}B_{n,m,k}^{\lambda,\mu,\alpha}\left[z_1,z_2,t\right] \quad \ldots(4.6)$$

also let

$$\theta_2\left[x,z_1,z_2\right]=\sum_{k=0}^{\infty}\Omega_k x^k\,\aleph_{U:W}^{0,n:V}\left(\begin{array}{c|c} z_1 & A:C \\ \vdots & \cdots\cdots \\ z_2 & B:D \end{array}\right) \quad \ldots(4.7)$$

then $$\sum_{n=0}^{\infty}C_n^{\lambda,\mu,\alpha}\left[x,z_1,z_2,t\right]t^n=(1+\eta)^{\alpha+1}\theta_2\left[x(-\eta)^m,z_1(1+\eta)^{\epsilon_1},z_2(1+\eta)^{\epsilon_2}\right], \quad \ldots(4.8)$$

where $\eta=\frac{t}{1-t}$ and $\eta(0)=0$ ...(4.9)

Corollary 7. Suppose

$$C_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,z_2,t\right]=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l\,t^l}{(\mu+n)_l\,l!}\aleph_{U_{11}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1 & (-\lambda-wk-n-l;\epsilon_1,\epsilon_2),A:C \\ \vdots & \cdots\cdots \\ z_2 & (-\lambda-wk-mk;\epsilon_1,\epsilon_2),B:D \end{array}\right) \quad \ldots(4.10)$$

and $$D_n^{\lambda,\mu,\alpha}\left[x,z_1,z_2,t\right]=\sum_{k=0}^{[n/m]}\binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\frac{(-)^{mk}\,\Omega_k x^k}{(n-mk)!}C_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,z_2,t\right] \ldots(4.11)$$

then $$\sum_{n=0}^{\infty}D_n^{\lambda,\mu,\alpha}\left[x,z_1,z_2,t\right]t^n=(1+\eta)^{\alpha+1}\theta_2\left[x(-\eta)^m(1+n)^w,z_1(1+\eta)^{\epsilon_1},z_2(1+\eta)^{\epsilon_2}\right], \ldots(4.12)$$

where η is defined by (2.9) and θ_2 is defined by (2.7)

Corollary 8. Suppose

$$E_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,z_2,t\right]=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l\,t^l}{(\mu+n)_l\,l!}\aleph_{U_{11}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1 & \left(-\lambda-wk-n-l;\epsilon_1,\epsilon_2\right),A:C \\ \vdots & \cdots\cdots \\ z_2 & \left(-\lambda-wk-mk;\epsilon_1,\epsilon_2\right),B:D \end{array}\right) \quad \ldots(4.13)$$

and $$F_n^{\lambda,\mu,\alpha}\left[x,z_1,z_2,t\right]=\sum_{k=0}^{[n/m]}\binom{\mu+n-1}{n-mk}^{-1}\binom{\alpha+n-1}{n-mk}\frac{(-)^{mk}\,\Omega_k x^k}{(n-mk)!}E_{n,m,k}^{\lambda,\omega,\mu,\alpha}\left[z_1,z_2,t\right] \ldots(4.14)$$

also let $$\theta_3\left[x,z_1,z_2\right]=\sum_{k=0}^{\infty}\Omega_k x^k\,\aleph_{U_{11}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1 & \left(1-\lambda-wk-mk;\epsilon_1,\epsilon_2\right),A:C \\ \vdots & \cdots\cdots \\ z_2 & \left(-\lambda-wk-mk;\epsilon_1,\epsilon_2\right),B:D \end{array}\right) \quad \ldots(4.15)$$

then $$\sum_{n=0}^{\infty}F_n^{\lambda,\mu,\alpha}\left[x,z_1,z_2,t\right]t^n=(1+\eta)^{\lambda}\,\theta_3\left[x(\eta)^m(1+\eta)^{\omega},z_1(1+\eta)^{\epsilon_1},z_2(1+\eta)^{\epsilon_2}\right], \ldots(4.16)$$

where η is defined by (2.9).

5. Conclusion

Due to general nature of the Aleph-function of several variables, our formulas are capable to be reduced into many known and new generating functions for special functions of one and several variables.

References

[1] Ayant, F.Y., An integral accociated with the Aleph-functions of several variable. International Journal of Mathematics Trends and Technology (IJMTT). 2016 Vol 31 (3), page 142-154

[2] Jaimini B.B. and Saxena H. On certains classes of generating functions associated with the H-function of one and serveral variables. J. Rajasthan Acad Phy Sci, 2009 Vol 8, no 3, page 255-264.

[3] Raina R.K. On a reduction formula involving combinatorial coefficients and hypergeometric function. Boll. Un. Math.Ital A (ser. 7), 1990 (4) page 183-189.

[4] Sharma C.K. and Ahmad S.S. : On the multivariable I-function. Acta ciencia Indica Math, 1994 vol 20, no 2, page 113-116.

[5] Sharma K. On the integral representation and applications of the generalized function of two variables, International Journal of Mathematical Engineering and Sciences, Vol 3, issue 1 (2014), page 1-13.

[6] Srivastava H.M. and Panda R. Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975), page 119-137.

[7] Srivastava H.M. and Raina R.K. New generating functions for certain polynomial systems associated with the H-function, Hokkaido. Math. Journal. 10, (1981), page 34-45.

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Multistability, Chaos and Complexity in Kraut Model

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Abstract

Evolutionary behavior of a nonlinear prototype multistability model has been investigated to study various stable states as well as multistability, chaos and complexity appearing within the system. The system has been represented by 2-dimensional coupled discrete equations. Bifurcation diagram of the system has been obtained by varying a parameter while keeping other parameters same and proper analysis has been performed. For a certain set of values of parameters, the system shows chaos. Numerical values of Lyapunov exponents are calculated to confirm regular and chaotic evolution. To investigate complexity of the system, topological entropies have also been. These measurable quantities have been represented graphically which help in meaningful discussion of evolution properties. Finally, possible correlation dimension of the chaotic attractor has been calculated.

Key Words: Chaos, Bifurcation, Lyapunov characteristic exponent (LCE), Topological entropy, Correlation dimension.

AMS Subject Classification: 34H10, 34C23, 37L30, 37B40

1. Introduction

Evolutionary state of a nonlinear system can bedetermined by its initial conditions and the parameters involved in it. Chaos is a state of the system showing unpredictability and sensitivity to the initial condition. Real systems are mostly of complex structure and during evolution, one observes properties like bifurcation, chaos, multistability etc. depending on the structure of the mathematical model representing the system. Present trends of researches in nonlinear system are also to understand complexity behavior of the system arising due to interaction between components within it. Unpredictability observed is due to chaos and complexity in the system.Complexity of different type has been explained extendedly through some important articles [1- 4]. Following are the measurable quantities for any complex systems: (i) Lyapunov exponents, (LCEs), positive value guarantees the presence of chaos and (ii) the topological entropies which provide the presence of complexity; more the topological entropy signifies the system is more complex, and also, (iii) the correlation dimension for the chaotic attractor for chaotic evolution of the system.

The objective of the present work is to investigate complexity and chaos phenomena a nonlinear dynamic model proposed recently, [5, 6], which show its multistability character. Here, analyzing the stability of steady state and the bifurcation phenomena we further proceed to obtain various measures of complexity such as LCEs, topological entropies and correlation dimensions. Then, we present these measured quantities graphically and explain their appearances.

2. Dynamic Model

$$\begin{aligned} x_{n+1} &= 1-\alpha x_n^2 + \gamma(x_n - y_n) \\ y_{n+1} &= 1-\alpha y_n^2 + \gamma(x_n - y_n) \end{aligned} \qquad \ldots(1)$$

(a) **Fixed Points and their stability :**

In general, the system (1) has four fixed points obtained as

$P_1^*\left(-\frac{1+\sigma}{2\alpha},-\frac{1+\sigma}{2\alpha}\right)$, $P_2^*\left(-\frac{1-\sigma}{2\alpha},-\frac{1-\sigma}{2\alpha}\right)$, $P_3^*\left(-\frac{\beta+\varsigma}{2\alpha},-\frac{\beta-\varsigma}{2\alpha}\right)$ and $P_4^*\left(-\frac{\beta-\varsigma}{2\alpha},-\frac{\beta+\varsigma}{2\alpha}\right)$ where

$\sigma = \sqrt{1+4\alpha}$, $\beta = 1+2\gamma$ $\varsigma = \sqrt{1+4\alpha-4\gamma^2}$

For $\gamma = 0.29$ & $\alpha = 0.74$ coordinates of above fixed points are respectively obtained as P_1*(0.6689, 0.6689), P_2*(-2.02025, -2.02025), P_3*(-2.35377, 0.21863) and P_4*(0.21863, -2.35377). Using proper stability analysis, it has been observed that only the fixed points P_1* and P_2* are stable and P_3*, P_4* are unstable. Possibility of occurring of **chaotic saddle** near P_3*, P_4* may happen. As in this case orbits initiating close to P_1* and P_2* are also stable and the system becomes regular. In Fig. 1, we have drawn the time series graphs for $\gamma = 0.29$ & $\alpha = 0.74$ with initial point (0.5, 0.5), nearby P_1*. The graphs are of periodic nature.

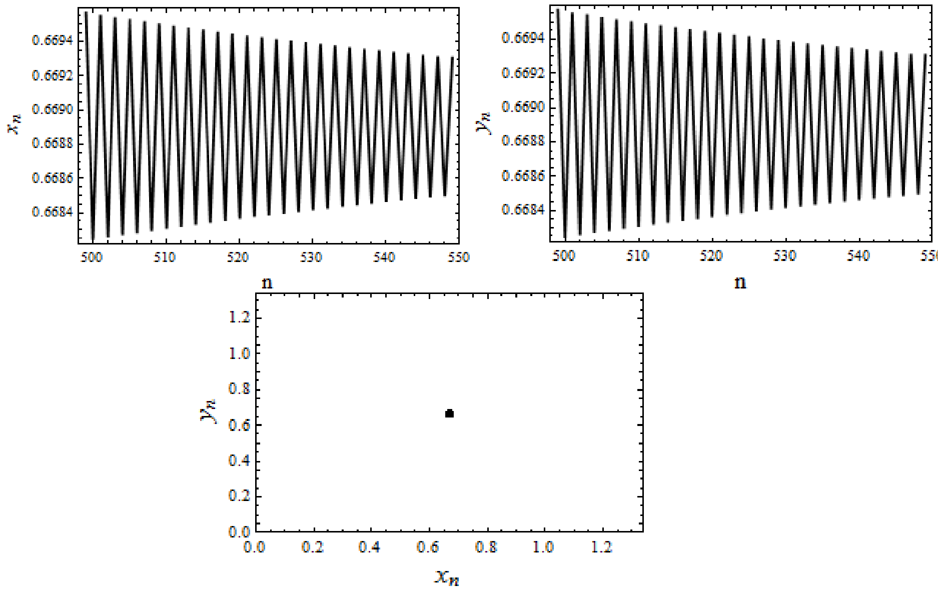


Fig. 1: Time series graphs and phase plot of system (1) for $\gamma = 0.29$ & $\alpha = 0.74$ with initial point (0.5, 0.5).

(b) **Bifurcation Diagrams**:

Bifurcation in a dynamical system is said to occur when phase portrait changes its topological structure with variation of parameter. During the processes of variation of selected parameter of the system while keeping other parameters constant, a stable steady state solution first bifurcates into two. Continuing this process, one may observes four stable steady solutions, then eight etc. Finally, one may get emergence of chaos. In many ecosystems, a special type of bifurcation called Hopf bifurcation, results during the eigenvalues leading to formation of limit cycle. A limit cycle may subsequently undergo a bifurcation resulting in 2-Torus, 3-Torus etc. Creation of tori forms another route for transition from stationary to chaotic behavior; a turbulence.

In the present case, we have obtained bifurcation scenario of the system (1) by varying the parameter α while keeping γ, $\gamma = 0.29$, constant, (see Fig. 2).

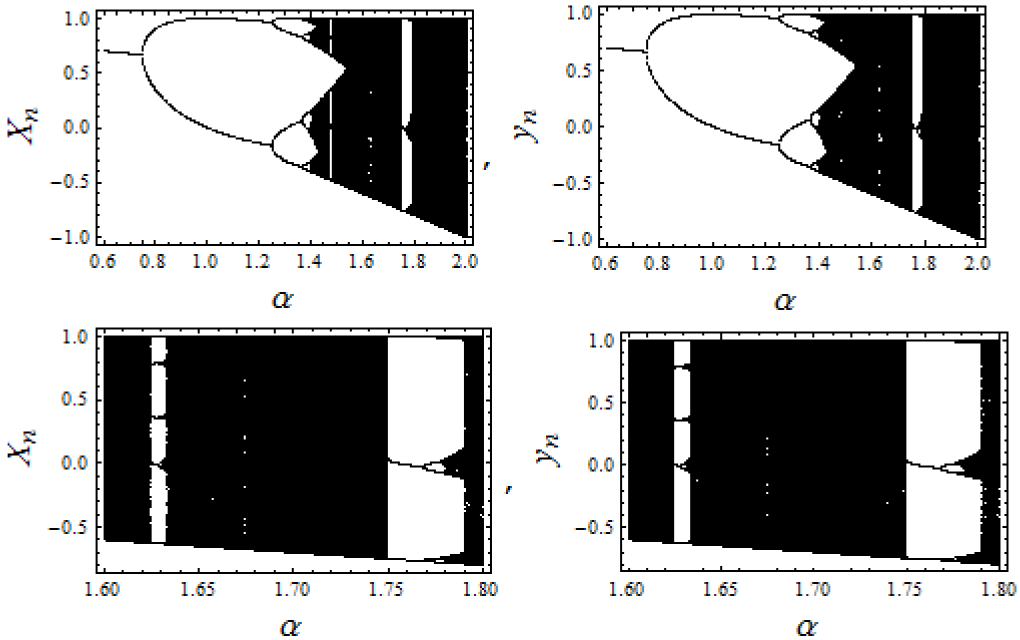
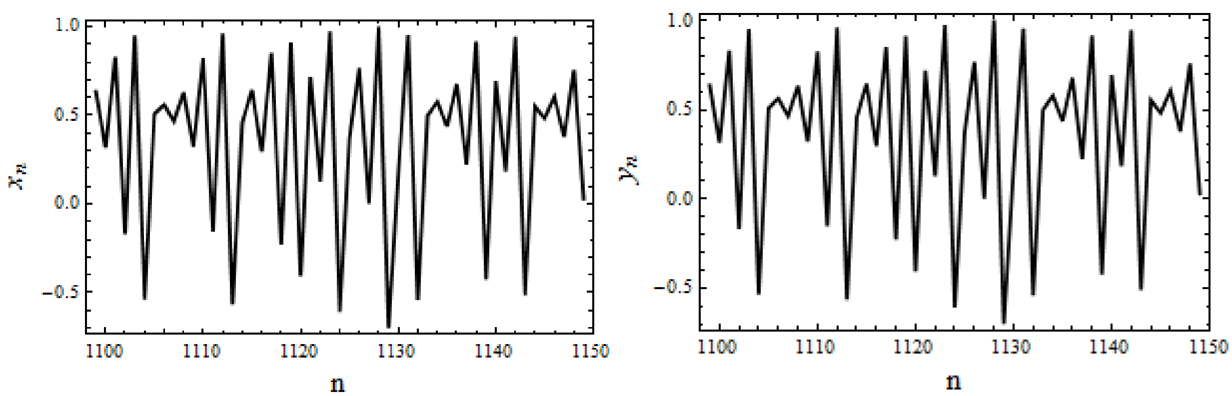


Fig. 2: Bifurcations along x and y axes of the system (1). Lower figures are for close range of γ showing periodic windows.

(c) **Chaotic Motion and Chaotic Attractors**:

System (1) evolve chaotically for $\gamma = 0.29$ and $\alpha = 1.7$ and one finds chaotic attractors. In Fig. 3, time series plots and plot of chaotic line form attractor are given.



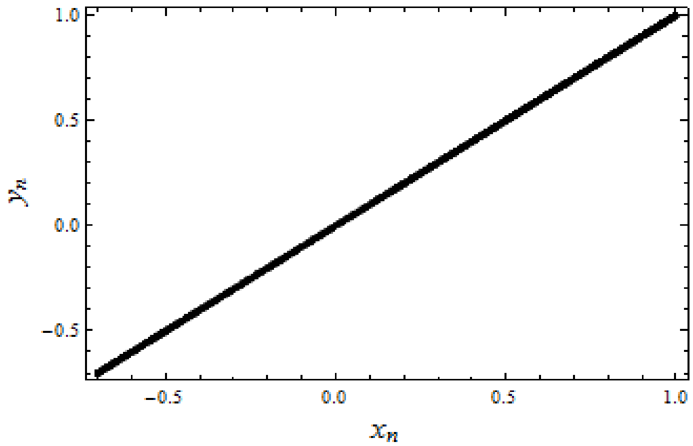


Fig. 3: Chaotic time series plots and plot of line shaped chaotic attractors for $\gamma = 0.29$ and $\alpha = 1.7$.

3. Measuring Chaos and Complexity

As stated, most real systems are of complex structure and during motion they exhibit chaos, complexity and irregularity. Chaos is measured by Lyapunov exponents (LCEs), such that any LCE < 0 signifies system is regular and, on the other hand if LCEs > 0, the system is chaotic. Degree of chaos lies in in the fact that how large LCE is.

(a) Measuring Lyapunov Exponents, (LCEs):

Chaotic attractor exhibits during evolution for $\gamma = 0.29$ and $\alpha = 1.7$ and then the system shows sensitive dependence on initial conditions. At this stage, two trajectories starting together with nearby locations will rapidly diverge from each other and, therefore, have totally different futures. The practical implication is that long-term prediction becomes impossible in a system where small uncertainties are amplified enormously fast. Lyapunov characteristic exponents [7- 11], is very effective tool for measuring regular and chaotic motions since this measures the degree of sensitivity to initial condition in a system. Systematic analytic description for derivation of LCEs be given in some recent book of nonlinear dynamics, e.g. [12 - 15].

In Fig. 4 we have drawn plots of LCEs for its chaotic motion with above stated parameter values.

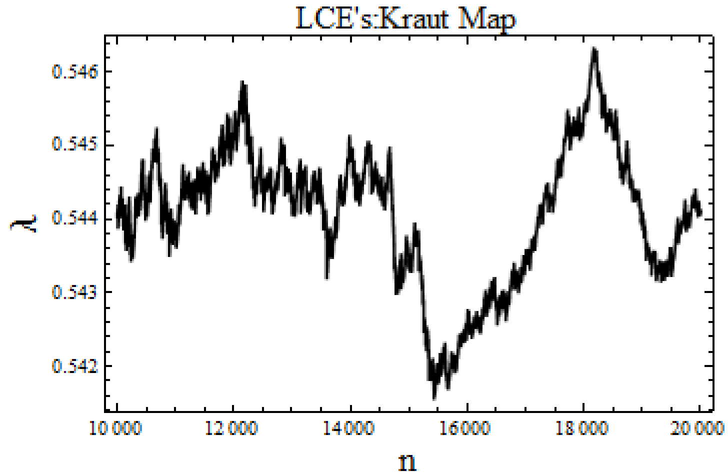


Fig. 4: Plot of Lyapunov exponents (LCEs) of system (1) for $\gamma = 0.29$ and $\alpha = 1.7$.

(b) **Topological Entropies**:

Topological entropy, a non-negative number, provides a perfect way to measure complexity of a dynamical system. For a system, more topological entropy means the system is more complex. Actually, it measures the exponential growth rate of the number of distinguishable orbits as time advances [16, 17].

For the system (1) with parameters $\gamma = 0.29$ and (a) $0.5 \le \alpha \le 1.0$ & (b) $0.72 \le \alpha \le 0.755$, we have calculated topological entropies and plotted as shown in Fig. 5.In these regions system does not have chaotic motion but here, we find significant topological entropy.

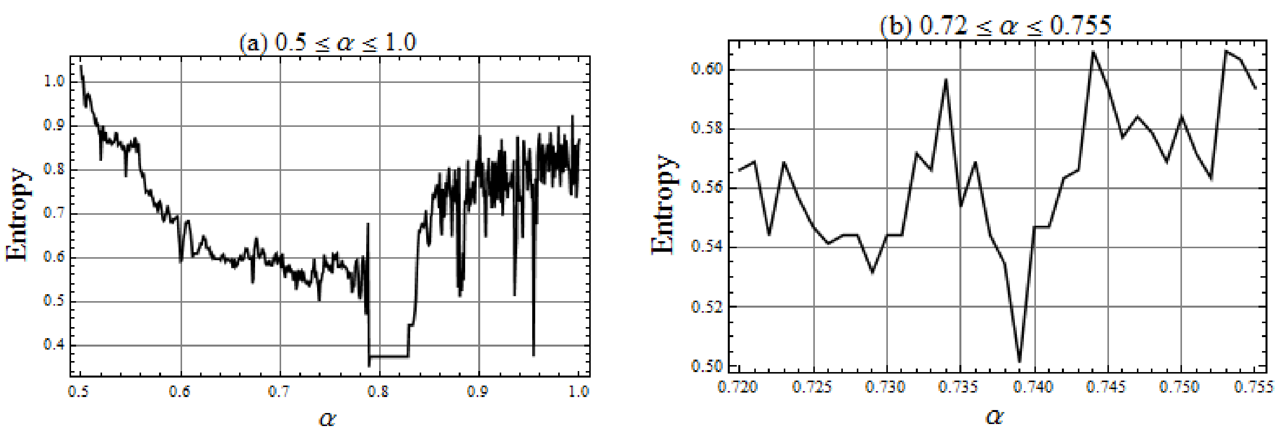


Fig. 5: Plots of topological entropies for $\gamma = 0.29$ and $0.72 \le \alpha \le 0.755$.

(c) **Correlation Dimension D_C**:

Correlation dimension provide the dimensionality of the system. If the system evolve chaotically, its correlation can be interpreted as the dimension of the chaotic attractor which has a fractal structure. To calculate correlation dimension D_C, here we have used the procedure described by Martelli, [15].

To obtain $\mathbf{D_C}$ for our system, first we have calculated correlation integral C(r) for an orbit of the system as explained in a recent article [18]. Then, a plot, Fig. 5, is obtained for certain data related to this C(r).

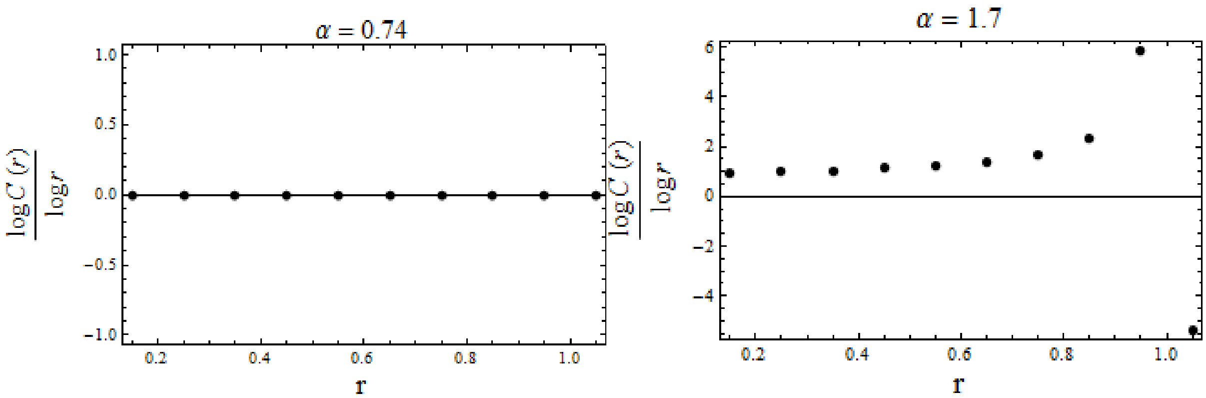


Fig. 6: Plots of data of correlation integral

For α = 0.74, correlation dimension is zero as it is a regular case but for α = 1.7 correlation integral data when plotted form a curve. In this case, for α = 1.7, when the correlation integral data used for linear fit, the linear fit line obtained as

$$Y = 1.61922 - 0.830034\ x$$

The y-intercept of this line is 1.61922 approximately. So, this analysis implies, [15], the correlation dimension obtained, approximately, as $\mathbf{D_C} \approx 1.62$.

4. Discussions

The studies made here on Kraut system, shows that during evolution this system evolve regularly as well as chaotically in some parameter range, Fig. 1 & Fig. 3. Also, the system shows enough complexity even when it is not chaotic. Bifurcation diagrams are drawn, Fig. 2, to demonstrate how such phenomena are evolving during bifurcation. Measure of chaosis given by plot of LCEs, in Fig. 4, and those of complexities is given by plot of topological entropies, Fig. 5. Finally, dimensionality of the chaotic attractor is obtained as the correlation dimension; which is, for case $\gamma = 0.29$ and α = 1.7, $D_C \approx 1.62$.

References

[1] J.R. Beddington, C.A. Free and J.H. Lawton, Dynamic complexity in predator–prey models framed in difference equations. *Nature*,**255**(1975),58-60.

[2] Y. Xiao,D. Cheng and S. Tang, Dynamic complexities in predator–prey ecosystem models with age-structure for predator, *Chaos, Solitons and Fractals*,**14** (2002), 1403-1411.

[3] V Kaitala, M. Heino, Complex non-unique dynamics in ecological interactions,*Proc R Soc London B*,**263**(1996),1011-1015.

[4] S. Tang, L. Chen, A discrete predator–prey system with age-structure for predator and natural barriers for prey, *Math. Model.Numer. Anal*.,**35**(2001), 675-690

[5] Suso Kraut and Ulrike Fendel: Multistability, noise and attractor-hopping: The crucial role of chaotic saddle. Phys Rev E Stat Nonlin Soft Matter Phys. 2002 Jul;66(1 Pt 2):015207. Epub 2002 Jul 26.

[6] Suso Kraut, Ulrike Fendel and C. Grebogi, Preference of attractors in noisy multistablesystems. Phys. Rev. E59 (1999): 5253 - 5260.

[7] G. Benettin, L.Galgani, A. Giorgilli, J.M.Strelcyn (1980): Lyapunov Characteristic Exponents for smooth dynamical systems and for Hamiltonian systems; a method for computing all of them. Part 1& II: Theory, Meccanica,**15**: 9 - 30.

[8] P. Bryant, R. Brown, H.Abarbanel (1990): Lyapunov exponents from observed time series, *Phys.Rev. Lett.*,**65,** No.13 :1523-1526.

[9] R. Brown,P. Bryant, H. Abarbanel (1991) Computing the Lyapunov spectrum of a dynamical system from an observed time series, Phys. Rev. A,**43**, No.6: 2787-2806.

[10] P. H. Bryant (1993): Existential singularity dimensions for strange attractors, *Physics Letters A*,**179**, No.3: 186-190.

[11] C. Skokos (2010) : The Lyapunov characteristic exponents and their computation, *Lect. Notes. Phys.*,**790**(2010), 63-135.

[12] Nagashima, H. and Baba, Y. (2005): Introduction to chaos: Physics and Mathematics of Chaotic Phenomena. IOP Publishing Ltd, Bristol.

[13] Lynch, S. (2007): Dynamical Systems with Applications using Mathematica. Birkhäuser, Boston.

[14] Drazin, P. G. (1992): *Nonlinear Systems*, 1st ed. Cambridge University Press, Cambridge.

[15] Martelli M. (1999): Introduction to Discrete Dynamical Systems and Chaos. John Wiley & Sons, Inc., 1999, New York.

[16] Adler, R L Konheim, A G McAndrew, M H. (1965): Topological entropy, Trans. Amer. Math. Soc. 114: 309-319

[17] R. Bowen, R. (1973): Topological entropy for noncompact sets, Trans. Amer. Math. Soc. 184: 125-136.

[18] L. M. Saha, Til Prasad Sarma and Purnima Dixit (2016): Study of Complexities in Bouncing Ball Dynamical System. J. Appl. Comp. Sci. Math., 10 (21): 46 – 50.

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EOQ Model of Instantaneous Deteriorating Items with Controllable Deterioration Rate with Selling Price and Advertisement Dependent Demand with Partial Backlogging

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Abstract

This paper deals with an economic order quantity (EOQ) model for deteriorating items with price and advertisement dependent demand. In this model, shortages are allowed and partially backlogged. The backlogging rate is dependent on the length of the waiting time for the next replenishment. The purpose of this paper to develop an inventory model for instantaneous deteriorating items with the considerations of facts that the deterioration rate can be controlled by using the preservation technology (PT) and the holding cost is a linear function of time which was treated as constant in most of the deteriorating inventory models. This paper aids the retailer in minimizing the total inventory cost by finding the optimal order quantity. The proposed model is effective as well as efficient for the business organization that uses the preservation technology to reduce the deterioration rate of the instantaneous deteriorating items of the inventory.

KEYWORDS *Deteriorating items, Holding cost, Inventory, Partial Backlogging, Preservation technology, Shortages*

1. Introduction

Deterioration is defined as decay, damage, spoilage, evaporation, loss of utility or loss of marginal value of a commodity that results in decreased usefulness. Commonly inventory models deal with non-deteriorating items (i.e. items that never deteriorate) and instantaneous deteriorating items (i.e. as soon as they enter the inventory they are subject to deterioration). Most physical goods undergo decay or deterioration over time, example being medicines, volatile liquids, blood banks and so on. So decay or deterioration of physical goods in stock is a very realistic factor and there is a big need to consider this in inventory modeling.

Researcher in the field of inventory control have suggested various model taking into consideration different demands and deterioration. The first attempt to describe the optimal ordering policies for such items was made by Ghare and Schrader (1963). They presented EOQ model for an exponentially decaying inventory. Dave and Patel (1981) developed the first deteriorating inventory model with linear trend in demand. They considered demand as a linear function of time. Goyal and Giri (2001) gave recent trends of modelling in deteriorating items inventory. They classified inventory models on the basis of demand variations and various other condition or constrains.

Ouyang, Wu and Chang (2005) developed an inventory model for deteriorating items with exponential declining demand and partial backlogging.

Alamri and Balkhi (2007) studied the effect of learning and forgetting on the optimal production lot size for deteriorating items with time varying demand and deterioration rates. In (2008) Roy Ajanta developed a deterministic inventory model when the deterioration rate is time proportional demand rate is function of selling price and holding cost is time dependent. Skouri, Konstantaras, Papachristos and Canas (2009) developed an inventory model with ramp type demand rate partial backlogging and Weibull's deterioration rate. Mishra and Singh (2010) developed a deteriorating inventory model with partial backlogging when demand and deterioration rate is constant. They made Abad (1996, 2001) more realistic and applicable in practise.

Hung (2011) gave an inventory model with generalized type demand, deterioration and backorder rate. Mishra and Singh (2011) developed deteriorating inventory model for time dependent demand and holding cost with partial backlogging. Leea and Dye (2012) formulate a deteriorating inventory model with stock dependent demand by allowing preservation technology cast as a decision variable in conjunction with replacement policy. Maihami and V Kamlabadi (2012) developed a joint pricing and inventory control system for non-instaneous deteriorating items and adopt a price and time dependent demand function. Sarkar and Sarkar (2013) developed an improved inventory model with partial backlogging time varying deterioration and stock dependent demand. Tam and Weng (2013) developed the discrete-in-time deteriorating inventory model with time varying demand, variable deterioration rate and waiting time dependent partial backlogging. Sanjay kumar singh (2014) developed an inventory model with optimum ordering interval for deteriorating items with selling price dependent demand and random deterioration. Chauhan A and Singh AP (2014) developed an optimal replenishment and ordering policy for time dependent demand and deteriorating with discounted cash flow analysis. In (2015) developed an improved inventory model with an integrated production inventory model with back order and lot for lot policy in fuzzy sense. In this paper, he extended Banerjee (1986) and Mehta (2005) models with the assumption that the backorders for buyer is allowed. In that model, he considered the integrated inventory model with fuzzy order quantity and fuzzy shortage quantity.

SR Singh (2016) developed an inventory model with multivariate demands in different phases with customer returns and inflation. He discussed the impact of customer returns on inventory system of deteriorating items under inflationary environment and partial backlogging.

Kousar Jaha begum (2016) developed an EPQ model for deteriorating items with generalizes Pareto decay having selling price and time dependent demand. The deterioration rate of items in the above mentioned papers is viewed as an exogenous variable which is not subject to control. In practise the deterioration rate of products can be controlled and reduced through various effects such as procedural changes and specialized equipment acquisition. The consideration of PG is important due to rapid social changes and the fact that PT can reduce the deterioration rate significantly. BY the efforts of investing in preservation technology we can reduce the deterioration rate.

In the present work an EOQ model of instaneous deteriorating items with controllable deterioration rate for selling price and advertisement dependent demand pattern over a finite horizon is proposed in which the inflation and time value of money are considered. Shortages are allowed and partially backlogging in this model. Holding cost in linear function of time. So in this paper we made the model of Mishra and Singh (2011) more realistic by considering the fact that the use preservation technology can reduce the deterioration rate significantly which helps the retailers to reduce their economic losses.

2. Assumptions and Notations

The given mathematical model is based on following assumptions and notations

2.1 Notations

1. A is ordering cost per order, C is purchase cost per order, H(t) is the inventory holding cost per unit per unit time.
2. $\prod_b$ backordered cost per unit short per time unit.
3. $\prod_1$ cost of lost sale per unit.
4. £ is the preservation technology (PT) cost per reducing deterioration rate in order to presence the product, £>0.

5. θ is deterioration rate and m(£) is the reduced deterioration rate due to use of preservation technology.
6. $\mathrm{D}_r = \mathrm{Q} - \mathrm{m}(£)$ is the resultant deterioration rate
7. IM is the maximum inventory level during [0,T]
8. IB is the maximum inventory level during shortage period
9. Q = (IM+IB) is the order quantity during a cycle of $TC(t_1, t_2, £)$ is total cost per time unit.
10. $T = (t_1 + t_2)$ is the length of the cycle time, where $t_1$is the time at which the inventory level reaches zero, $t_1 \geq 0$ and t_2 is the length of the period during which shortages are allowed $t_2 \geq 0$.
11. $I_1(t)$is the level of positive inventory at time t, $0 \leq t \leq t_1$.
12. $I_2(t)$is the level of negative inventory at time t, $t_1 \leq t \leq t_1 + t_2$.
13. $B_r(t)$ is the Backlogging rate and μ is backlogging parameter.

2.2 Assumptions

1. The demand rate D is a deterministic function of selling price, *s,* and advertisement cost A_c per unit item i.e. $D(A_c, s) = A_c^n a s^{-b}$ $a > 0,$ $b > 1, 0 \leq n < 1$, *a* is the scaling factor, *b* is the index of price elasticity and n is the shape parameter.
2. The lead time is Zero.
3. No replacement or repair of deteriorated items takes place in a given cycle..
4. Shortages are allowed and partially backlogged.
5. The effects of inflation and time value of money are considered.
6. The replenishment takes place at an infinite rate.
7. During the fixed period μ, the product has no deterioration. After that, it will deteriorate with a constant rate θ, 0 < θ < 1.

   During stock out period, the backlogging rate is variable and is dependent on the length of the waiting time for next replenishment so that the backlogging rate for negative inventory is $B_r(t) = \frac{1}{1+£(T-t)}$, where £ is backlogging parameter 0 ≤ £ ≤ 1 and (T- t) is waiting time.

3. Mathematical Modelling and Analysis

The rate of change of inventory during positive stock period $[0, t_1]$ occur due to demand & resultant deterioration rate D_r and in shortage period $[t_1, T]$ occur due to demand & a fraction of demand is backlogged & backlogging rate is $B_r(t)$. Hence the inventory level at any time during $[0, t_1]$ and during $[t_1, T]$ is governed by the differential equations.

$$\frac{dI_1(t)}{dt} + D_r I_2(t) = -D; \qquad 0 \le t \le t_1 \qquad \text{... (1)}$$

$$\frac{dI_2(t)}{dt} = \frac{-D}{1+\mu(T-t)}; \qquad t_1 \le t \le T \qquad \text{... (2)}$$

with boundary condition $I_1(t) = I_2(t) = 0$ at $t = t_1$ and $I_1(t) = IM$ at $t = 0$.

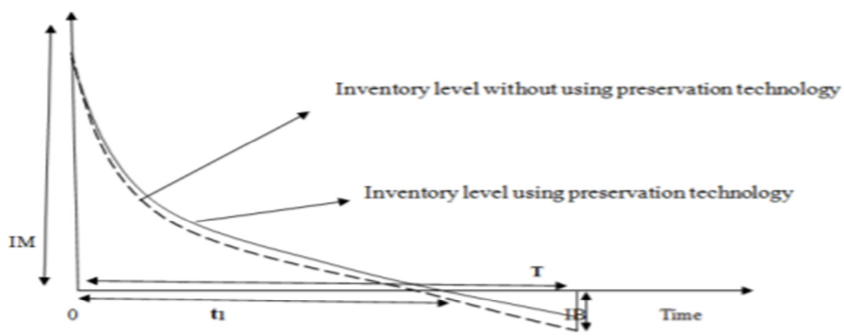

Figure 1. Graphical Representation of Inventory System

4. Analytical Solution

Case 1: Inventory level without shortage

During the period $[0, t_1]$ the inventory depletes due to the deterioration and demand. Hence the inventory level at any time during $[0, t_1]$ is described by differential equation.

$$\frac{dI_1(t)}{dt} + \theta I_1(t) = -D; \quad 0 \le t \le t_1 \qquad \text{... (3)}$$

with the boundary conditions $I_1(t_1) = 0$ at $t = t_1$. the solution of equation (3) is

$$I_1(t) = \frac{A_c^n a s^{-b}}{\theta - m(£)} \; e^{(\theta - m(£))(t_1 - t)} - 1 \qquad \text{... (4)}$$

Case 2: Inventory level with shortages

During the interval $[t_1, T]$ the inventory level depends on demand and a fraction of demand is backlogged. The state of inventory during $[t_1, T]$ can be represented by the differential equation.

$$\frac{dI_2(t)}{dt} = \frac{-D}{1+\mu(t_1+t_2-t)}; \qquad t_1 \le t \le t_1 + t_2 \qquad \text{... (5)}$$

with the boundary conditions $I_2(t_1) = 0\ at\ t = t_1$. the solution of equation (5) is

$$I_2(t) = \frac{A_c^n as^{-b}}{\mu}\left[log\frac{1+\mu(t_1+t_2-t)}{1+\mu t_2}\right] \qquad \text{... (6)}$$

Therefore the total cost per replenishment cycle consists of the following components.

(1) Inventory holding cost per cycle

$$IHC = \int_0^{t_1} H(t)\ \ I_1(t)dt = \int_0^{t_1}(\alpha+\beta t)\, I_1(t)dt$$

$$IHC = \left[\frac{-D}{(\theta-\mathrm{m}(£))^2}[\alpha\left(e^{(\theta-\mathrm{m}(£))(t_1-t)}+\left(\theta-\mathrm{m}(£)\right)t\right)\right.$$
$$+\beta e^{(\theta-\mathrm{m}(£))(t_1-t)}\left((1+t)-\left(\theta-\mathrm{m}(£)\right)\frac{t^2}{2}\right)]] \qquad \text{... (7)}$$

(2) Backordered cost per cycle

$$BC = \prod\nolimits_b \int_{t_1}^{(t_1+t_2} -I_2(t)dt$$

$$BC = -\prod\nolimits_b D\left[\left(\left(\frac{t_1t_2-{t_2}^2}{2}+\frac{\mu {t_2}^3}{6(1+\mu t_2)^3}\right)\right)\right] \qquad \text{... (8)}$$

(3) Lost Sales cost per cycle

$$LS = \prod\nolimits_1 \int_{t_1}^{(t_1+t_2}\left[1-\frac{D}{1+\mu(t_1+t_2-t)}dt\right]$$

$$= \prod\nolimits_1\left[t_2+\frac{D}{\mu}\log(1+\mu t_2)\right] \qquad \text{... (9)}$$

(4) Purchase cost per cycle = (purchase cost per unit)* (order quantity in one cycle)

$$\boxed{\mathrm{P.C} = \mathrm{C} * \mathrm{Q}}$$

when t = 0, the level of inventory is maximum and it is denoted by IM = $IM = (\,I_1(0))$ then from the equation (4)

$$IM = \frac{A_c^n as^{-b}}{\theta-\mathrm{m}(£)}\left[e^{(\theta-\mathrm{m}(£))(t_1)}-1\right] \qquad \text{... (10)}$$

The maximum backordered inventory is obtained at $T = (t_1+t_2)$ then from the equation (6)

$$\boxed{IB = -I_2(t_1+t_2)}$$

$$IB = \frac{A_c^n as^{-b}}{\mu}\left[\log(\frac{1}{1+\mu t_2})\right] \quad ...(11)$$

Thus the order size during total time interval [0, T]

$$\boxed{Q = IM + IB}$$

now from equation (10) & (11)

$$Q = \frac{A_c^n as^{-b}}{\theta - \mathrm{m}(£)}\left[e^{(\theta-\mathrm{m}(£))(t_1)} - 1\right] + \frac{A_c^n as^{-b}}{\mu}\left[\log(\frac{1}{1+\mu t_2})\right] \quad ...(12)$$

$$\mathrm{P.C} = \mathrm{C} * \ A_c^n as^{-b} \ \left[\frac{e^{(\theta-\mathrm{m}(£))(t_1)}-1}{\theta-\mathrm{m}(£)}\right] + \frac{1}{\mu}\left[\log\left(\frac{1}{1+\mu t_2}\right)\right] \quad ...(13)$$

(5) Ordering cost(O. C) = A ... (14)

Therefore the total cost (TC) per time unit is given by $(t_1, t_2, £)$

$$\mathrm{TC}\ (t_1, t_2, £) = \frac{1}{(t_1+t_2)}[Ordering\ cost + Carrying\ cost + backordering\ cost + total\ sales\ cost + purchase\ cost]$$

$$\text{i.e. TC}\ (t_1, t_2, £) = \frac{1}{(t_1+t_2)}[O.C + I.H.C + B.C + L.S\ + P.C]$$

$$\mathrm{TC}\ (t_1, t_2, £) = \frac{1}{(t_1+t_2)}[A - \frac{D}{(\theta-\mathrm{m}(£))^2}\left[\alpha\left(e^{(\theta-\mathrm{m}(£))(t_1-t)} + (\theta - \mathrm{m}(£))t\right) + \beta e^{(\theta-\mathrm{m}(£))(t_1-t)}\left((1+t) - (\theta - \mathrm{m}(£))\frac{t^2}{2}\right)\right] - \prod_b D\left[\left(\left(\frac{t_1t_2-t_2{}^2}{2} + \frac{\mu t_2{}^3}{6(1+\mu t_2)^3}\right)\right)\right] + \prod_1\left[t_2 + \frac{D}{\mu}\log(1+\mu t_2)\right] + \mathrm{C} * A_c^n as^{-b}\ \left[\frac{e^{(\theta-\mathrm{m}(£))(t_1)}-1}{\theta-\mathrm{m}(£)}\right] + \frac{1}{\mu}\left[\log(\frac{1}{1+\mu t_2})\right] \quad ...(15)$$

Differentiate the equation (15) with respect to $t_1, t_2\ and\ £$ then we get

$\frac{\partial(TC)}{\partial t_1}, \frac{\partial(TC)}{\partial t_2}\ and\ \frac{\partial(TC)}{\partial £}$. To minimize the total cost $TC(t_1, t_2, £)$ per unit time, the optimal value of $t_1, t_2\ and\ £$ can be obtained by solving the following equations

$$\frac{\partial(TC)}{\partial t_1} = 0, \frac{\partial(TC)}{\partial t_2} = 0\ and\ \frac{\partial(TC)}{\partial £} = 0 \quad ...(16)$$

The H-Matrix of the function $TC(t_1, t_2, £)$ is defined as $H = \begin{bmatrix} \frac{\partial^2 TC}{\partial t_1{}^2} & \frac{\partial^2 TC}{\partial t_1 \partial t_2} & \frac{\partial^2 TC}{\partial t_1 \partial £} \\ \frac{\partial^2 TC}{\partial t_2 \partial t_1} & \frac{\partial^2 TC}{\partial t_2{}^2} & \frac{\partial^2 TC}{\partial t_2 \partial £} \\ \frac{\partial^2 TC}{\partial £ \partial t_1} & \frac{\partial^2 TC}{\partial £ \partial t_2} & \frac{\partial^2 TC}{\partial £^2} \end{bmatrix}$ provided the determinant of principal minor (H-Matrix) of $TC(t_1, t_2, £)$ is positive definite. i.e. $\det(H_1) > 0,\ \det(H_2) > 0,\ \det(H_3) > 0,$ where $H_1,\ H_2,\ H_3$ is the principal minor of the H Matrix.

5. Conclusion

In this paper, an EOQ model for instantaneous deteriorating items with controllable deterioration rate with selling price and advertisement dependent demand with partial backlogging. This paper considered demand as the increasing function of the selling price and advertising parameter. Also shortages is allowed and it can be partially backlogged where the backlogging rate is dependent on the time of waiting for the next replenishment. It is discussed over the finite planning horizon. The purpose of this study is to present an inventory model involving controllable deterioration rate to extend the traditional EOQ model. The product with high deterioration rate are always crucial to the retailers business. In real markets, the retailer can reduce the deterioration rate of product by making effective capital investment in storehouse equipment. To reduce the deterioration rate retailer invested in the PT cost and a solution procedure has presented to determine an optimal replenishment cycle, shortages period, order quantity and preservation technology cost such that the total inventory cost per unit time has minimized. This model is very practical for the retailers who use the preservation technology in their warehouses to control the deterioration rate under other assumptions of this model. This model can further be extended by taking more realistic assumptions such as probabilistic demand rate. This model is also very useful in retail business segment such as that of fashionable cloths, domestic goods and other daily products.

References

[1] Abad, P.L (1996).Optimal pricing and lot sizing under condition of perishability and partial backordering. *Management Science, 42*, 1093-1104.

[2] Abad, P.L (2001).Optimal price and order size for a reseller under partial backlogging. *Computers and Operation Research, 28,* 53-65.

[3] Alamri, A.A, & Balkhi, Z.T (2007). The effects of learning and forgetting on the optimal production lot size for deteriorating items with time varying demand and deterioration rates. *International Journal of Production Economics, 107*, 125-138.

[4] Banerjee, A.(1986). A joint economic lot size model for purchaser and vendor. *Decision Sciences, Vol. 13,No.3*, 361-391.

[5] Begum, K.J & Devendra,P.(2016). EPQ model for deteriorating items with Generalizes Parato Decay having selling price and time dependent demand.*Vol.4(1),*1-11.

[6] Chauhan, A.& Singh, A.P.(2014). optimal replenishment and ordering policy for time dependent demand and deterioration with discounted cash flow analysis. *Int. J. Mathematics in Operational research, Vol.6,No.4*.407-436.

[7] Dave, U., & Patel, L.K. (1981). (T, Si) – Policy inventory model for deteriorating items with time proportional demand. *Journal of Operational Research Society, 32,* 137-142.

[8] Ghare, P.M, & Schrader, G.F (1963). A model for an exponentially decaying inventory. *Journal of Industrial Engineering, 14,* 238-243.

[9] Goyal, S.K.&Giri, B.C (2001).Recent trend in modelling of deteriorating inventory. *European Journal of Operational Research, 134*,1-16.

[10] Hung, K.C (2011). An inventory model with generalized type demand, deterioration and backorder rates. *European Journal of Operation Research, 208(3),* 239-242.

[11] Kumar, S., Agarwal, N., Sharma, A.& Kumar, S.(2014). optimum ordering interval for deteriorating items with selling price dependent demand and random deterioration. *Aryabhatta Journal of Mathematics & Informatics Vol. 6,No.1,* 165-168.

[12] Leea Y.P & Dye, C.Y (2012). An inventory model for deteriorating items under stock dependent demand and controllable deterioration rate. *Computers & Industrial Engineering, 63(2)*, 474-482.

[13] Mahata, G.C.(2015).An integrated production-inventory model with backorder and lot for policy in fuzzy sense. *Int. J. Mathematics in Operational research, Vol. 7,*No.1, 69-102.

[14] Mahata, G. C.,Goswami, A.& Gupta, D,K(2005). A joint economic lot size model for purchaser and vendor in fuzzy sense. *Computers and Mathematics with applications, Vol.50,*1767-1790.

[15] Maihami, R.,& Kamalabadi, I.N.(2012). Joint pricing and inventory control for non-instantaneous deteriorating items with partial backlogging and time and price dependent demand. *International journal of Production Economics,136(1),*116-122.

[16] Mishra, V.K,. & Singh, L.S.(2011).deteriorating inventory model for time dependent demand and holding cost with partial backlogging. *International Journal of Management Science and Engineering Management,6(4),*267-271.

[17] Mishra, V.K,. & Singh, L.S.(2010).Deteriorating inventory model with time dependent demand and partial backlogging. *Applied Mathematical Sciences,4(72)*, 3611-3619.

[18] Quyang, L.-Y.,K.-S., & Cheng, M.-C.(2005). An inventory model for deteriorating items with exponential declining demand and partial backlogging. *Yugoslav Journal of Operational Research,15(2),* 277-288.

[19] Roy, A,(2008). An inventory model for deteriorating items with price dependent demand and time varying holding cost. *Advanced Modelling and Optimization,10*,25-37.

[20] Sarkar, B.,& Sarkar, S. (2013). An improved inventory model with partial backlogging, time varying deterioration and stock-dependent demand. *Economic Modelling, 30*, 924-932.

[21] Singh,S.R.,Singh,S. & Sharma, S.(2016). Inventory model with multivariate demands in different phases with customer returns and inflation. *Int. J. Mathematics in Operational research, Vol.8*,No.4.466-489.

[22] Skouri, k.,Konstantaras, I., Papachristos, S.,& Ganas, I.(2009).Inventory models with ramp type demand rate, partial backlogging and Weibull deterioration rate. European *Journal of operational research,192,*79-92.

[23] Ten, Y., & Weng, M.X.(2013).A discrete- in- time deteriorating inventory model with time-varying demand, variable deterioration rate and waiting time dependent partial backlogging. *Internal System of Journal Sciences,44*,1483-1493.

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Fractional Differential Equation Model for Spread of Technological Innovations in Arbitrary Order

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Abstract

The present paper deals with the study of mathematical model for spread of technological innovations in companies or industries. Here we show the rate at which new innovations are adopted by the company using fractional differential equation model. We have found solution in terms of Mittag-Leffler function.

AMS Classification: 33E12, 44A10, 26A33, 92B99.

Keywords: Generalized Mittag-Leffler function, Laplace transform, Caputo fractional derivative.

1. Introduction

Fractional calculus is three centuries old as the conventional calculus, but it's not that much popular among science and engineering community. In recent years, an increasing interest for the analysis and applications of fractional calculus has been investigated extensively. Recently, Nieto [3] obtained very important results and he also reported that fractional calculus is more accurate than classical calculus to describe the dynamic behaviour of many real-world physical systems including rheology, viscoelasticity, electrochemistry, electromagnetism etc. Also, it provides an excellent instrument for the description of memory and hereditary properties of various materials and processes which are neglected in classical integer-order models. Besides, many important mathematical models are described using fractional calculus

- *Dedicated to Prof. M. A. Pathan on his 75th Birthday.*

approach. Recently, considerable amount of work has been done in the area of fractional differential equations and a large collection of analytical and numerical methods were developed and employed for obtaining the solution.

2. Prerequisites

Definition : The *Mittag-Leffler function*[8] with two parameters is defined as,

$$E_{\alpha,\beta}\left(z\right)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma\left(\alpha n+\beta\right)} \quad , \qquad \ldots(0)$$

where $\alpha,\beta\in C; Re\left(\alpha\right)>0$, $Re(\beta)>0$.

Definition : *Caputo's* definition of fractional derivative is given by

$${}_{0}^{C}D_{t}^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{t}\frac{f^{n}\left(\tau\right)}{\left(t-\tau\right)^{\alpha-n+1}}d\tau, \qquad \ldots(2)$$

where $\alpha\in R$, is order of fractional derivative, $n-1<\alpha\leq n$ *and* $n\in N=\{1,2,3,\ldots\}$, $f^{n}\left(\tau\right)=\frac{d^{n}}{dt^{n}}f\left(\tau\right)$ *and* $\Gamma(.)$ is Euler Gamma function defined by

$$\Gamma\left(z\right)=\int_{0}^{\infty}x^{z-1}e^{-x}dx. \qquad \ldots(3)$$

The Laplace transform of the Caputo fractional derivative with respect to time t.

$$\begin{aligned}\mathcal{L}\left\{{}_{0}^{C}D_{t}^{\alpha}u\left(t\right)\right\} &=\int_{0}^{\infty}e^{-st}\,{}_{0}D_{t}^{\alpha}u\left(t\right)dt\\ &=s^{\alpha}F\left(s\right)-\sum_{j=0}^{n-1}s^{\alpha-j-1}f^{k}\left(0\right).\end{aligned} \qquad \ldots(4)$$

The inverse Laplace transform requires the Mittag-Leffler function, which is defined as (1).

Laplace transform of the Mittag-Leffler function follows as,

$$\int_{0}^{\infty}e^{-sz}z^{\beta-1}E_{\alpha,\beta}\left(az^{\alpha}\right)dz=\frac{s^{\alpha-\beta}}{s^{\alpha}-a}, \qquad \ldots(5)$$

where $a,\alpha,\beta\in C, Re(\alpha)>0, Re(\beta)>0,$ so we have,

$$\mathcal{L}\left\{x^{\beta-1}E_{\alpha,\beta}\left(-ax^{\alpha}\right)\right\}=\frac{s^{\alpha-\gamma}}{s^{\alpha}+a}. \qquad \ldots(6)$$

Definition : Chauhan *et al.*[5] has obtained the following results,

$$\log_{E_{\alpha}}m\oplus\log_{E_{\alpha}}n=\log_{E_{\alpha}}\left(m\square\, n\right) \qquad \ldots(7)$$

$$\log_{E_\alpha} m \ominus \log_{E_\alpha} n = \log_{E_\alpha} (m \oplus n) \qquad \ldots(8)$$

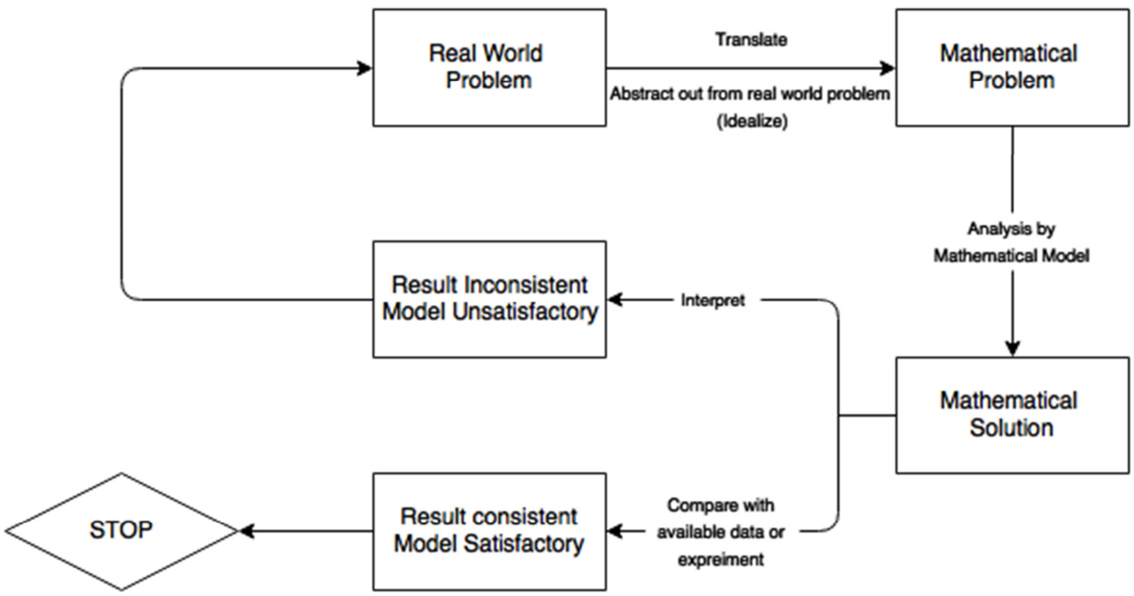
Definition: Jumarie[6] has given the following formula

$$\int \frac{d^\alpha x}{x} = Ln_\alpha \frac{x}{c}, \qquad x = E_\alpha \left(Ln_\alpha x \right). \qquad \ldots(9)$$

where C denotes a constant such that $\frac{x}{C} > 0$ and $Ln_\alpha x$ denotes the inverse function of the Mittag-Leffler function.

3. Mathematical Models

It is integrated process of translating real world problem into mathematical problem, which is based on mathematical concepts i.e. function, variables, constants, inequality, etc. taken from algebra, geometry, calculus and other branches of mathematics.



Innovation Model

Here, we are going to discuss about the process of adoption of a new technological innovation by a large population of companies or industries.Here we are considering a situation where new technological innovations are introduced in large population of companies which is capable of adopting the innovations through motivation.We divide the total employees of companies in two exclusive groups viz, susceptibles S and adoptives A .Susceptibles are those companies who are unaware of new innovations but are capable of adopting it through motivation.Adoptives are those companies who have already adopted the new innovations.

Here, we are considering both this groups i.e. susceptibles and adoptives as function of time t because the number of companies in each of these groups changes during the period of investigation.We consider the population of total companies or industries during the time of investigation remains constant (let N).

Mathematically we can say that,

$$S(t)+A(t)=N, \qquad for \;\; t>0. \qquad \dots(10)$$

The rate of change of adoptives is given by,

$$\frac{dA}{dt}=\beta AS, \qquad \dots(11)$$

where β is the rate at which the motivation for the adoption of innovation is provided by contact between A and S only.

By contact between A and S as well as by mass communication sources like TV, radio, newspaper, etc.

In above equation β and β' are positive constants called the adoption rate due to contact and mass communication respectively.

We discuss two models and give the formulation of models as given below,

1. The direct contact (DC).
2. The mixed contact (MC).

4. Direct contact model

Here the motivation for adoption of new innovation is provided only by contact between A and S .

If at $t=0,\ A_0$ companies have adopted the innovation, then we have initial value problem follows from equation (10) and (11),

$$\frac{dA}{dt}=\beta A(N-A), \qquad \dots(12)$$

The solution of (12) is given by,

$$A(t)=\left\{\frac{N}{1+\left(\frac{n}{A_0}-1\right)}e^{-\beta Nt}\right\}. \qquad \dots(13)$$

So, we can interpret from above equation that the adoption process $A'(t)$ is maximum when $A=\frac{N}{2}$. In other words, the adoption process accelerates up to the point at which half community has adopted the innovations and thereafter the process decelerates.

5. Fractional Differential Equation for DC Model

Here we are motivated to study the fractional differential equation for the direct contact model and obtain the solution for the said model.

On writing (12) with arbitrary order α, as,

$$\frac{d^{\alpha}A}{dt^{\alpha}}=\beta A(N-A), where\, 0<\alpha\leq 1 \qquad \ldots(14)$$

$$\Rightarrow \int\frac{d^{\alpha}A}{A}+\int\frac{d^{\alpha}A}{N-A}=N\beta\int dt^{\alpha}. \qquad \ldots(15)$$

On using the (7), the solution of (15) is given by,

$$\log_{E_\alpha} A \ominus \log_{E_\alpha}(N-A)=\frac{N\beta}{\Gamma(2-\alpha)}\int t^{1-\alpha}dt^{\alpha}+c, \qquad \ldots(16)$$

$$\Rightarrow \frac{I}{N-A}=CE_{\alpha}\left[\frac{N\beta}{\Gamma(2-\alpha)}\int t^{1-\alpha}dt^{\alpha}\right], \qquad \ldots(17)$$

where $0<\alpha<1$ and $\lim_{\alpha\to 1}\log_{E_\alpha} x=\log_e x.$

Initially, when time $t=0$ we have,

$A(0)=A_0$, we get $C=\frac{A_0}{N-A_0}$, i.e.

$$\frac{A}{N-A}=\frac{A_0}{N-A_0}E_{\alpha}\left[\frac{N\beta}{\Gamma(2-\alpha)}\int t^{1-\alpha}dt^{\alpha}\right] \qquad \ldots(18)$$

$$\Rightarrow A=\frac{N}{1+\frac{N-A_0}{A_0}\left[E_{\alpha}\{\frac{N\beta}{\Gamma(2-\alpha)}\int t^{1-\alpha}dt^{\alpha}\}\right]^{-1}} \qquad \ldots(19)$$

when $\alpha\to 1$ and $t\to\infty$, gives $A=N$.

6. The Mixed Model

Here, we consider the situation where no companies have adopted the innovation at time $t = 0$, then we have following initial value problem,

$$\frac{dA}{dt} = (\beta A + \beta')(N - A),\ A(0) = 0. \qquad \ldots(20)$$

7. Fractional Differential Equation for MC Model

Here, we solve the given model in the form of fractional differential equation.

$$\frac{dI}{dt} = \beta A(\gamma - A), \qquad A(0) = A_0,\ \gamma = \left(N - \frac{\beta'}{\beta}\right). \qquad \ldots(21)$$

For developing fractional differential equation model we write (20) as,

$$\frac{d^{\alpha} A}{dt^{\alpha}} = \beta A(\gamma - A), A(0) = A_0. \qquad \ldots(22)$$

As shown in the previous model, the solution of this fractional differential equation can also be given by

$$\log_{E_\alpha} A \ominus \log_{E_\alpha} (\gamma - A) = \frac{\gamma\beta}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^{\alpha} + c \qquad \ldots(23)$$

$$\frac{A}{\gamma - A} = CE_{\alpha}\left[\frac{\gamma\beta}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^{\alpha}\right], \qquad \ldots(24)$$

where $0 < \alpha < 1$ and $\lim_{\alpha \to 1} \log_{E_\alpha} x = \log_e x.$ Initially when time $t = 0$, we have,

$A(0) = A_0$, and further simplification gives $C = \frac{A_0}{\gamma - A_0}$. Thus, we arrive at,

$$\frac{A}{\gamma - A} = \frac{A_0}{\gamma - A_0} E_{\alpha}\left[\frac{\gamma\beta}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^{\alpha}\right], \qquad \ldots(25)$$

$$or \quad A = \frac{\gamma}{1 + \frac{\gamma - A_0}{A_0}\left[E_{\alpha}\left\{\frac{\gamma\beta}{\Gamma(2-\alpha)} \int t^{1-\alpha} dt^{\alpha}\right\}\right]^{-1}}. \qquad \ldots(26)$$

when $\alpha \to 1$ and $t \to \infty$ this yields $A = N - \frac{\beta'}{\beta}$.

8. Conclusion

We are interested to obtain the analytical solution of fractional differential equation model for spread of technological innovations. This work may be useful for computational study and statistical survey purpose.

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References

[1] A.M. Mathai and H.J.Haubold, *Special Functions for Applied Scientists*, Springer, New York, 2010.

[2] I.Podlubny, *Fractional Differential Equations,* Technical University of Kosice, Academic Press, 1999.

[3] J.J.Nieto, *Maximum principles for fractional differential equations derived from Mittag-Leffler functions,* Appl.Math. Lett. 23, 2010, 1248-1251.

[4] J.N.Kapur, *Mathematical Modeling,* New Age International, New Delhi, India, 1988.

[5] J.P.Chauhan,R.K. Jana andA.K.Shukla, *A solution of logarithmic properties of* $E_\alpha(x)$, due for submission.

[6] G. Jumarie, *Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions,* App. Math. Lett., 22, 2009, 378-385.

[7] J.Li,Y.Zhao andH.Zhu, *Bifurcation of an SIS model with nonlinear contact rate*, J. Math. Anal. Appl.432,2015,1119-1138.

[8] K.S.MillerandB.Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,* John Wiley & Sons, New York, 1993.

[9] L.DebnathandD.Bhatta, *Integral Transforms and their Applications,* Chapman & Hall, CRC Press, New York, 2007.

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New Information Inequalities and Resistor-Average Distance

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Abstract

In this paper, we have established an upper and lower bounds of well-known information divergence measure in terms of Relative Jensen-Shannon divergence measure using a new f-divergence measure and new information inequalities. The Relation between Relative Jensen-Shannon divergence measure and Resister average distance has also studied.

Keywords: - Chi-square divergence, Jenson-Shannon's divergence, Resister Average distance, Triangular discrimination etc.

AMS Classification 62B-10, 94A-17,26D15

1. Introduction

Let $\Gamma_n = \left\{ P = (p_{1,} p_{2,} .... p_n) \middle| p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2$ be the set of all complete finite discrete probability distributions. Jain & Saraswat [10, 11] introduced new f-divergence measure given by

$$S_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i + q_i}{2q_i}\right) \qquad \ldots(1.1)$$

where $f : (1/2, \infty) \to \mathbb{R}_+$ is a convex function and $P, Q \in \Gamma_n$.

The new f -divergence is a general class of divergence measures that includes several divergences used in measuring the distance between two probability distributions. This class has introduced on convex function f , normalized functions f(1)=0 and defined on (1/2, ∞). An important property of this divergence is that many known divergences can be obtained from this measure by appropriately defining the convex function f .

Proposition 1.1 Let $f:[0,\infty)\to\mathbb{R}$ be convex and $P,Q\in\Gamma_n$ then we have the following inequality

$$S_f(P,Q)\geq f\left(1\right) \qquad \ldots(1.2)$$

Equality holds in (1.2) iff

$$p_i = q_i \;\; \forall\, i = 1,2,..,n \qquad \ldots(1.3)$$

Corollary 1.1.1 (Non-negativity of new f-divergence measure) Let $f:[0,\infty)\to\mathbb{R}$ be convex and normalized, i.e.

$$f(1)=0 \qquad \ldots(1.4)$$

Then for any $P,Q\in\Gamma_n$ from (1.2) of proposition 1.1 and (1.4), we have the inequality

$$S_f(P,Q)\geq 0 \qquad \ldots(1.5)$$

If f is strictly convex, equality holds in (2.5) iff

$$p_i = q_i \;\; \forall\, i\in\left[i,2,...............n\right] \qquad \ldots(1.6)$$

and

$$S_f(P,Q)\geq 0 \;\text{ and }\; S_f(P,Q)=0 \;\text{ iff }\; P=Q \qquad \ldots(1.7)$$

Proposition 1.2 Let $f_1\,\&\,f_2$ are two convex functions and $g=a\,f_1+b\,f_2$ then $S_g(P,Q)=a\,S_{f_1}(P,Q)+b\,S_{f_2}(P,Q)$, where $a\,\&\,b$ are constants and $P,Q\in\Gamma_n$

There are some examples of divergence measures in the category of Csiszar's f-divergence divergence measure which can be obtained using new f-divergence measure like as Bhattacharya divergence [1], Triangular discrimination [5], Relative J-divergence [7], Hellinger discrimination [8], Chi-square divergence [12], Relative Jensen-Shannon divergence [13], Relative arithmetic-geometric divergence measure [14], Unified relative Jensen-Shannon and arithmetic-geometric divergence measure[14] which are following:

- If $f(t)=(t-1)^2$ then Chi-square divergence measure is given by

$$S_f(P,Q)=\frac{1}{4}\left[\sum_{i=1}^{n}\frac{p_i^2}{q_i}-1\right]=\frac{1}{4}\chi^2(P,Q) \qquad \text{… (1.8)}$$

- If $f(t)=-\log t$ then relative Jensen-Shannon divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n}q_i\log\left(\frac{2q_i}{p_i+q_i}\right)=F(Q,P) \qquad \text{… (1.9)}$$

- If $f(t)=t\log t$ then relative arithmetic-geometric divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right)=G(Q,P) \qquad \text{… (1.10)}$$

- If $f(t)=\frac{(t-1)^2}{t},\ \forall t>0$ then Triangular discrimination is given by

$$S_f(P,Q)=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{2(p_i+q_i)}=\frac{1}{2}\Delta(P,Q) \qquad \text{… (1.11)}$$

- If $f(t)=(t-1)\log t$ then Relative J-divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n}\left(\frac{p_i-q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right)=\frac{1}{2}J_R(P,Q) \qquad \text{… (1.12)}$$

- If $f(t)=(1-\sqrt{t})$ then Hellinger discrimination is given by

$$S_f(P,Q)=\left[1-B\left(\frac{P+Q}{2},Q\right)\right]=h\left(\frac{P+Q}{2},Q\right) \qquad \text{…(1.13)}$$

- If $f(t)=\begin{cases}[\alpha(\alpha-1)]^{-1}\left[t^{\alpha}-1\right], \alpha\neq 0,1 \\ -\log t & if\,\alpha=0 \\ t\log t & if\,\alpha=1\end{cases}$

Then Unified relative Jensen-Shannon and Arithmetic-Geometric divergence measure of type α is given by

$$S_f(P,Q)=\Omega_\alpha(Q,P)=\begin{cases} FG_\alpha(Q,P)=\left[\alpha(\alpha-1)\right]^{-1}\left[\sum_{i=1}^{n} q_i\left(\frac{p_i+q_i}{2q_i}\right)^{\alpha}-1\right], & \alpha\neq 0,1 \\ F(Q,P)=\sum_{i=1}^{n} q_i\log\left(\frac{2q_i}{p_i+q_i}\right), & \alpha=0 \\ G(Q,P)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right), & \alpha=1 \end{cases} \quad \ldots (1.14)$$

2. New Information Inequalities

The following theorem concerning inequalities among new f-divergence measure and Relative Jensen-Shannon divergence measure. The results are on similar lines to the result presented by Dragomir [6] and Jain & Saraswat [9].

Theorem 2 Let $f:(0,\infty)\to\mathbb{R}$ is normalized mapping i.e. $f(1)=0$ and satisfy the assumptions.

(i) f is twice differentiable on (r, R), where $0\le r\le 1\le R\le\infty$

(ii) there exist constants m, M such that

$$m\le t^2 f''(t)\le M \qquad \ldots(2.1)$$

If P, Q are discrete probability distributions satisfying the assumptions

$$r<\frac{1}{2}\le r_i=\frac{p_i+q_i}{2q_i}\le R,\ \forall i\in\{1,2,..........n\} \qquad \ldots(2.2)$$

Then we have the inequality

$$m\,F(Q,P)\le S_f(P,Q)\le M\,F(Q,P) \qquad \ldots(2.3)$$

Proof: - Define a mapping $F_m:(0,\infty)\to\mathbb{R}, F_m(t)=f(t)-\left[-m\log t\right]$. Then $F_m(.)$ is normalized, twice differentiable and since

$$F_m''(t)=f''(t)-\frac{m}{t^2}=\frac{1}{t^2}\left[t^2 f''(t)-m\right]\ge 0 \qquad \ldots(2.4)$$

For all $t\in(r,R)$, it follows that $F_m(.)$ is convex on (r,R). Applying non-negativity property of f-divergence functional for $F_m(.)$ and proposition 2.2, we may state that

$$0\le S_{F_m}(P,Q)=S_f(P,Q)-m\,S_{(-\log t)}(P,Q)=S_f(P,Q)-m\,F(Q,P)$$

$$\Rightarrow \qquad 0\le S_f(P,Q)-m\,F(Q,P) \qquad \ldots(2.5)$$

From where the first inequality of (3.3)

Now we again Define a mapping $F_M : (0,\infty) \to \mathbb{R}, F_M(t) = M\left[-\log t\right] - f(t)$, which is obviously normalized, twice differentiable and by (3.1), convex on (r,R). Applying non-negativity property of f-divergence measure for $F_M(.)$ and the linearity property, we obtain the second part of (3.3) i.e.

$$0 \le M\, F(Q,P) - S_f(P,Q) \qquad \ldots(2.6)$$

From results (2.5) and (2.6) give result (2.3)

Remark.1 If we have strict inequality “>” in (2.3) for any $t \in (r,R)$ then the mapping $F_m(.)$ and $F_M(.)$ are strictly convex and equality holds in (2.3) iff $P = Q$

Remark.2 It is important note that f is twice differentiable on $(0,\infty)$ and $m \le t^2 f''(t) \le M < \infty, \forall t \in (0,\infty)$, then inequality (2.1) holds for any probability distributions P and Q.

3. Some Particular Cases

In this section we established bounds of particular well known divergence measures in terms of Relative Jensen-Shannon divergence and Jensen-Shannon divergence measure using inequality of (2.3) of Theorem 2 which may be interested in Information Theory and Statistics.

The results are on similar lines to the result presented by Dragomir [6] and Jain & Saraswat [10].

Proposition 3.1:-Let $P,Q \in \Gamma_n$ be two probability distributions satisfying (3.2) then we have the following inequalities

$$r^2\; F(Q,P) \le \frac{1}{8}\chi^2(P,Q) \le R^2\; F(Q,P) \qquad \ldots(3.1)$$

$$r^2\; I(P,Q) \le \frac{1}{16}\Psi(P,Q) \le R^2\; I(P,Q) \qquad \ldots(3.2)$$

Proof:-Consider the mapping $f : (r,R) \to \mathbb{R}$.

$$f(t) = (t-1)^2,\; f'(t) = 2(t-1),\; f''(t) = 2 > 0,\; \forall t > 0$$

$f''(t) > 0$ and $f(1) = 0$, So function f is convex and normalized.

Define $\quad g(t) = t^2 f''(t) = t^2(2) = 2t^2$

Then obviously

$$M = \sup_{t\in[r,R]} g(t) = 2R^2,\ m = \inf_{t\in[r,R]} g(t) = 2r^2 \qquad \ldots(3.3)$$

Since $S_f(P,Q) = \frac{1}{4}\chi^2(P,Q)$ from (2.8)

From equation (2.8), (3.3) & (3.3) prove of the result (3.1)

Now Interchange $P \to Q$ we have

$$r^2\ F(P,Q) \le \frac{1}{8}\chi^2(Q,P) \le R^2\ F(P,Q)$$

(3.4) adding inequalities (3.1) & (3.4) prove of the result (3.2).

Proposition 3.2:-Let $P,Q \in \Gamma_n$ be two probability distributions satisfying (3.2) then we have the following inequalities

$$r\,F(P,Q) \le G(P,Q) \le R\,F(P,Q) \qquad \ldots(3.5)$$

$$r\,I(P,Q) \le T(P,Q) \le R\,I(P,Q) \qquad \ldots(3.6)$$

Proof:-Consider the mapping $f:(r,R)\to\mathbb{R}$.

$$f(t) = t\log t,\ f'(t) = 1+\log t,\ f''(t) = \frac{1}{t} > 0,\ \forall t > 0$$

$f''(t) \ge 0$ and $f(1) = 0$, So function f is convex and normalized.

Define $$g(t) = t^2 f''(t) = t^2\left(\frac{1}{t}\right) = t$$

Then obviously

$$M = \inf_{t\in[r,R]} g(t) = R,\ m = \sup_{t\in[r,R]} g(t) = r \qquad \ldots(3.7)$$

Also $S_f(P,Q) = G(Q,P)$ from (2.10)

From equation (2.10), (3.3) & (3.7)

$$r\,F(Q,P) \le G(Q,P) \le R\,F(Q,P) \qquad \ldots(3.8)$$

Interchange $P \to Q$ of (3.8) prove of the result (3.5)

Adding inequalities (3.5) & (3.8) prove of the result (3.6).

Proposition 3.3:-Let $P, Q \in \Gamma_n$ be two probability distributions satisfying (3.2) then we have the following inequalities

$$(1+r)\ F(Q,P) \le \frac{1}{2} J_R(P,Q) \le (1+R)\ F(Q,P) \qquad \ldots(3.9)$$

$$(1+r)\ I(P,Q) \le \frac{1}{4} J(P,Q) \le (1+R)\ I(P,Q) \qquad \ldots(3.10)$$

Proof:-Consider the mapping $f:(r,R) \to \mathbb{R}$.

$$f(t) = (t-1)\log t,\ f'(t) = \left(1-\frac{1}{t}\right) + \log t,\ f''(t) = \frac{(1+t)}{t^2} > 0,\ \forall t > 0$$

$f''(t) \ge 0$ and $f(1) = 0$, So function f is convex and normalized.

Define $g(t) = t^2 f''(t) = t^2\left(\frac{1+t}{t^2}\right) = (1+t)$

Then obviously

$$M = \sup_{t\in[r,R]} g(t) = (1+R),\ \ m = \inf_{t\in[r,R]} g(t) = (1+r) \qquad \ldots(3.11)$$

Since $S_f(P,Q) = \frac{1}{2} J_R(P,Q)$ from (2.12)

From equation (2.12), (3.3) & (3.11) give the result (3.9).

Now Interchange $P \to Q$ then we have

$$(1+r)\ F(P,Q) \le \frac{1}{2} J_R(Q,P) \le (1+R)\ F(P,Q) \qquad \ldots(3.12)$$

Adding inequalities (3.9) & (3.12) prove of the result (3.10).

Proposition 3.4:-Let $P, Q \in \Gamma_n$ be two probability distributions satisfying (3.2) then we have the following inequalities

$$\frac{1}{R} F(P,Q) \le \frac{1}{4}\Delta(P,Q) \le \frac{1}{r} F(P,Q) \qquad \ldots(3.13)$$

$$\frac{1}{R} I(P,Q) \le \frac{1}{4}\Delta(P,Q) \le \frac{1}{r} I(P,Q) \qquad \ldots(3.14)$$

Proof:-Consider the mapping $f:(r,R) \to \mathbb{R}$.

$$f(t)=\frac{(t-1)^2}{t}=\left(t+\frac{1}{t}-2\right),\ f'(t)=\left(1-\frac{1}{t^2}\right),\ f''(t)=\frac{2}{t^3}$$

$f''(t)\geq 0$ and $f(1)=0$, So function f is convex and normalized.

Define $g(t)=t^2 f''(t)=t^2\left(\frac{2}{t^3}\right)=\frac{2}{t}$

Then obviously

$$M=\sup_{t\in[r,R]} g(t)=\frac{2}{r},\ m=\inf_{t\in[r,R]} g(t)=\frac{2}{R} \qquad \ldots (3.15)$$

Since $S_f(P,Q)=\frac{1}{2}\Delta(P,Q)$ from (2.11)

From equation (2.11), (3.3) & (3.8)

$$\frac{1}{R}F(Q,P)\leq\frac{1}{4}\Delta(P,Q)\leq\frac{1}{r}F(Q,P) \qquad \ldots (3.16)$$

Now interchange $P\rightarrow Q$ of (3.16) then give the result (3.13).

Adding inequalities (3.13) & (3.16) prove of the result (3.13).

Proposition 3.5:-Let $P,Q\in\Gamma_n$ be two probability distributions satisfying (3.2) then we have the following inequality

$$r^{\alpha}F(P,Q)\leq\Omega_{\alpha}(P,Q)\leq R^{\alpha}F(P,Q) \qquad \ldots (3.17)$$

Proof:-Consider the mapping $f:(r,R)\rightarrow\mathbb{R}$.

$$f(t)=\left[\alpha(\alpha-1)\right]^{-1}\left[t^{\alpha}-1\right],\ \alpha\neq 0\,\&\,1,\ f'(t)=\left[\alpha-1\right]^{-1}t^{\alpha-1},\ f''(t)=t^{\alpha-2}>0,\ \forall t>0$$

$f''(t)\geq 0$ and $f(1)=0$, So function f is convex and normalized.

Define $g(t)=t^2 f''(t)=t^{\alpha}$

Then obviously

$$M=\sup_{t\in[r,R]} g(t)=R^{\alpha},\ m=\inf_{t\in[r,R]} g(t)=r^{\alpha} \qquad \ldots (3.18)$$

Since $S_f(P,Q)=\Omega_{\alpha}(Q,P)$ from (2.13)

From equation (2.13), (3.3) & (3.18)

$$r^{\alpha}F(Q,P)\leq\Omega_{\alpha}(Q,P)\leq R^{\alpha}F(Q,P) \qquad \ldots (3.19)$$

Interchange $P\rightarrow Q$ and proved of the result (3.17)

Corollary 3.5.1:-For $\alpha = \frac{1}{2}$ and Let $P, Q \in \Gamma_n$ be two probability distribution satisfying (3.2) then we have the following inequalities

$$\frac{\sqrt{r}}{4} F(P,Q) \le \left[1 - B\left(P, \frac{P+Q}{2}\right)\right] \le \frac{\sqrt{R}}{4} F(P,Q) \qquad \ldots(3.20)$$

and
$$\frac{\sqrt{r}}{4} F(P,Q) \le h\left(P, \frac{P+Q}{2}\right) \le \frac{\sqrt{R}}{4} F(P,Q) \qquad \ldots (3.21)$$

Proof: - Consider the mapping $f : (r, R) \to \mathbb{R}$. If $\alpha = \frac{1}{2}$ of equation (3.17)

$$f(t) = 4(1 - t^{\frac{1}{2}}), f'(t) = -2t^{-\frac{1}{2}}, f''(t) = t^{-\frac{3}{2}} > 0, \forall t > 0$$

$f''(t) \ge 0$ and $f(1) = 0$, So function f is convex and normalized.

Define $g(t) = t^2 f''(t) = \sqrt{t}$

Then obviously

$$M = \sup_{t \in [r,R]} g(t) = \sqrt{R},\ m = \inf_{t \in [r,R]} g(t) = \sqrt{r} \qquad \ldots (3.22)$$

Since $S_f(P,Q) = 4\left[1 - B\left(P, \frac{P+Q}{2}\right)\right] = 4h\left(P, \frac{P+Q}{2}\right)$ from (2.13) (3.23)

From equation (2.13), (3.3) & (3.22) & (3.23) give the results (3.20) & (3.21).

Corollary 3.5.2 - The results for $\alpha = 0 \& 1$ of result (3.17) are already proved in results (3.1) and (3.2).

4. Resistor-Average Distance

We use the Resistor-Average distance as a measure of dissimilarity between two probability densities it is defined as

$$D_{RAD}(P,Q) = \left[F(P,Q)^{-1} + F(P,Q)^{-1}\right]^{-1}$$

Relative Jensen-Shannon divergence measure from which is derived, it is non-negative and equal to zero iff p(x) ≡ q(x), but unlike it, it is symmetric. Another important property of the Resistor-Average distance is that when two classes of patterns C_p and C_q are distributed according to, respectively, p(x) and q(x), it is instructive to consider two special cases: when divergences in both directions between two pdfs are approximately equal and when one of them is much greater than the other:

$* D_F(P,Q) \approx D_F(P,Q) \approx D$

$$D_{RAD}(P,Q) \approx D$$

$* D_F(P,Q) \approx D_F(P,Q) \approx D$

$$D_F(P,Q) \approx D_F(Q,P) \ or \ D_F(P,Q) \approx D_F(Q,P)$$

$$D_{RAD}(P,Q) \approx \min D_F(P,Q) \ or \ D_F(Q,P)$$

References

[1] Bhattacharya A., “Some analogues to amount of information and their uses in statistical estimation”, Sankhya8 (1936) 1-13

[2] Csiszar I. “Information measure: A critical survey. Trans.7th prague conf. on info. Th. Statist. Decius. Funct, Random Processes” and 8th European meeting of statist Volume B. Acadmia Prague, 1978, PP-73-86

[3] Csiszar I., “Information-type measures of difference of probability functions and indirect observations” studia Sci.Math.hunger.2 (1961).299-318

[4] Csiszar I. and Korner,J. “Information Theory: Coding Theorem for Discrete Memory-Less Systems”, Academic Press, New York, 1981

[5] Dacunha-Castella D., “Ecole d’Ete de probabilities de Saint-Flour VII-1977 Berline, Heidelberg”, New-York:Springer 1978

[6] Dragomir S.S., “Some inequalities for (m,M)-convex mappings and applications for the Csiszar’s Φ-divergence in information theory”. Math. J. Ibaraki Univ. (Japan) 33 (2001), 35-50.

[7] Dragomir S.S., V. Gluscevic and C.E.M. Pearce, “Approximation for the Csiszar f –divergence via mid-point inequalities, in inequality theory and applications- Y. J. Cho, J. K. Kim and S.S. dragomir (Eds), nova science publishers, inc., Huntington, new York, vol1, 2001, pp.139-153.

[8] Hellinger E., “Neue Begrundung der Theorie der quadratischen Formen Von unendlichen vielen veranderlichen”, J.Rein.Aug.Math,136(1909),210-271

[9] Jain K. C. and R. N. Saraswat , “Some bounds of Csiszar’s f-divergence measure in terms of the well known divergence measures of Information Theory” International Journal of Mathematical Sciences and Engineering Applications, Vol. V No.5 September 2011.

[10] Jain K. C. and R. N. Saraswat “Some New Information Inequalities and its Applications in Information Theory” International Journal of Mathematics Research, Volume 3, Number 3 (2012), pp. 295-307.

[11] Jain K. C. and R. N. Saraswat “A New Information Inequality and its Application in Establishing Relation among Various f-Divergence Measures” Journal of Applied Mathematics, Statistics & Informatics (JAMSI), Volume 8, Number 1, 2012

[12] Pearson K., "On the criterion that a give system of deviations from the probable in the case of correlated system of variables in such that it can be reasonable supposed to have arisen from random sampling", Phil. Mag., 50(1900),157-172.

[13] Sibson R., Information Radius, Z, Wahrs. undverw.geb. (13) (1969), 139-160

[14] Taneja I.J., "New Developments in generalized information measures", Chapter in: Advances in imaging and Electron Physics, Ed. P. W. Hawkes 91 (1995),37-135

[15] Taneja I. J. and Pranesh Kumar, "Generalized non-symmetric divergence measures and inequalities" (2000) The Natural Science and Engineering Research Council's Discovery grant to Pranesh Kumar

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On Unified Finite Integrals Involving the Generalized Legendre's Associated Function, the Generalized Polynomials and the $\overline{H}$-function

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Abstract

In the present paper we first evaluate a basic finite integral involving the products of the generalized Legendre's associated function $P_\gamma^{\alpha,\beta}$ and the $\overline{H}$-function. Further we evaluate two more general integrals involving the products of $P_\gamma^{\alpha,\beta}$, the multivariable polynomials $S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}$ and the $\overline{H}$-function. All the evaluated integrals are believed to be new and reduce to a large number of simple integrals lying scattered in the literature. We mention two special cases of the second integral, which are also new and of interest by themselves. A known integral given by Anandani also follows as special case of the first main integral.

Keywords: generalized Legendre's associated function; generalized polynomials $S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}$; Laguerre polynomials ; $\overline{H}$ -function

Mathematics Subject Classification (2010): 33C45, 33C47, 33C60,

1. Introduction

In this paper, we shall define and represent the $\overline{H}$ -function in the following manner [2]

$$\overline{H}_{P,Q}^{M,N}[z] \equiv \overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \bar{\phi}(\xi)\, z^{\xi}\, d\xi \qquad \ldots(1.1)$$

where

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\ \prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\ \prod_{j=N+1}^{p}\Gamma(a_j-\alpha_j\xi)} \qquad \ldots(1.2)$$

The nature of contour L, the convergence conditions of the integral given by (1.1), its special cases and other details of the $\overline{H}$ -function can be referred in the paper by Gupta and Soni [6].

Also, the generalized polynomials $S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}[x_1 \cdots x_s]$ occurring here in will be defined and represented in the following form which differs slightly from that given by Srivastava [12,p.185,eqn.(7)]

$$S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}[x_1,\cdots,x_s] = \sum_{k_1=0}^{\left[\frac{n_1}{m_1}\right]}\cdots\sum_{k_s=0}^{\left[\frac{n_s}{m_s}\right]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_s)_{m_sk_s}}{k_s!}A[n_1,k_1;\cdots;n_s,k_s]x_1^{k_1},\cdots,x_s^{k_s} \qquad \ldots(1.3)$$

where $n_i = 0,1,2,\ldots;\ m_i \neq 0\ \ [i=1,\ldots,s],\ m_i$ is an arbitrary positive integer and the coefficients $A[n_1,k_1;\cdots;n_s,k_s]$ are arbitrary constants, real or complex.

If we take $s=1$ in the equation (1.3) the generalized polynomials $S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}[x_1 \cdots x_s]$ reduces to the well known general class of polynomials $S_n^m[x]$ introduced by Srivastava [13, p.1, eqn. (1)].

Finally, the generalized Legendre's associated function $P_{\gamma}^{\alpha,\beta}(x)$ [10,p.560,eqn.(3);5,p.81,eqn.(1.1)] occurring in this paper will be defined and represented as follows:

$$P_\gamma^{\alpha,\beta}(x) = \frac{(1+x)^{\beta/2}}{(1-x)^{\alpha/2}\ \Gamma(1-\alpha)}\ {}_2F_1\left[\gamma-\frac{\alpha-\beta}{2}+1, -\gamma-\frac{\alpha-\beta}{2}; 1-\alpha; \frac{1-x}{2}\right] \qquad \ldots(1.4)$$

where β and γ are unrestricted and α is not a positive integer . Further details about this function including its particular cases can be found in the papers of Kuipers et al. and Kuipers [8, 9].

2. Main Integrals

First integral

$$\int_o^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}\, P_\gamma^{\alpha,\beta}(x)\ \overline{H}\left[z\, x^u (1-x)^v\right] dx$$

$$= \sum_{t=0}^{\infty} \frac{(\gamma-\frac{\alpha-\beta}{2}+1)_t\,(-\gamma-\frac{\alpha-\beta}{2})_t}{2^t\, t!\ \Gamma(1-\alpha+t)}$$

$$\overline{H}_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}(1-\rho,u;1),(1-\sigma+\frac{\alpha}{2}-t,v;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\rho-\sigma+\frac{\alpha}{2}-t,u+v;1)\end{matrix}\right.\right] \qquad \ldots(2.1)$$

The integral (2.1) is valid under the following conditions :

(i) α is not a positive integer, $u \geq 0, v \geq 0$.

(ii) $\mathrm{Re}(\rho) + u \min\limits_{1\leq j\leq M}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right] > 0$,

$\mathrm{Re}(\sigma-\frac{\alpha}{2}) + v \min\limits_{1\leq j\leq M}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right] > 0$.

Second integral:

$$\int_o^1 x^{\rho-1}(1-x)^{\sigma-1}\ (1+x)^{-\beta/2}\ P_\gamma^{\alpha,\beta}(x)\ S_{n_1,\cdots,n_s}^{m_1,\cdots,m_s}\begin{bmatrix} e_1\, x^{u_1}\,(1-x)^{v_1} \\ \vdots \\ e_s\, x^{u_s}\,(1-x)^{v_s}\end{bmatrix} \overline{H}\left[z\, x^u (1-x)^v\right] dx$$

$$=\sum_{t=0}^{\infty}\sum_{k_1=0}^{\left[\frac{n_1}{m_1}\right]}\cdots\sum_{k_s=0}^{\left[\frac{n_s}{m_s}\right]}A[n_1,k_1;\cdots;n_s,k_s]\left(\prod_{j=1}^{s}\frac{(-n_j)_{m_jk_j}\,e_j^{k_j}}{k_j!}\right)\frac{(\gamma-\frac{\alpha-\beta}{2}+1)_t\,(-\gamma-\frac{\alpha-\beta}{2})_t}{2^t\,t!\;\Gamma(1-\alpha+t)}$$

$$\overline{H}_{P+2,Q+1}^{M,N+2}\left[z\left|\begin{matrix}(1-\rho-u_1k_1-\cdots-u_sk_s,u;1),(1-\sigma+\frac{\alpha}{2}-t-v_1k_1-\cdots-v_sk_s,v;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\rho-\sigma+\frac{\alpha}{2}-t-(u_1+v_1)k_1-\cdots-(u_s+v_s)k_s,\,u+v;1)\end{matrix}\right.\right] \qquad \ldots(2.2)$$

The integral (2.2) is valid under the following conditions :

(i) α is not a positive integer, $u\geq 0, v\geq 0$; $u_j\geq 0, v_j\geq 0$, j=1,…,s.

(ii) $\mathrm{Re}(\rho)+u\min\limits_{1\leq j\leq M}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right]>0$,

$\mathrm{Re}(\sigma-\frac{\alpha}{2})+v\min\limits_{1\leq j\leq M}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right]>0.$

Third integral:

$$\int_{-1}^{+1}(1-x)^{\rho}\,(1+x)^{\sigma}\,P_{\gamma}^{\alpha,\beta}(x)\,S_{n_1,\cdots,n_s}^{m_1,\cdots,m_s}\begin{bmatrix}e_1\,(1-x)^{u_1}\,(1+x)^{v_1}\\ \vdots\\ e_s\,(1-x)^{u_s}\,(1+x)^{v_s}\end{bmatrix}\overline{H}\left[z\,(1-x)^{u}\,(1+x)^{v}\right]dx$$

$$=\frac{2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}}{\Gamma(1-\alpha)}\sum_{t=0}^{\infty}\sum_{k_1=0}^{\left[\frac{n_1}{m_1}\right]}\cdots\sum_{k_s=0}^{\left[\frac{n_s}{m_s}\right]}2^{\sum_{j=1}^{s}(u_j+v_j)k_j}\;A[n_1,k_1;\cdots;n_s,k_s]$$

$$\left(\prod_{j=1}^{s}\frac{(-n_j)_{m_jk_j}\,e_j^{k_j}}{k_j!}\right)\frac{(\gamma-\frac{\alpha-\beta}{2}+1)_t\,(-\gamma-\frac{\alpha-\beta}{2})_t}{(1-\alpha)_t\;t!}$$

$$\overline{H}_{P+2,Q+1}^{M,N+2}\left[z2^{u+v}\left|\begin{matrix}(\frac{\alpha}{2}-\rho-t-u_1k_1-\cdots-u_sk_s,u;1),(-\sigma-\frac{\beta}{2}-v_1k_1-\cdots-v_sk_s,v;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\rho-\sigma-\frac{\beta-\alpha}{2}-t-1-(u_1+v_1)k_1-\cdots-(u_s+v_s)k_s,\,u+v;1)\end{matrix}\right.\right] \qquad \ldots(2.3)$$

The integral (2.3) is valid under the following conditions :

(i) α is not a positive integer, $u \geq 0, v \geq 0$; $u_j \geq 0, v_j \geq 0$, j=1,…,s.

(ii) $\mathrm{Re}(1+\rho-\frac{\alpha}{2})+u \min\limits_{1\leq j\leq M}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right]>0$,

$\mathrm{Re}(1+\sigma+\frac{\beta}{2})+v \min\limits_{1\leq j\leq M}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right]>0.$

Proofs:

To establish the integral (2.1), we first express the generalized Legendre's associated function occurring in its left hand side in terms of ${}_2F_1$ with the help of (1.4) and the $\overline{H}$-function in terms of Mellin-Barnes contour integral by (1.1), Now we interchange the order of x and ξ integrals (which is permissible under the conditions stated with (2.1)) in the result thus obtained and get after a little simplification the left hand side of (2.1) (say Δ) as

$$\Delta=\frac{1}{\Gamma(1-\alpha)}\frac{1}{2\pi i}\int_L \overline{\phi}(\xi)\, z^{\xi}\{\int_o^1 x^{\rho+u\xi-1}(1-x)^{\sigma-\frac{\alpha}{2}+v\xi-1}$$

$${}_2F_1\left[\gamma-\frac{\alpha-\beta}{2}+1,-\gamma-\frac{\alpha-\beta}{2};1-\alpha;\frac{1-x}{2}\right]dx\}d\xi \qquad \ldots(2.4)$$

on evaluating the x-integral occurring on the right hand side of (2.4) with the help of a known result [11,p. 60,eqn.(2.16(ii))] and expressing the function ${}_3F_2$ so obtained in terms of series and interchanging the order of summations and integrations (which is permissible under the conditions stated with(2.1)),the equation (2.4) takes the following form after a little simplification

$$\Delta=\sum_{t=0}^{\infty}\frac{(\gamma-\frac{\alpha-\beta}{2}+1)_t(-\gamma-\frac{\alpha-\beta}{2})_t}{2^t\, t!\ \Gamma(1-\alpha+t)}\{\frac{1}{2\pi i}\int_L \overline{\phi}(\xi)\ z^{\xi}\ \frac{\Gamma(\rho+u\xi)\Gamma(\sigma-\frac{\alpha}{2}+t+v\xi)}{\Gamma(\rho+\sigma-\frac{\alpha}{2}+t+(u+v)\xi)}\}d\xi \qquad \ldots(2.5)$$

Finally, on reinterpreting the multiple Mellin-Barnes contour integral occurring in the right hand side of (2.5) in terms of the $\overline{H}$-function, we easily arrive at the desired result (2.1).

To prove (2.2), we first express the generalized polynomials $S_{n_1,\dots,n_s}^{m_1,\dots,m_s}[x_1 \cdots x_s]$ occurring in the left hand side of (2.2) in series form with the help of (1.3) and then interchange the order of summations and integration (which is permissible under the conditions stated with (2.2)).Now on evaluating the integral so obtained with the help of the integral (2.1), we easily obtain the desired result (2.2).

To evaluate the integral (2.3), we make use of the following integral

$$\int_{-1}^{+1} (1-x)^{\rho} (1+x)^{\sigma} P_{\gamma}^{\alpha,\beta}(x)\, dx = \frac{2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}\, \Gamma(1+\rho-\frac{\alpha}{2})\, \Gamma(1+\sigma+\frac{\beta}{2})}{\Gamma(1-\alpha)\, \Gamma(2+\rho+\sigma+\frac{\beta-\alpha}{2})}$$

$$ {}_3F_2\left[\begin{matrix}\gamma-\frac{\alpha-\beta}{2}+1, -\gamma-\frac{\alpha-\beta}{2}, 1+\rho-\frac{\alpha}{2}; 1\\ 1-\alpha,\, 2+\rho+\sigma+\frac{\beta-\alpha}{2};\end{matrix}\right] \qquad \ldots(2.6)$$

where α is not a positive integer, $\mathrm{Re}(1+\rho-\frac{\alpha}{2})>0$, $\mathrm{Re}(1+\sigma+\frac{\beta}{2})>0$ and proceed in a manner similar to that given earlier in proofs of (2.1)and (2.2).

3. Special Cases

(i) If we reduce $S_{n_1,\dots,n_s}^{m_1,\dots,m_s}$ occurring in(2.2) to Laguerre polynomials $L_{n_1}^{\theta}$ [4, p.999, eqn. (8.704); 15, p.159, eqn.(1.8)]. We arrive at the following integral after a little simplification

$$\int_{o}^{1} x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\alpha/2} P_{\gamma}^{\alpha}(x)\, L_{n_1}^{\theta}\left[e_1 x^{u_1} (1-x)^{v_1}\right] \overline{H}\left[z\, x^{u}(1-x)^{v}\right] dx$$

$$= \sum_{t=0}^{\infty} \sum_{k_1=0}^{[n_1]} \binom{n_1+\theta}{n_1} \frac{e_1^{k_1} (-n_1)_{k_1} (\gamma+1)_t (-\gamma)_t}{(\theta+1)_{k_1}\, k_1!\; 2^t\, t!\, \Gamma(1-\alpha+t)}$$

$$\overline{H}_{P+2,Q+1}^{M,N+2}\left[z \left| \begin{matrix} (1-\rho-u_1k_1, u; 1), (1-\sigma+\frac{\alpha}{2}-t-v_1k_1, v; 1), (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q}, (1-\rho-\sigma+\frac{\alpha}{2}-t-(u_1+v_1)k_1,\, u+v; 1) \end{matrix} \right. \right] \qquad \ldots(3.1)$$

The conditions of existence of (3.1) can be easily obtained with the help of the conditions stated with (2.2).

(ii) Now we give an interesting special case of (2.2) involving **g** function connected with a certain class of Feymman integrals [6, p. 98, eq.(1.3)]

$$\int_o^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2} P_\gamma^{\alpha,\beta}(x)\, S_{n_1,\cdots,n_s}^{m_1,\cdots,m_s}\begin{bmatrix} e_1 x^{u_1}(1-x)^{v_1} \\ \vdots \\ e_s x^{u_s}(1-x)^{v_s}\end{bmatrix} g[\zeta,\eta,\tau,p; z\,x^u(1-x)^v]\,dx$$

$$= \frac{K_{d-1}\Gamma(p+1)\Gamma(\frac{1}{2}+\frac{\tau}{2})}{(-1)^p 2^{2+p}\pi^{\frac{1}{2}}\Gamma(\zeta)\Gamma(\zeta-\frac{\tau}{2})}\sum_{t=0}^{\infty}\sum_{k_1=0}^{\left[\frac{n_1}{m_1}\right]}\cdots\sum_{k_s=0}^{\left[\frac{n_s}{m_s}\right]} A[n_1,k_1;\cdots;n_s,k_s]$$

$$\left(\prod_{j=1}^{s}\frac{(-n_j)_{m_j k_j} e_j^{k_j}}{k_j!}\right)\frac{(\gamma-\frac{\alpha-\beta}{2}+1)_t(-\gamma-\frac{\alpha-\beta}{2})_t}{2^t\, t!\,\Gamma(1-\alpha+t)}$$

$$\overline{H}_{5,4}^{1,5}\left[-z\left|\begin{matrix}(1-\rho-u_1k_1-\cdots-u_sk_s,u;1),(1-\sigma+\frac{\alpha}{2}-t-v_1k_1-\cdots-v_sk_s,v;1),(1-\zeta,1;1),(1-\zeta+\frac{\tau}{2},1;1),(1-\eta,1;1+p)\\(0,1),(\frac{\tau}{2},1;1),(-\eta,1;1+p),(1-\rho-\sigma+\frac{\alpha}{2}-t-(u_1+v_1)k_1-\cdots-(u_s+v_s)k_s,\,u+v;1)\end{matrix}\right.\right]$$

(3.2)

where $K_d \equiv 2^{1-d}\pi^{d/2}/\Gamma(d/2)$ [7,p.4121,,eq.(1.5)] and conditions easily obtainable from (2.2) are satisfied.

(iii) If we take A_j (j=1,…,N)= B_j (j=M+1,…Q)=1in(2.1),the $\overline{H}$ -function occurring there in reduces to the Fox H-function[3,14] and we get an integral given by Anandani [1 ,p.343,eqn.(2.2)]

References

[1] P.Anandani, Proc. Nat. Acad. Sci. India, Sect. A.,**39(III)** (1969),341-348.

[2] R.G.Buschman and H.M.Srivastava,J.Phys. A; Math.Gen.,**23**(1990),4707-4710.

[3] C.Fox, Trans. Amer. Math. Soc.,**98**(1961),395-429.

[4] I.S. Gradshteyn and I.M.Ryzhik, Table of integrals, Series and Products, Academic Press, Inc. New York (1980).

[5] K.C.Gupta and Pawan Agrawal, Ganita Sandesh Vol.**7**,No.**2**, 81-87(1993).

[6] K.C.Gupta and R.C.Soni, Kyungpook Math. J.,**41(1)**(2001),97-104.

[7] A.A.Inayat-Hussain,J.Phys.A:Math.Gen.,**20**(1987),4119-4128.

[8] L.Kuipers and B.Meulenbeld, Nederl. Akad. Van. Watensch Amsterdum Proc. Ser. A.,**60(4)** (1957),436-450.

[9] L.Kuipers, Nederl. Akad. Van. Watensch Amsterdum Proc. Ser. A.,**62(2)** (1959),148-152.

[10] B.Meulenbeld, Nederl. Akad. Van. Watensch Amsterdum Proc. Ser. A.,**61(5)** (1958),557-563.

[11] I.N.Sneddon, Special Functions of Mathematical Physics and Chemistry, Oliver and Boyd, NewYork (1961)

[12] H.M.Srivastava, Pacific J.Math.,**117**(1985),183-191.

[13] H.M.Srivastava, Indian J.Math.,**14**(1972),1-6.

[14] H.M.Srivastava, K.C.Gupta and S.P.Goyal,The H-functions of One and Two Variables with Applications, South Asian Publishers, NewDelhi, Madras,(1982).

[15] H.M.Srivastava and N.P.Singh,Rend.Circ.Mat.Palermo Ser.2,**32**(1983),157-187.

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An EOQ Model With Stock and Selling Price Dependent Demand and Weibull Distribution Deterioration Under Partial Backlogging and Inflation

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Abstract

The purpose of this study is to reflect the real life situation effect of inflation in EOQ models. It is assumed that the rate of deterioration is time dependent and two parameter Weibull function. The demand is stock and selling price dependent. Holding cost is constant what this paper presents. The demand is partially backlogged and proposed model considered allows shortages. We solved this model by maximizing the total inventory profit. The result is illustrated with the help of numerical examples. The effect of changes in various parameters used in model on the optimum solution is shown by using sensitivity analysis. We can use the model in optimizing the total inventory profit for business enterprises.

Keywords: EOQ, Weibull deterioration, Stock depended demand, holding cost inflation.

1. Introduction

We can see in the markets that demand for a certain product can increase or decrease according to its availability. More is the product, huge is the demand. As it attracts the customers to buy it. Also if that product is less available then the customers think that product is not so popular it has become old. For many years, researchers and practitioners have come to know that the demand for certain products can be depending upon its inventory level on display. **Gupta and Vrat (1986)** were the first who developed models for stock dependent consumption rate. **Baker and Urban (1988)** established an economic order quantity model for a power form inventory-level-dependent demand pattern. **Mandal and Phaujdar (1989)** introduced an economic production quantity model for deteriorating items with constant production rate linearly stock-dependent demand. Researchers like **Pal et al. (1993), Giri et.al (1996), Ray et al. (1998), Uthaya Kumar and Parvathi (2006), Roy and Choudhuri (2008), Choudhury et al. (2013)** and many others worked on it. **Soni and Shah (2008)** introduced the optimal ordering policy for an inventory model with stock dependent demand. **Wu et. Al (2006)** was the first who developed an inventory model for non-instantaneous deteriorating items with stock-dependent demand. **Chang et al. (2010)** established an optimal replenishment policy for non-instantaneous deteriorating items with stock- dependent demand.**Sana (2010)** established an EOQ model for perishable items with stock-dependent demand;**Gupta et al. (2013)** introduced optimal ordering policy for stock-dependent demand inventory model with non-instantaneous deteriorating items.**Vipin Kumar, S. R. Singh, &Dhir Singh (2011)** was developed an Inventory Model For Deteriorating Items With Permissible Delay In Payment Under Two-Stage Interest Payable Criterion And Quadratic Demand"**Mishra and Tripathy (2012)** gave an idea on an inventory model for time dependent Weibull deterioration with partial backlogging,**Vipin Kumar, GopalPathak, C.B.Gupta (2013)** derived a Deterministic Inventory Model for Deteriorating Items with Selling Price Dependent Demand and Parabolic Time Varying Holding Cost under Trade Credit"**Palanivel and Uthayakumar (2014)** established model for non-instantaneousdeterioratingproducts with time dependent two variable Weibull deterioration rate, where demand rateis power function of time and permitting partial backlogging.**Vipin Kumar, Anupama Sharma, C.B.Gupta (2014)**established an EOQ Model For Time Dependent Demand and Parabolic Holding Cost With Preservation Technology Under Partial Backlogging For Deteriorating Items. **Farughi et al. (2014)** modeled pricing and inventory control policy for non-instantaneous deteriorating items with price and time dependent demand permitting shortages with partial backlogging. **Vipin Kumar, Anupma Sharma , C.B.Gupta (2015)** worked on two-Warehouse Partial Backlogging Inventory Model For Deteriorating Items With Ramp Type Demand" .**While, Zhang et al. (2015)** developedpricing model for non-instantaneous deteriorating item by considering constant deterioration rate andstock sensitive demand. Further,**Vipin Kumar, Anupama Sharma, C.B.Gupta (2015)** "A Deterministic Inventory Model For Weibull Deteriorating Items with Selling Price Dependent Demand And Parabolic Time Varying Holding Cost**GopalPathak, Vipin Kumar, C.B.Gupta (2017)** A Cost Minimization Inventory Model for Deteriorating Products and Partial Backlogging under Inflationary Environment . **AditiKhanna, Aakanksha Kishore and Chandra K. Jaggi (2017)** Strategic production modeling for defective items with imperfect inspection process,rework, and sales return under two-level trade credit **GopalPathak, Vipin Kumar, C.B.Gupta (2017)**developed An Inventory Model for Deterioration Items with Imperfect Production and Price Sensitive Demand under Partial Backlogging,

Mashud et al. (2018) worked on non-instantaneous deteriorating item having different demand rates allowing partial backlogging.

In this paper, we have developed an inventory model for deteriorating items with partial backlogging under stock and price dependent demand. The presented model is developed with the effect of inflation. The concavity is also shown through the figure made by Mathematica-11 software. Numerical example is taken to test the validity, analytically and graphically. In last section sensitivity analysis is mentioned.

The rest paper is organized as follows section second in two subparts, the first part explain the assumptions and second part show the notations used throughout the study. In section II the analytical calculation of model is shown along with different costs and sales revenue. Next section provides the numerical illustration of the problem with a fig. to show the convexity. In section IV, we discussed the sensitivity and observations. In last section gives conclusions of the problem.

2. Assumption and Notations

This inventory model is developed on the basis of the following assumption and notations

2.1 Assumptions

i. Deterioration rate which follows a two parameter Weibull distribution, $\theta(t) = \alpha\beta t^{\beta-1}$, where $0 \leq \alpha \ll 1$ is the scale parameter, $\beta > 0$ is the shape parameter and $0 \leq \theta(t) << 1$.

ii. Demand rate is function of selling price and stock considered as $D\left(I(t), s\right) = \begin{cases} a + bI(t) - s, & 0 \leq t \leq v \\ a - s, & v \leq t \leq T \end{cases}$ where a, b are demand parameters and s is selling price. also $a > 0,\ 0 < b << 1$ and $a > s$

iii. Holding cost is constant as $h(t) = h$ where $h > 0$

iv. Replenishment rate is instantaneous

v. Lead time is zero

vi. The planning horizon is finite

vii. During the stock out period, the unsatisfied demand is backlogged; the rate of backlogging is variable and is dependent on the length of the waiting time for the next replenishment. For the negative inventory the backlogging rate is $B(T-t) = \dfrac{1}{1 + \delta(T-t)}$

2.2 Notations

i. A : Ordering cost.

ii. p :Purchasing cost

iii. s : Unit selling price

iv. C : Shortage cost per unit per unit time.

v. l : Lost sale cost per unit.

vi. Q_1 : Maximum inventory level during (0, T).

vii. Q_2 : Maximum inventory level during the shortage period.

viii. Q the retailer's order quantity

ix. v : the time at which the inventory level falls to zero (decision variable)

x. T : inventory cycle length (decision variable)

xi. $I_1(t)$: Inventory level at any time during $(0, v)$.

xii. $I_2(t)$: Inventory level at any time during (v , T).

xiii. $TC(v,T)$ the retailer total cost optimal value

3 Mathematical Formulation and solution

In this section the behavior of the inventory system as shown in fig. 1 as the inventory level decreases due to demand and deterioration in the interval $[0, v]$ and at the time v inventory level reaches zero and the shortages starts during the interval $[v, t]$ which is under the partial backlogging effect.

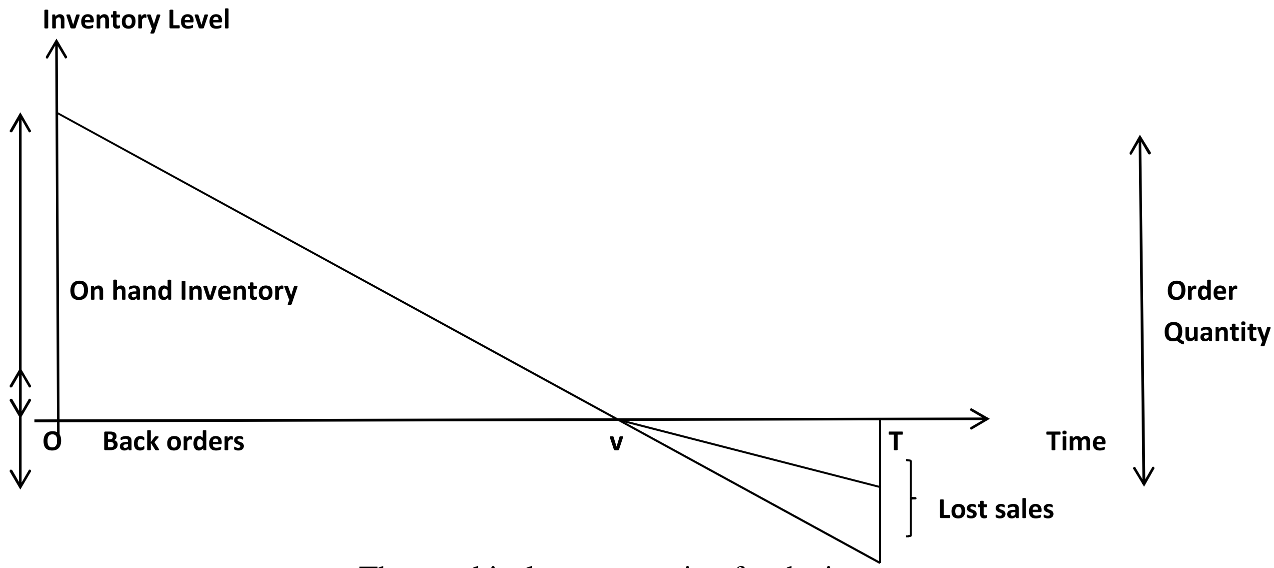


Fig. 1 The graphical representation for the inventory system

The instantaneous state of the system is given by

$$\frac{dI_1(t)}{dt} = -\theta(t)I_1(t) - D\big(I(t),s\big) \qquad 0 \le t \le v \qquad \text{….(1)}$$

$$\frac{dI_1(t)}{dt} = -B(T-t)D\big(I(t),s\big) \qquad v \le t \le T \qquad \text{….(2)}$$

With boundary conditions $I_1(v) = 0 = I_2(v)$

The solution of above differential equation are

$$I_1(t) = (a-s)\left[(v-t) + \frac{b}{2}(v^2 - t^2) + \frac{\alpha}{\beta+1}\left(v^{\beta+1} - t^{\beta+1}\right) - b\left(tv - t^2\right) - \alpha\left(vt^{\beta} - t^{\beta+1}\right)\right] 0 \le t \le v \qquad \text{….(3)}$$

$$I_2(t) = \frac{(a-s)}{\delta}\Big[\log\big(1+\delta(T-t)\big) - \log\big(1+\delta(T-v)\big)\Big] \; v \le t \le T \qquad \text{….(4)}$$

The maximum positive inventory is

$$Q_1 = I_1(0) = (a-s)\left[v + \frac{b}{2}v^2 + \frac{\alpha}{\beta+1}v^{\beta+1}\right] \qquad \text{….(5)}$$

The maximum backordered units are

$$Q_2 = -I_2(T) = \frac{(a-s)}{\delta}\log\left[1+\delta(T-v)\right] \qquad \text{....(6)}$$

Hence, order size during the time interval $[0,T]$

$$Q = Q_1 + Q_2 = (a-s)\left\{v + \frac{b}{2}v^2 + \frac{\alpha}{\beta+1}v^{\beta+1} + \frac{1}{\delta}\log\left[1+\delta(T-v)\right]\right\} \qquad \text{....(7)}$$

The total per cycle consists of the following components.

Ordering cost: Ordering cost per cycle is $OC = A$ ....(8)

Holding cost: Holding cost during the interval $[0,v]$

$$HC = h\int_0^v I_1(t)e^{-Rt}dt = h(a-s)\left[\frac{v^2}{2} + \frac{bv^3}{6} - \frac{Rv^3}{6} + \frac{v^{2+\beta}\alpha\beta}{(1+\beta)(2+\beta)}\right] \qquad \text{....(9)}$$

Shortage cost : The shortage cost during the interval $[v,T]$

$$SC = c_2\int_v^T -I_2(t)e^{-Rt}dt$$

$$= -c_2(a-s)\left[T - v - \frac{3RT^2}{4} + \frac{RTv}{2} + \frac{Rv^2}{4} + \frac{R(v-T)}{2\delta} + \left(\frac{R}{2\delta^2} + \frac{RT-1}{\delta}\right)\log\left[1+(T-v)\delta\right]\right] \text{....(10)}$$

Lost sales cost: The lost sale cost during the interval $[t_1,T]$

$$LSC = c_3\int_v^T \left(1 - B(T-t)\right)D(t)e^{-Rt}dt$$

$$= l(a-s)\left[T - v + \frac{R(v^2-T^2)}{2} + \frac{R(v-T)}{\delta} + \left[\frac{R}{\delta^2} + \frac{RT-1}{\delta}\right]\log\left[1+(T-v)\delta\right]\right] \qquad \text{....(11)}$$

Purchase cost: The purchasing costper cycle is

$$PC = pQ = p(a-s)\left[v + \frac{bv^2}{2} + \frac{\alpha v^{\beta+1}}{\beta+1} + \frac{\log\left(1+\delta(T-v)\right)}{\delta}\right]e^{-RT} \qquad \text{....(12)}$$

Sales revenue cost: The sales revenue is given by $SR = s\left[\int_0^v D(t)e^{-Rt}dt + \int_v^T D(t)e^{-Rt}dt\right]$

$$= s(a-s)\left(\begin{array}{l} T - v - \frac{1}{2}\left(T^2 - v^2\right)R \\ +\left(v + \frac{1}{24}bv^2\left(12 - 4Rv + bv(4-Rv) - \frac{12v^{\beta}\alpha\beta\left(Rv(1+\beta) - 2(3+\beta)\right)}{(1+\beta)(2+\beta)(3+\beta)}\right)\right)\end{array}\right) \quad ….(13)$$

Therefore the total profit per unit items is given by

$$P(s,v) = \frac{1}{T}\{SR - OC - HC - BC - LSC - PC\}$$

$$P(s,v) = \frac{1}{T}s(a-s)\left(\begin{array}{l} T - v - \frac{1}{2}\left(T^2 - v^2\right)R \\ +\left(v + \frac{1}{24}bv^2\left(12 - 4Rv + bv(4-Rv) - \frac{12v^{\beta}\alpha\beta\left(Rv(1+\beta) - 2(3+\beta)\right)}{(1+\beta)(2+\beta)(3+\beta)}\right)\right)\end{array}\right)$$

$$-\frac{1}{T}A - \frac{1}{T}h(a-s)\left[\frac{v^2}{2} + \frac{bv^3}{6} - \frac{Rv^3}{6} + \frac{v^{2+\beta}\alpha\beta}{(1+\beta)(2+\beta)}\right]$$

$$+\frac{1}{T}c_2(a-s)\left[T - v - \frac{3RT^2}{4} + \frac{RTv}{2} + \frac{Rv^2}{4} + \frac{R(v-T)}{2\delta} + \left(\frac{R}{2\delta^2} + \frac{RT-1}{\delta}\right)\log\left[1 + T\delta - v\delta\right]\right]$$

$$-\frac{1}{T}l(a-s)\left[T - v + \frac{R\left(v^2 - T^2\right)}{2} + \frac{R(v-T)}{\delta} + \left[\frac{R}{\delta^2} + \frac{RT-1}{\delta}\right]\log\left[1 + T\delta - v\delta\right]\right]$$

$$-\frac{1}{T}p(a-s)\left[v + \frac{bv^2}{2} + \frac{\alpha v^{\beta+1}}{\beta+1} + \frac{\log\left(1 + \delta(T-v)\right)}{\delta}\right]e^{-RT} \quad ….(14)$$

Our main objective is to maximize the Total profit function $P(s,v)$ the necessary condition for maximize the total inventory profit are $\frac{\partial P(s,v)}{\partial s} = 0$ and $\frac{\partial P(s,v)}{\partial v} = 0$ ….(15)

Using the software Mathematica 11, we can calculate the optimal value of s* and v* by equation (15).And the optimal value of the total Inventory cost is determined by equation (14). Theoptimal value of s*and v*, satisfy the sufficient conditions for maximizing the total inventory profit function $\frac{\partial^2 P(s,v)}{\partial s^2} < 0$, $\frac{\partial^2 P(s,v)}{\partial v^2} < 0$ and $\left(\frac{\partial^2 P(s,v)}{\partial s^2}\right)\left(\frac{\partial^2 P(s,v)}{\partial v^2}\right) - \left(\frac{\partial^2 P(s,v)}{\partial s \partial v}\right) > 0$

In addition, at $s = s^*$ and $v = v^*$ optimal value is $P(s,v) = P^*(s^*, v^*)$

4. Numerical Example

Considerthe followingnumerical values of parameters to illustrate the profit function $\alpha = 0.3$, $\beta = 6$, $h = 25$, $A = 500$, $a = 22$, $b = 30$, $\delta = 0.04$, $p = 10$, $c = 40$, $l = 50$, $R = 0.06$, $T = 3$Use Mathematica-11to obtain the optimal solution for v and s Based on the above numerical values of used parameters the optimal solution is $s^* = 19.519$ $v^* = 1.521$ $P(s^*, v^*) = 35493.68$

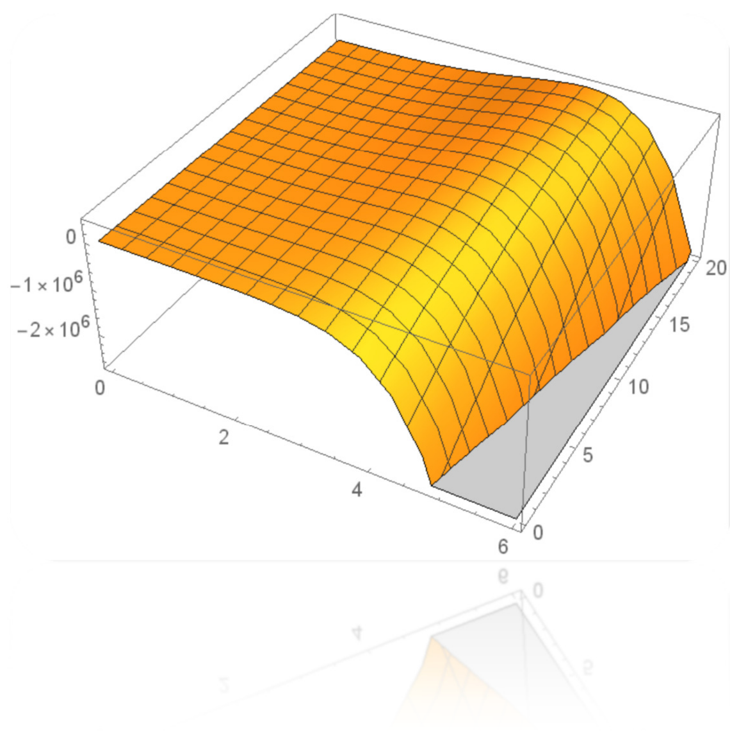


Fig.2 $P(s,v)$ **v/s** s **and** v

From fig. 2, observed that the total cost function is a strictly concave function. Thus, theoptimum values of s and v can be obtained with the help of the average net profit function of the model provided that the total profit per unit time of the inventory system is maximum.

5. Sensitivity Analysis

In this section, the effects of studying the changes in the optimal value of total profit cost per unit time and the optimal value of order quantity per cycle with respect to changes in parameters are discussed. Based on example, the sensitivity analysis is performed bychanging the value of each of the parameters by $\pm 25\%$ and $\pm 50\%$, taking one parameter at a time and keeping the remaining parameters unchanged.

Table 1

Parameter	**%**	**Value**	**Total Profit**
Alfa	50%	0.45	7658.08
	25%	0.375	7664.07
	0	0.3	7670.05
	-25%	0.225	7676.03
	-50%	0.15	7682.02

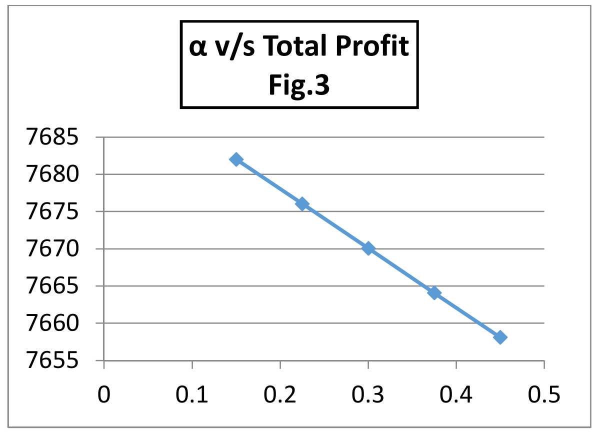


Table 2

Parameter	**%**	**Value**	**Total Profit**
Beta	50%	9	7631.05
	25%	7.5	7655.56
	0	6	7670.05
	-25%	4.5	7683.87
	-50%	3	7683.87

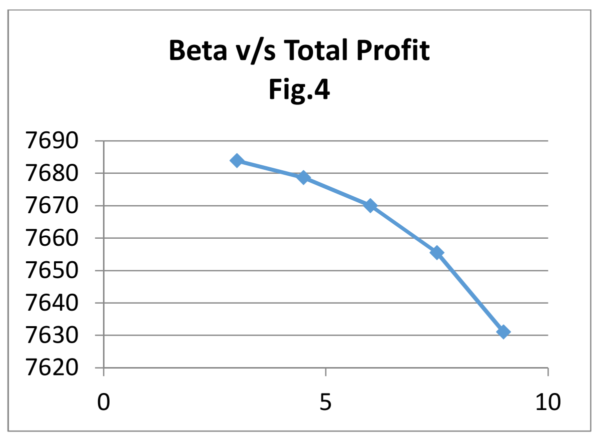
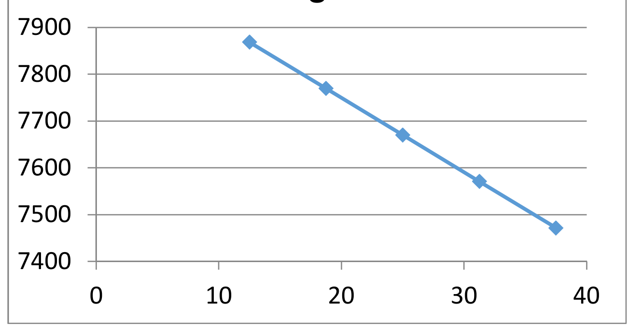


Table 3

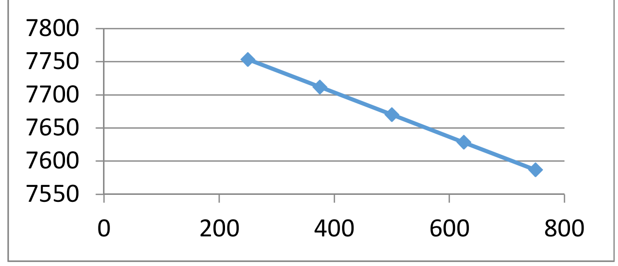
Parameter	%	Value	Total Profit
	50%	37.5	7471.55
H	25%	31.25	7570.8
	0	25	7670.05
	-25%	18.75	7769.3
	-50%	12.5	7868.55



h v/s Total Profit
Fig.5

Table 4

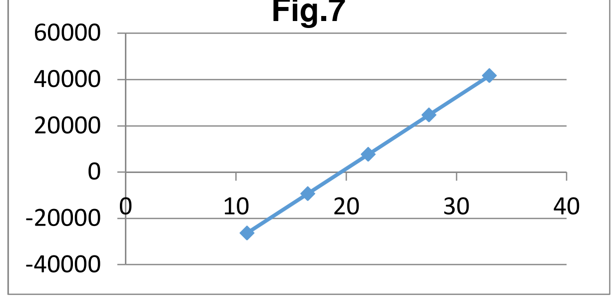
Parameter	%	Value	Total Profit
	50%	750	
	25%	625	
A	0	500	7670.05
	-25%	375	
	-50%	250	



A v/s Total Profit
Fig.6

Table 5

Parameter	%	Value	Total Profit
	50%	33	41676.7
	25%	27.5	24673.5
A	0	22	7670.05
	-25%	16.5	9333.29
	-50%	11	26336.6



a v/s Total Profit
Fig.7

Table 6			
Parameter	**%**	**Value**	**Total Profit**
B	50%	45	17779.5
	25%	37.5	12204.4
	0	30	7670.05
	-25%	22.5	4176.36
	-50%	15	1725.38

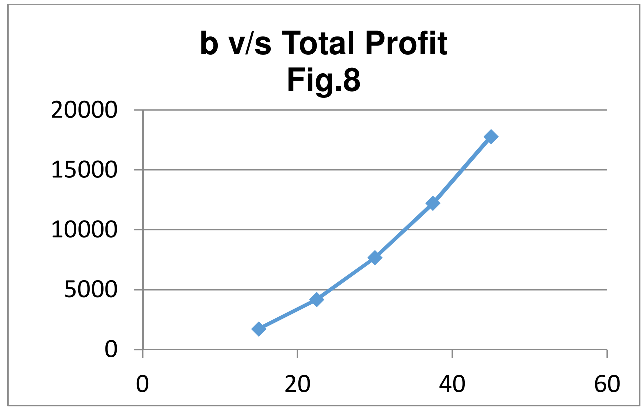


Table 7			
Parameter	**%**	**Value**	**Total Profit**
Delta	50%	0.6	7670.3
	25%	0.5	7670.18
	0	0.4	7670.05
	-25%	0.3	7669.92
	-50%	0.2	7669.79

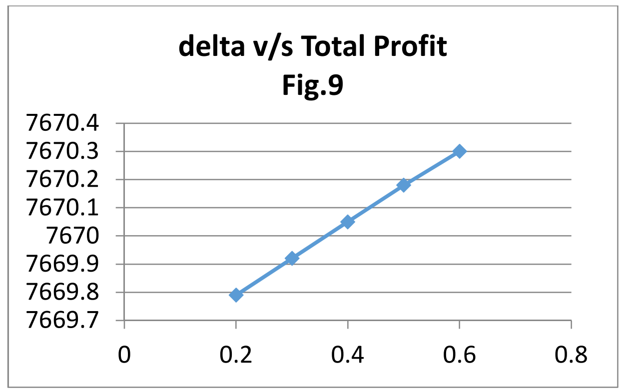


Table 8			
Parameter	**%**	**Value**	**Total Profit**
P	50%	15	7537.19
	25%	12.5	7603.62
	0	10	7670.05
	-25%	7.5	7736.5
	-50%	5	7802.91

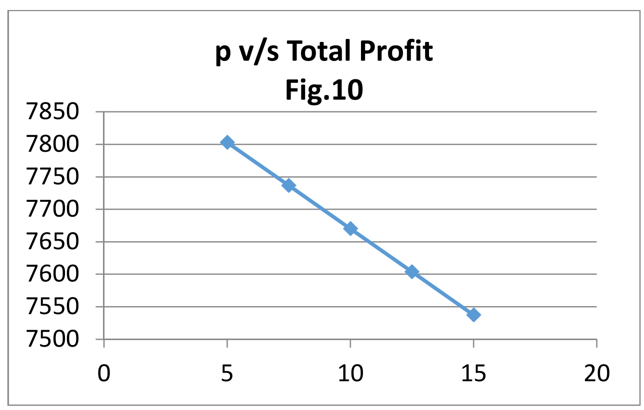


Table 9			
Parameter	**%**	**Value**	**Total Profit**
C	50%	60	7655.26
	25%	50	7662.66
	0	40	7670.05
	-25%	30	7677.44
	-50%	20	7684.84

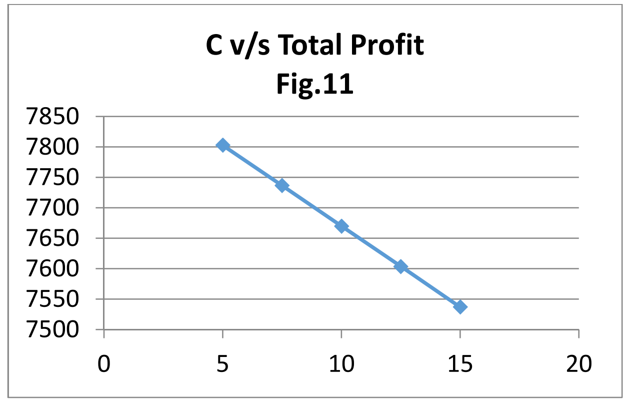


Table 10			
Parameter	**%**	**Value**	**Total Profit**
L	50%	45	7669.59
	25%	37.5	7669.82
	0	30	7670.05
	-25%	22.5	7670.28
	-50%	15	7670.51

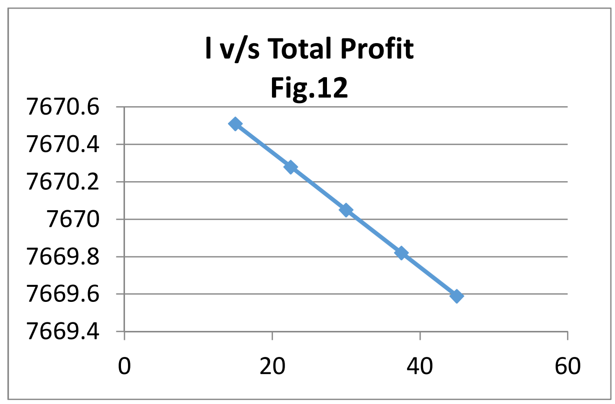


Table 11			
Parameter	**%**	**Value**	**Total Profit**
R	50%	0.09	7597.84
	25%	0.075	7634.02
	0	0.06	7670.05
	-25%	0.045	7705.36
	-50%	0.03	7740.1

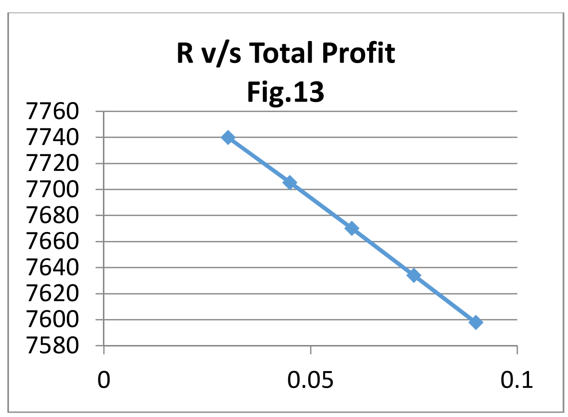
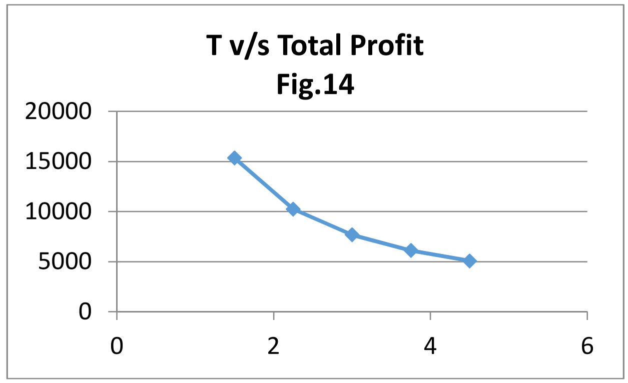


Table 12			
Parameter	**%**	**Value**	**Total Profit**
T	50%	4.5	5074.5
	25%	3.75	6117.43
	0	3	7670.05
	-25%	2.25	10239.1
	-50%	1.5	15344.2



6. Observations

Fromtables (1-12), the following facts are apparent

ii. With increment of a, b and δ, the total profit $P(s,v)$ shows increasing behavior

iii. If α, β, h, A, p, c, l, R, and T are increases then the total profit function $P(s,v)$ decreases.

7. Conclusion

In the above study an inventory model has been proposed in which demand rate is considered to be a function of price and stock where deterioration rate has been considered to follow two parameter Weibull function. The model has been applied to optimize the totalprofit for the business enterprises where demand is stock and price dependent and shortages are partially backlogged. The model is solved analytically by maximize the profit. Finally, the proposed model has been verified by the numerical and graphical analysis. This model can further be extended by taking more realistic assumptions such as probabilistic demand rate, other functions of holding costs, non-zero lead time etc.

References

[1] AditiKhanna, Aakanksha Kishore and Chandra K. Jaggi* Strategic production modeling for defective items with imperfect inspection process,rework, and sales return under two-level trade credit International Journal of Industrial Engineering Computations 8 (2017) 85–118

[2] Baker, R.C., Urban, T.L., (1988). A deterministic inventory system with an inventory level-dependent demand rate. Journal of the Operational Research Society 39, 823-831.

[3] Chang, C.T., Teng, J.T., Goyal S.K., (2010). Optimal replenishment policies for non-instantaneous deteriorating items with stock-dependent demand, International Journal of Production Economics, Volume 123, 62-68.

[4] Choudhury, D.K., Karmakar, B., Das, M., Datta,T.K., (2013). An inventory model for deteriorating items with stock-dependent demand, time-varying holding cost and shortages. OPSEARCH, 10.1007/s12597-013-0166-x.

[5] Giri, B.C., Pal, S., Goswami, A., Chaudhuri, K.S., (1996). An inventory model for deteriorating items with stock-dependent demand rate. European Journal of Operational Research 95, 604-610.

[6] GopalPathak, Vipin Kumar, C.B.Gupta (2017) A Cost Minimization Inventory Model for Deteriorating Products and Partial Backlogging under Inflationary Environment Global Journal of Pure and Applied Mathematics, Volume 13, 5977-5995

[7] GopalPathak, Vipin Kumar, C.B.Gupta (2017) An Inventory Model for Deterioration Items with Imperfect Production and Price Sensitive Demand under Partial Backlogging International Journal of Advance Research in Computer Science and Management Studies Volume 5, Issue 8, (27-36)

[8] Gupta, R., Vrat, P., 1986. Inventory model with multi-items under constraint systems for stock dependent consumption rate. Operations Research 24, 41–42.

[9] Khanlarzade, N., Yegane, B., Kamalabadi, I., &Farughi, H. (2014). Inventory control withdeteriorating items: A state-of-the-art literature review. *International Journal of IndustrialEngineering Computations*, *5*(2), 179-198.

[10] Mandal, B.N., Phaujdar, S., 1989. An inventory model for deteriorating items and stock dependent consumption rate. Journal of the Operational Research Society 40, 483-488.

[11] Mashud, A., Khan, M., Uddin, M., & Islam, M. (2018). A non-instantaneous inventory model havingdifferent deterioration rates with stock and price dependent demand under partially backloggedshortages. *Uncertain Supply Chain Management*, *6*(1), 49-64.

[12] Mishra, U., Tripathy, C.K., 2012. An inventory model for time dependent Weibulldeterioration with partial backlogging. American journal of operational research, 2(2): 11-15.

[13] Pal, S., Goswami, A., Chaudhuri, K.S., 1993. A deterministic inventory model for deteriorating items with stock-dependent demand rate. International Journal of Production Economics 32, 291-299.

[14] Palanivel, M., &Uthayakumar, R. (2014). An EOQ model for non-instantaneous deteriorating items with power demand, time dependent holding cost, partial backlogging and permissible delay in payments. World Academy of Science, Engineering and Technology, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 8(8), 1127-1137.

[15] Ray, J., Goswami, A., Chaudhuri, K.S., 1998. On an inventory model with two levels of storage and stock-dependent demand rate. International Journal of Systems Science 29, 249-254.

[16] Roy, A., 2008. An inventory model for deteriorating items with price dependent demand and time - varying holding cost. AMO-Advanced Modeling and Optimization, 10, 2008, 25-36.

[17] Sana, S.S., 2010. An EOQ model for perishable item with stock dependent demand and price discount rate. American Journal of Mathematical and Management Sciences. Volume 30, Issue 3-4, 229-316.

[18] Soni, H., Shah, N.H., 2008. Optimal ordering policy for stock dependent demand underprogressive payment scheme. European Journal of Operational Research 184:91-10.

[19] Uthayakumar, R. Parvathi, P., 2006. A deterministic inventory model for deteriorating items with partially backlogged and stock and time dependent demand under trade credit. International Journal of Soft Computing 1(3):199-206.

[20] Vipin Kumar, S. R. Singh, &Dhir Singh (2011)" An Inventory Model For Deteriorating Items With Permissible Delay In Payment UnderTwo-Stage Interest Payable Criterion And Quadratic Demand" International Journal Of Mathematics & Applications Vol. 4, No. 2, (December 2011), Pp. 207-219

[21] Vipin Kumar, GopalPathak, C.B.Gupta (2013)"A Deterministic Inventory Model for Deteriorating Items with Selling Price Dependent Demand and Parabolic Time Varying Holding Cost under

Trade Credit" International Journal of Soft Computing and Engineering (IJSCE), Volume-3, Issue-4, (33-37)

[22] Vipin Kumar, Anupama Sharma, C.B.Gupta (2014) "An EOQ Model For Time Dependent Demand and Parabolic Holding Cost With Preservation Technology Under Partial Backlogging For Deteriorating Items International Journal of Education and Science Research Review Volume-1, Issue-2 (170-185)

[23] Vipin Kumar, Anupma Sharma , C.B.Gupta (2015) "Two-Warehouse Partial Backlogging Inventory Model For Deteriorating Items With Ramp Type Demand" Innovative Systems Design and Engineering Vol.6, No.2, (86-97)

[24] Vipin Kumar, Anupama Sharma, C.B.Gupta (2015) "A Deterministic Inventory Model For Weibull Deteriorating Items with Selling Price Dependent Demand And Parabolic Time Varying Holding Cost" International Journal of Soft Computing and Engineering (IJSCE) , Volume-5 Issue-1, (52-59)

[25] Vipin Kumar, GopalPathak, C.B.Gupta (2017) An EPQ Model with Trade Credit for Imperfect Items International Journal of Advance Research in Computer Science and Management Studies Volume 5, Issue 7, (84-93)

[26] Wu, K. S., Ouyang, L. Y. and yang, C. T. 2006. An optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand & partial backlogging, International Journal of Production Economics, 101, 369-384.

[27] Zhang, J., Wang, Y., Lu, L., & Tang, W. (2015). Optimal dynamic pricing and replenishment cycle fornon-instantaneous deterioration items with inventory-level-dependent demand. InternationalJournal of Production Economics, *170*, 136-145.

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Multistage Stochastic Decision Making in Dynamic Multi-level Distribution System Using Recourse Model

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Abstract

The decision taken at the beginning i.e. in stage 0 is called the initial decision, whereas decisions taken in succeeding stages are called recourse decisions. In multi-objective, multi-level optimization problems, there often exist conflicts (contradictions) between the different objectives to be optimized simultaneously. Two objective functions are said to be in conflict if the full satisfaction of one, results in only partial satisfaction of the other. Multistage decision making under uncertainty involves making optimal decisions for a T-stage horizon before uncertain events (random parameters) are revealed while trying to protect against unfavorable outcomes that could be observed in the future.

The prime objective of this research work is to present some contributions for constructing general Multi-stage Stochastic multi-criteria Decision making models. With the help of Numerical problems and LINGO software (can solve linear, nonlinear and integer multistage stochastic programming problems), we tried to prove that Recourse decisions provide latitude for obtaining improved overall solutions by realigning the initial decision with possible realizations of uncertainties in the best possible way as compare with the heuristic model for the decision making.

Keywords: Recourse decision, Multi-criteria dynamic decision making, optimization, decision tree, LINGO

1. Introduction

Decision making is the most important task of a manager and it is often a very difficult one. The domain of decision analysis models falls between two extreme criterions. These depends upon the degree of knowledge we have about the outcome of our actions. One "pole" on this scale is deterministic. The opposite "pole" is pure uncertainty.

Modeling related to decision making problems involves two distinct parties—one is the decision maker and the other is the model builder known as the analyst. The analyst is to assist the decision maker in his/her decision making process. Therefore, the analyst must be equipped with more than a set of analytical methods. Specialists in model building are often tempted to study a problem, and then go off in isolation to develop an elaborate mathematical model for use by the manager (i.e., the decision maker).

In deterministic models, a good decision is judged by the outcome alone. However, in probabilistic models, the decision maker is concerned not only with the outcome value but also with the amount of risk each decision carries. As an example of deterministic versus probabilistic models, consider the past and the future. Nothing we can do can change the past, but everything we do influences and changes the future, although the future has an element of uncertainty. Managers are captivated much more by shaping the future than the history of the past.

Uncertainty is the fact of life and business. Probability is the guide for a "good" life and successful business. The concept of probability occupies an important place in the decision making process, whether the problem is one faced in business, in government, in the social sciences, or just in one's own everyday personal life. In very few decision making situations is perfect information—all the needed facts—available. Most decisions are made in the face of uncertainty. Probability enters into the process by playing the role of a substitute for certainty—a substitute for complete knowledge. Ahmed (2000) presented several examples having decision dependent uncertainties that were formulated as MILP problems and solved by LP-based branch & bound algorithms. Moreover, Viswanath et al. (2004) and Held and Woodruff (2005) addressed the endogenous uncertainty problems where decisions can alter the probability distributions.

Recently, few practical applications that involve multistage stochastic programming with endogenous uncertainty have been addressed. Goel and Grossmann (2004) and Goel et al. (2006) dealt with the gas field development problem under uncertainty in size and quality of reserves where decisions on the timing of field drilling yield an immediate resolution of the uncertainty. Solak (2007) considered the project portfolio optimization problem that deals with the selection of research and development projects and determination of optimal resource allocations under decision dependent uncertainty where uncertainty resolved gradually. Colvin and Maravelias (2008, 2010) presented several theoretical properties, specifically for the problem of scheduling of clinical trials having uncertain outcomes in the pharmaceutical R&D pipeline, and developed a branch and cut framework to solve these MSSP problems

2. Stochastic programming

A stochastic program (SP) is a mathematical program (linear, nonlinear or mixed-integer) in which some of the model parameters are not known with certainty, and the uncertainty can be expressed with known probability distributions Birge and Louveaux (1997). This area is receiving increasing attention given the limitations of deterministic models. Applications arise in a variety of industries: Financial portfolio planning over multiple periods for insurance and other financial companies, in face of uncertain prices, interest rates, and exchange rates, Exploration planning for petroleum companies, Fuel purchasing when facing uncertain future fuel demand, Fleet assignment: vehicle type to route assignment in face of uncertain route demand, Electricity generator unit commitment in face of uncertain demand, Hydro management and flood control in face of uncertain rainfall, lending in face of uncertain input scrap qualities, Product planning in face of future technology uncertainty, Stochastic programs fall into two major categories: a) multistage stochastic programs with recourse, and b) chance-constrained programs. LINGO's capabilities are extended to solve models in the first category, namely multistage stochastic recourse models. The term *stochastic program* (SP) refers to a multistage stochastic model with recourse. The term *stage* is an important concept in this paper. Usually it means the same as 'time period', however there are situations where a stage may consist of several time periods. The terms *random*, *uncertain* and *stochastic* are used interchangeably.

3. Multistage Decision Making under Uncertainty

Multistage decision making under uncertainty involves making optimal decisions for a T-stage horizon before uncertain events (random parameters) are revealed while trying to protect against unfavorable outcomes that could be observed in the future.

In general form, a multistage decision process with T+1 stages follows an alternating sequence of random events and decisions:

i. in stage 0, we make a decision x_0, taking into account that………
ii. at the beginning of stage 1, "Nature" takes a set of random decisions ω_1, leading to realizations of all random events in stage 1, and…

iii. at the end of stage 1, having seen nature's decision, as well as our previous decision, we make a recourse decision $x_1(\omega_1)$, taking into account that …

iv. at the beginning of stage 2, "Nature" takes a set of random decisions ω_2, leading to realizations of all random events in stage-2, and…

v. at the end of stage 2, having seen nature's decision, as well as our previous decisions, we make a recourse decision $x_2(\omega_1, \omega_2)$, taking into account that …

.

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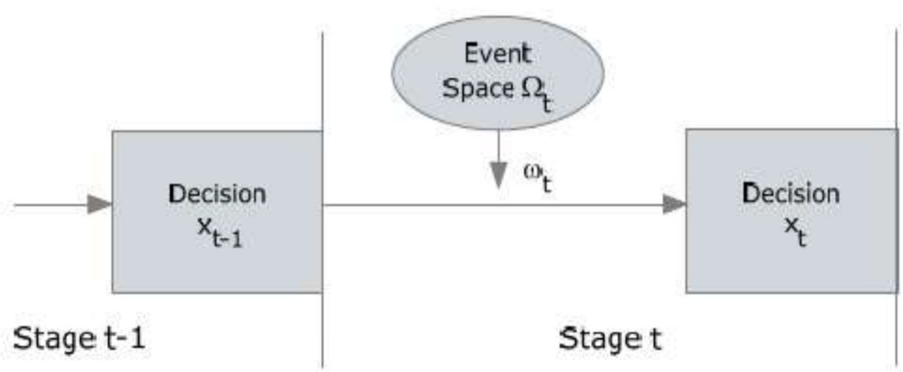
T_0: At the beginning of stage T, "Nature" takes a random decision, ω_T, leading to realizations of all random events in stage T, and…

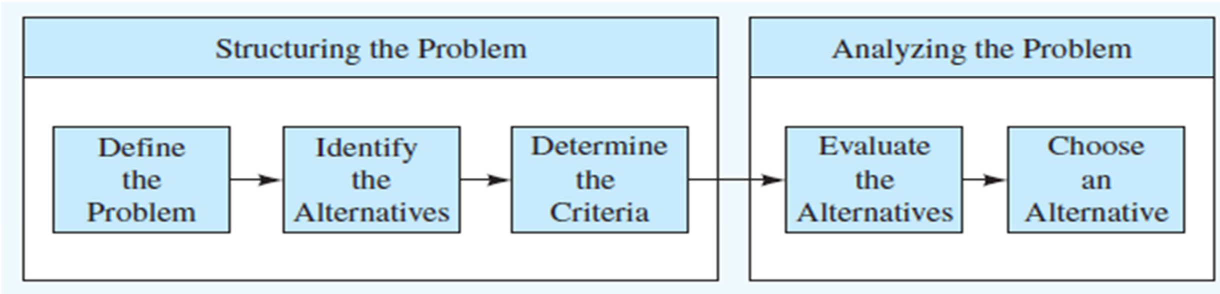
T_1: at the end of stage T, having seen all of nature's T previous decisions, as well as all our previous decisions, we make the final recourse decision $x_T(\omega_1,\ldots,\omega_T)$.

The relationship between the decision variables and realizations of random data:



Each decision, represented with a rectangle, corresponds to an uninterrupted sequence of decisions until the next random event. And each random observation corresponds to an uninterrupted sequence of random events until the next decision point.

The relationship between problem solving and decision making:



Recourse Model:

The decision taken in stage 0 is called the *initial decision*, whereas decisions taken in succeeding stages are called *recourse decisions*. Recourse decisions are interpreted as corrective actions that are based on the actual values the random parameters realized so far, as well as the past decisions taken thus far. Recourse decisions provide latitude for obtaining improved overall solutions by realigning the initial decision with possible realizations of uncertainties in the best possible way. The multistage stochastic program with (T+1) stages is:

Minimize (or maximize): $c_0x_0 + E_1[c_1x_1 + E_2[c_2x_2 .... + E_T[c_1x_1].......]]$
such that:

$A_{00}x_0$				$\sim b_0$
$A(\omega_1)_{10}x_0$	$+A(\omega_1)_{11}x_1$			$\sim b(\omega_1)_1$
$A(\omega_1,\omega_2)_{20}x_0$	$+A(\omega_1,\omega_2)_{21}x_1$	$+A(\omega_1,\omega_2)_{22}x_2$		$\sim b(\omega_1,\omega_2)_2$
…	…	…		…
$A(\omega_1,\ldots,\omega_T)_{T0}x_0$	$+A(\omega_1,\ldots,\omega_T)_{T1}x_1 +$	…	$+A(\omega_1,\ldots,\omega_T)_{TT}x_T$	$\sim b(\omega_1,\ldots,\omega_T)_T$
L_0	$\le x_0$	$\le U_0$		
$L(\omega_1)_1$	$\le x_1$	$\le U(\omega_1)_1$		
…	…	…		
$L(\omega_1,\ldots,\omega_T)_T$	$\le x_T$	$\le U(\omega_1,\ldots,\omega_T)_T$		

where,

$(\omega_1,\ldots,\omega_t)$ represents random outcomes from event space $(\Omega_1,\ldots,\Omega_t)$ up to stage-t,

$A(\omega_1,..., \omega_t)_{tp}$ is the coefficient matrix generated by outcomes up to stage-t for all p=1… t, t=1… T,

$c(\omega_1,..., \omega_t)_t$ is the objective coefficients generated by outcomes up to stage-t for all t=1… T,

$b(\omega_1,..., \omega_t)_t$ is the right-hand-side values generated by outcomes up to stage-t for all t=1… T,

$L(\omega_1,..., \omega_t)_t$ and $U(\omega_1,..., \omega_t)_t$ are the lower and upper bounds generated by outcomes up to stage-t for all t=1… T,

'~' is one of the relational operators '≤', '=', or '≥'; and

x_0 and $x_t \equiv x(\omega_1,..., \omega_t)_t$ are the decision variables (unknowns) for which optimal values are sought. The expression being optimized is called the cost due to initial-stage plus the expected cost of recourse.

Setting up Stochastic Programming Model of the Problem:

There are four steps to setting up an Stochastic Programming Model:

Step I - Defining the Core Model:

The main/core model is the same optimization model we would construct if all the random variables were known with certainty. There is nothing in the core model that addresses the stochastic nature of the model. For our current example. This model is formulated as follows:

A quilt shop must come up with a plan for its quilt purchases under uncertain weather conditions. The demand for the current period (period 1) is known and is 100 units. The demand for the upcoming period is not known with certainty and will depend on how cold the weather is. There are three possible outcomes for the weather: normal, cold and very cold. Each of these outcomes are equally likely. The following table lists the costs and demands under the three outcomes

Outcome	Probability	Quilt Cost/Unit (Rs.)	Units Demand
Normal	1/3	200.00	100
Cold	1/3	300.00	150
Very Cold	1/3	500.00	180

Quilt for the current period is bought now and delivered directly to the customers at a cost of Rs. 200 per unit. Quilt in the upcoming period can either be bought now and held in storage for period 2 use, or it can be purchased in period 2 at a price that will depend on the weather as per the table above. Storing quilt bought in period 1 for use in period 2 costs the company Rs.10 per unit. The question the shopkeeper is faced with is: How much Quilt should be bought in periods 1 and 2 to meet total customer demand at minimal expected cost?

Step II - Identifying the Random Variables:

The next step in building our SP model is to identify the random variables. The random variables are the variables that are stochastic by nature and whose values are not known before we must make our initial decisions. In the above problem, there are two random variables, the second period cost and demand.

Step III - Identifying the Initial Decision and Recourse Variables:

The next step is to identify the initial decision variables and the recourse variables. Unlike the random variables, which are under Mother Nature's control, the initial decision and recourse variables are under our control. The initial decision variables occur at the very outset, before any of the random variables become known, and are always assigned to stage 0. The recourse variables are the subsequent decisions we make after learning of the outcomes of the random variables. Recourse variables that are decided after the stage *N* random variables become known are assigned to stage *N* as well. In the given problem, there is one initial decision, which is *PURCHASE_1,* the amount of quilt to purchase in period 1. The weather then reveals itself and our recourse variable is *PURCHASE_2,* the amount to purchase in period 2.

Step IV - Declare Distributions

The last step is to declare the joint probability distribution for the random variables COST_2 and DEMAMD_2. In this case, we will be using an outcome table distribution, and in order to declare the distribution, make use of the scalar-based functions. Now, to be able to actually apply the distribution to random variables, we need to declare an instance of the distribution. By doing things this way, it's possible to reuse the same outcome table on more than one set of random variables.

Our last step is to associate, or bind, the random variables to the instance of the distribution. Specifically, we wish to bind the cost and demand random variable.

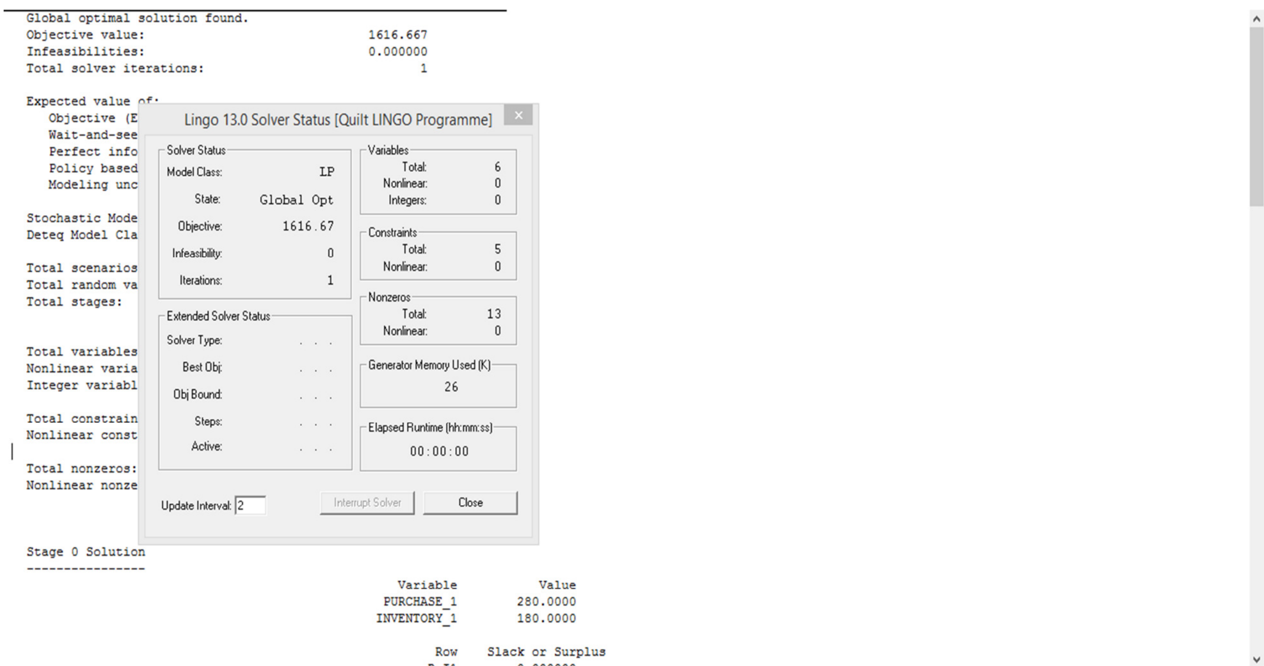
LINGO Model of the Problem

```
!QUILT PROBLEM [MSDDMDS] FOR THE MODEL;
Model:
! Minimize Total Cost = Purchases + Holding;
[R_OBJ] MIN= PURCHASE_COST + HOLD_COST;
! Compute purchase cost;
[R_PC] PURCHASE_COST = 5 * PURCHASE_1 + COST_2 * PURCHASE_2;
! Compute holding cost;
[R_HC] HOLD_COST = INVENTORY_1 + INVENTORY_2;
! Compute inventory levels;
[R_I1] INVENTORY_1 = PURCHASE_1 - 100;
[R_I2] INVENTORY_2 = INVENTORY_1 + PURCHASE_2 - DEMAND_2;
! *** STEP 2 *** - Define Random Variables;
!The random variables are period 2's demand and cost.;
@SPSTGRNDV( 1, COST_2);
@SPSTGRNDV( 1, DEMAND_2);
! *** STEP 3 *** - Define initial decision and recourse
variables;
!The initial decision is how much to purchase in period 1;
@SPSTGVAR( 0, PURCHASE_1);
!Period 2 purchases are a recourse variable after
the weather reveals itself;
@SPSTGVAR( 1, PURCHASE_2);
! *** STEP 4 *** - Assign distributions to the random
variables;
!Declare a discrete distribution called 'DST_DMD' with
three outcomes and two jointly distributed variables
(i.e., Demand and Cost);
@SPTABLESHAPE( 'DST_DMD', 3, 2);
!Load the three equally likely outcomes into 'DST_DMD';
!Dist Name Probability Cost Demand;
@SPTABLEOUTC( 'DST_DMD', 1/3, 200.0, 100);
```

```
@SPTABLEOUTC( 'DST_DMD', 1/3, 300.0, 150);
@SPTABLEOUTC( 'DST_DMD', 1/3, 500.0, 180);
!Declare a specific instance of the 'DST_DMD' distribution,
naming the instance 'DST_DMD_1';
@SPTABLEINST( 'DST_DMD', 'DST_DMD_1');
!Bind Period 2 Cost and Demand to the distribution instance;
@SPTABLERNDV( 'DST_DMD_1', COST_2, DEMAND_2);!The random variables are
period 2's demand and cost.;
!STOP;
```

Solution

i) Solution window



ii) Complete Solution:

```
Global optimal solution found.
Objective value:                              1616.667
Infeasibilities:                              0.000000
Total solver iterations:                             1
```

```
Expected value of:
   Objective (EV):                                          1616.667
   Wait-and-see model's objective (WS):                     1360.000
   Perfect information (EVPI = |EV - WS|):                  256.6667
   Policy based on mean outcome (EM):                       8152.222
   Modeling uncertainty (EVMU = |EM - EV|):                 6535.556

Stochastic Model Class:        SINGLE-STAGE STOCHASTIC
Deteq Model Class:                                  LP

Total scenarios/leaf nodes:                          3
Total random variables:                              2
Total stages:                                        1

                                   Core          Deteq
Total variables:                      6             18
Nonlinear variables:                  0              0
Integer variables:                    0              0

Total constraints:                    5             17
Nonlinear constraints:                0              0

Total nonzeros:                      13             47
Nonlinear nonzeros:                   0              0

Stage 0 Solution
----------------
                                               Variable           Value
                                             PURCHASE_1        280.0000
                                            INVENTORY_1        180.0000
                                                    Row    Slack or Surplus
                                                   R_I1        0.000000
Staging Report
--------------
                                        Random Variable           Stage
                                                 COST_2               1
```

```
                                                DEMAND_2              1

                                                Variable           Stage
                                           PURCHASE_COST              1*
                                               HOLD_COST              1*
                                              PURCHASE_1              0
                                              PURCHASE_2              1
                                             INVENTORY_1              0*
                                             INVENTORY_2              1*

                                                     Row           Stage
                                                   R_OBJ              1*
                                                    R_PC              1*
                                                    R_HC              1*
                                                    R_I1              0*
                                                    R_I2              1*

                                                   (*) Stage was inferred
  Random Variable Distribution Report
 -----------------------------------
                      Sample        Sample
Random Variable         Mean        StdDev      Distribution
         COST_2     333.3333      124.7219     DST_DMD,DST_DMD_1,1
       DEMAND_2     143.3333      32.99832     DST_DMD,DST_DMD_1,2

  Scenario: 1   Probability: 0.3333333   Objective: 1660.000
  ----------------------------------------------------------
                                         Random Variable           Value
                                                  COST_2        200.0000
                                                DEMAND_2        100.0000

                                                Variable           Value
                                           PURCHASE_COST        1400.000
                                               HOLD_COST        260.0000
                                              PURCHASE_1        280.0000
                                              PURCHASE_2        0.000000
```

```
                                       INVENTORY_1        180.0000
                                       INVENTORY_2        80.00000

                                               Row           Value
                                              R_PC        0.000000
                                              R_HC        0.000000
                                              R_I1        0.000000
                                              R_I2        0.000000

Scenario: 2   Probability: 0.3333333   Objective: 1610.000
----------------------------------------------------------

                                   Random Variable           Value
                                            COST_2        300.0000
                                          DEMAND_2        150.0000

                                          Variable           Value
                                     PURCHASE_COST        1400.000
                                         HOLD_COST        210.0000
                                        PURCHASE_1        280.0000
                                        PURCHASE_2        0.000000
                                       INVENTORY_1        180.0000
                                       INVENTORY_2        30.00000

                                               Row           Value
                                              R_PC        0.000000
                                              R_HC        0.000000
                                              R_I1        0.000000
                                              R_I2        0.000000

Scenario: 3   Probability: 0.3333333   Objective: 1580.000
----------------------------------------------------------

                                   Random Variable           Value
                                            COST_2        500.0000
                                          DEMAND_2        180.0000
```

Variable	Value
PURCHASE_COST	1400.000
HOLD_COST	180.0000
PURCHASE_1	280.0000
PURCHASE_2	0.000000
INVENTORY_1	180.0000
INVENTORY_2	0.000000

Row	Value
R_PC	0.000000
R_HC	0.000000
R_I1	0.000000
R_I2	0.000000

Solution summary:

The stage-0 solution lists the values for all variables and rows that are part of the initial decision. These values are of pressing importance, in that they must be implemented currently. For this reason, they are displayed in their own separate section near the top of the report. In the case of our quilt company, the optimal initial decision to minimize expected cost is to purchase 280 units of quilt in period 1, storing 180 units in inventory. If period 2 is normal the company can fulfill demand entirely from inventory, otherwise it must make up the difference through additional purchases in period 2. The remainder of the solution report contains sub-reports for each of the scenarios. Information regarding the each scenario's probability, objective value and variable values are displayed.

Expected Value of Objective (EV) - is the expected value for the model's objective over all the scenarios, and is the same as the reported objective value for the model. [Calculated value is Rs. 1616.667]

Expected Value of Wait-and-See Model's Objective (WS) - reports the expected value of the objective if we could wait and see the outcomes of all the random variables before making our decisions. Such a policy would allow us to always make the best decision regardless of the outcomes for the random variables, and, of course, is not possible in practice. For a minimization, it's true that WS <= EV, with the converse holding for a maximization. Technically speaking, WS is a relaxation of the true SP model, obtained by dropping the non-anticipativity constraints. [Calculated value is Rs. 1360.000]

Expected Value of Perfect Information (EVPI) - is the absolute value of the difference between EV and WS. This corresponds to the expected improvement to the objective were we to obtain perfect information about the random outcomes. As such, this is a expected measure of how much we should be willing to pay to obtain perfect information regarding the outcomes of the random variables. [Calculated value is Rs. 256.6667]

Expected Value of Policy Based On Mean Outcome (EM) - is the expected true objective value if we (mistakenly) assume that all random variables will always take on exactly their mean values. EM is computed using a two-step process. First, the values of all random variables are fixed at their means, and the resulting deterministic model is solved to yield the optimal values for the stage 0 decision variables. Next, a) the stage 0 variables are fixed at their optimal values from the previous step, b) the random variables are freed up, c) the non-anticipativity constraints are dropped, and d) this wait-and-see model is solved. EM is the objective value from this WS model. [Calculated value is Rs. 8152.222].

Expected Value of Modeling Uncertainty (EVMU) - is the absolute value of the difference EV - EM. It is a measure of what we can expect to gain by taking into account uncertainty in our modeling analysis, as opposed to mistakenly assuming that random variables always take on their mean outcomes. [Calculated value is Rs. 6535.556]

7. Conclusions and Future Research

The key contributions of this paper are the following:

- We have proposed a multi-stage stochastic programming formulation for a expansion problem under uncertainty.
- A reformulation scheme has been developed by exploiting special sub-structure of decision problem.
- We have presented computational results demonstrating the effectiveness of the reformulation of recourse model for the multi-stage decision problem under uncertainty. The results in this paper pave the way for a number of future research avenues.

References:

[1] Ali Haghani and S. Oh. Formulation and solution of a multi-commodity, multi-modal network flow model for disaster relief operations. *Transportation Research A*, 30(3):231–250, 1996.

[2] Ben-Haim, Y. 2001. *Information-gap Decision Theory: Decisions under Severe Uncertainty*. San Diego: Academic Press.

[3] Boland, N., Dumitrescu, I., Froyland, G., 2008. A multistage stochastic programming approach to open pit mine production scheduling with uncertain geology.

[4] Constantine Toregas, Ralph Swain, C.ReVelle, and L.Bergman. The location of emergency service facilities. *Operations Research*, 19(6):1363–1373, 1971.

[5] Dentcheva, D., Wolfhagen, E.: Optimization with multivariate stochastic dominance constraints. SIAM Journal on Optimization 25(1), 564–588 (2015).

[6] Dupˇacov´a, J.: Risk objectives in two-stage stochastic programming models. Kybernetika 44(2), 227–242 (2008). http://dml.cz/dmlcz/135845

[7] Fiedrich, F. Gehbauer, and U. Rickers. Optimized resource allocation for emergency response after earthquake disasters. *Safety Science*, 35:41–57, 2000.

[8] G. Barbarosoglu and Y. Arda. A two-stage stochastic programming framework for transportaion planningin disaster response. *Journal of the Operational Research Society*, 55:43–53, 2004.

[9] G. Cornuejols, G.L. Nemhauser, and L.A. Wolsey. The uncapacitated facility location problem. *In:Discrete Location Theory (P. Mirchandani, R. Francis, eds.)Wiley, New York*, (119-171), 1990.

[10] Goel, V., Grossmann, I. E., 2006. A class of stochastic programs with decision dependent uncertainty. Mathematical Programming 108 (2-3, Ser. B), 355-394.

[11] Gutjahr, W.J., Nolz, P.C.: Multicriteria optimization in humanitarian aid. European Journal of Operational Research 252(2), 351 – 366 (2016)

[12] Held, H., Woodruff, D. L., 2005. Heuristics for multi-stage interdiction of stochastic networks. Journal of Heuristics 11 (5-6), 483-500.

[13] K. Hoyland and S. W. Wallace. Generating scenario trees for multi-stage decision problems. To appear in *Management Science*, 2000.

[14] K¨uchler, C., & Vigerske, S. (2010). Numerical evaluation of approximation methods in stochastic programming. Optimization, 59, 401-415.

[15] Kweku Muata(Noel) Bryson, Harvey Millar, Anito Joseph, and Ayodele Mobolurin. Using formal ms/or modeling to support disaster recovery planning. *European Journal of Operational Research*, 141:679–688, 2002.

[16] Linet Ozdamar, Ediz Ekinci, and Beste Kucukyazici. Emergency logistics planning in natural disasters. *Annals of Operations Research*, 129:217–245, 2004.

[17] Maier, H.R., Ascough II, J.C., Wattenbach, M., Renschler, C.S., Labiosa, W.B., Ravalico, J.K., 2008. Chap. 5: Uncertainty in environmental decision making: Issues, challenges, and future directions. In: Jakeman, A.J., Voinov, A.E., Rizzoli, A.E., Chen, S. (Eds.), Environmental Modelling and Software and Decision Support – Developments in Integrated Environmental Assessment (DIEA), vol. 3. Elsevier, The Netherlands, pp. 69–85.

[18] Ming-Hua Zeng, A Multistage Multiobjective Model for Emergency Evacuation Considering ATIS, Hindawi Publishing Corporation Mathematical Problems in Engineering Volume 2016.

[19] O. Kariv and S.L. Hakimi. An algorithmic approach to network location problems. part ii: The pmedians. *SIAM Journal on Applied Mathematics*, 37:539–560, 1979.

[20] Suleyman Tufekci and William A. Wallace. The emerging area of emergency management and engineering. *IEEE Transactions on Engineering Management*, 45(2), 1998.

[21] Wei Yi and Linet Ozdamar. Fuzzy modeling for coordinating logistics in emergencies. *International Scientific Journal of Methods and Models of Complexity-Special Issue on Societal Problems in Turkey*, 7(1), 2004.

[22] Z-L. Chen, S. Li, and D. Tirupati. A scenario based stochastic programming approach for technology and capacity planning. To appear in *Computers & Operations Research*, 2001.

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On The Sondow's Formula For π

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Abstract

We obtain a compact expression for $\sum_{k=1}^{\infty} \frac{1}{4\,k^2-x^2}$ *which allows to deduce the value of ζ(2), and also the Sondow's formula for π via the Wallis product.*

Keywords: Sondow's expression for π, Wallis product, ζ(2).

1. Introduction

Sondow [1-3] published the relation:

$$\pi \sum_{k=1}^{\infty} \frac{1}{4\,k^2-1} = \prod_{n=1}^{\infty} \left(1+\frac{1}{4\,n^2-1}\right); \qquad \dots (1)$$

in Sec 2 we employ the Wallis product [4-6] and known Lanczos relations [7] for $\cot(\pi x)$ and $\tan\left(\frac{\pi}{2}x\right)$, to give an elementary deduction of (1).

2. Sondow's Formula

Lanczos [7] uses interpolation techniques to obtain the expressions:

$$\cot(\pi x) = \frac{2}{\pi}x\left(\frac{1}{2x^2}-\frac{1}{1-x^2}-\frac{1}{4-x^2}-\frac{1}{9-x^2}-\ \dots\right), \qquad \tan\left(\frac{\pi}{2}x\right) = \frac{4}{\pi}x\left(\frac{1}{1-x^2}+\frac{1}{9-x^2}+\frac{1}{25-x^2}+\ \dots\right), \qquad \dots (2)$$

- *Dedicated to Professor M. A. Pathan on his 75th birth anniversary* then it is immediate that

$$\sum_{k=1}^{\infty} \frac{1}{4\,k^2-x^2} = \frac{1}{2\,x^2} - \frac{\pi}{4\,x}\left[2\cot(\pi x) + \tan\left(\frac{\pi}{2}x\right)\right] = \frac{2-\pi\,x\,cot(\frac{\pi}{2}x)}{4\,x^2}. \qquad \dots (3)$$

From (3) when $x \longrightarrow 0$ and the Bernoulli-Hôpital rule we deduce the following value of the Riemann zeta function [8]:

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}\ ; \qquad \dots (4)$$

and (3) with $x = 1$ implies the formula:

$$\sum_{k=1}^{\infty} \frac{1}{4\,k^2-1} = \frac{1}{2}, \qquad \dots (5)$$

which is a telescoping sum [3] because:

$$\sum_{k=1}^{\infty} \frac{1}{4\,k^2-1} = \frac{1}{2}\sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \ \dots\right) = \frac{1}{2}. \qquad \dots (6)$$

On the other hand, we have the Wallis product [4-6]:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4\,n^2}{4\,n^2-1} = \prod_{n=1}^{\infty}\left(1 + \frac{1}{4\,n^2-1}\right), \qquad \dots (7)$$

hence the Sondow's expression (1) is consequence from (5) and (7).

References

[1] J. Sondow, *A symmetric formula for pi,* Math. Horizons **5** (1997) 32 and 34

[2] http://home.earthlink.net/~jsondow/

[3] http://mathworld.wolfram.com/PiFormulas.html

[4] J. Wallis, *Arithmetica infinitorum,* Oxford, England (1656)

[5] https://en.wikipedia.org/wiki/Wallis_product

[6] https://en.wikipedia.org/wiki/List_of_formulae_involving_π

[7] C. Lanczos, *Linear differential operators,* SIAM, Philadelphia, USA (1996) Chap. 1

[8] V. Barrera-Figueroa, R. Cruz-Santiago, J. López-Bonilla, *On some expressions for* $\zeta(k), k = 2, 3, 4,$ Prespacetime Journal **7**, No. 12 (2016) 1674-1676

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I, Dr. R.P. Sharma, hereby declare that the particulars given above are true to the best of my knowledge and belief.

Dated : 30th December, 2017 ***Dr. R. P. Sharma***

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GAṆITA SANDESH

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The Sequence

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