Volume 30 (June & Dec. 2016) ISSN 0970-9169

GANIṬA SANDESH

गणित संदेश

(UGC approved)

A Half Yearly International Research Journal
of

Rajasthan Gaṇita Parishad

Registered Head Office
Department of Mathematics
SPC Government College, AJMER - 305 001 (*INDIA*)
(*NAAC Accredited "A" Grade College*)

Website : www.rgp.co.in
E-mail : rgp@rgp.co.in

Issued April, 2018

E-mail : editor@rgp.co.in
E-mail : cmtr@rgp.co.in

GANITA SANDESH

गणित संदेश

Notes For Contributors

1. The editors will be pleased to receive contributions from all parts of India / abroad in any area of Mathematics (Research / Teaching etc.).
2. Manuscripts for publication should be sent through E-Mail directly to the editor@rgp.co.in or **rgp@rgp.co.in** along with hard copy in Triplicate duly computerised with double spacing preferably in Adobe Pagemaker 6.5, with text in Times New Roman font 11 pts. and Mathematical symbols (Math Type, Equation editor or Corel Equation).
3. Authors should provide abstract and identify 4 to 5 key words for subject classification.
4. Unduly long papers and papers with many diagrams/tables will not be normaly accepted. In general, length of the accepted paper should not exceed 10 printed pages.
5. The contributors are required to meet the partial cost of publication at the rate of ₹ 200/- or equivalent US $ per printed page size A4 (even number of pages) payable in advance.

Membership Fees / Subcription Rates

	Period	In India (₹)	Outside India (US $)
❑ Admission / Enrolment Fee	First Time Only	**100**	**100** (or equivalent)
❑ Life Membership		**2,000**	**2,000** (or equivalent)
❑ Annual Membership fee for Teachers (Colleges / Universities), T.R.F. Registered Research Scholars	Financial year	**250**	**250** (or equivalent)
❑ Educational / Research Institutions	Calendar year	**500**	**500** (or equivalent)

❈ All payments must be made by Bank **D.D.** in favour of ***Rajasthan Ganita Parishad*** payable at **AJMER** or online **State Bank of India Account No. 10200807636, IFSC Code : SBIN0000603** under intimation to the **Treasurer, Rajasthan Ganita Parishad**, *Department of Mathematics*, SPC Government College, AJMER-305 001 (INDIA)

GanitaSandesh Vol. 30 No.(1& 2) 2016 pp. 1 – 12 ISSN : 0970-9169

SOME PROPERTIES OF EXTENDED HYPERGEOMETRICFUNCTION AND ITS APPLICATIONS

[1]APARNA CHATURVEDI AND [2]PRAKRITI RAI

Department of Mathematics,

Amity Institute of Applied Sciences, Amity University, **NOIDA (INDIA)**

[1]chaturvedi.aparna.tirwa@gmail.com, [2]prai@amity.edu

ABSTRACT

In the present paper, we haveextended the generalized Apostol-Bernoulli polynomials and the generalized Apostol- Euler polynomials. The main results provide interesting extensions of a representation for the generalized Apostol-Bernoulli polynomials and the generalized Apostol- Euler polynomials, using extended Generalized Gauss hypergeometric functions and its applications.

Keywords:*Extended Apostol-Bernoulli polynomials and Extended Apostol- Euler polynomials; extended Gauss hypergeometricfunctions,Stirling number of second kind.*

1.INTRODUCTION, DEFINITIONS AND PRELIMINARIES

With a view of describing and illustrating a technique involving an integral representation of extended Gauss hypergeometric function given by LuoMinjie [5] and the definition of a special analytic function given by H. M. Srivastava, R. K. Parmar and P. Chopra [18] for obtaining properties and representations of extended Gamma and extended Beta functions, we begin this paper by presenting some useful definitions.

In 2012, H. M. Srivastava, R. K. Parmar and P. Chopra [18] gave the following definition

Definition 1.1:[18]: Let a function $\Theta\left(\{k_l\}_{l\in ¥_0};z\right)$ be analytic within the disk $|z| < R$ $(0 < R < \infty)$ and let its Taylor-Maclaurincoeficients be explicitly denoted by the sequence $\{k_l\}_{l\in ¥_0}$. Suppose also that the function $\Theta\left(\{k_l\}_{l\in ¥_0};z\right)$ can be continued analytically in the right half-plane $\Re(z) > 0$ with the asymptotic property given as follows:

$$\Theta(k_l;z) \equiv \Theta\left(\{k_l\}_{l\in ¥_0};z\right) = \begin{cases} \sum_{l=0}^{\infty} k_l \dfrac{z^l}{l!} & (|z| < R;\ 0 < R < \infty;\ k_0 = 1) \\ M_0 z^{\omega} \exp(z)\left[1 + O\left(\dfrac{1}{z}\right)\right] & (\Re(z) \to \infty; M_0 > 0;\ \omega \in C) \end{cases}$$

for some suitable constantsM_0 and ω depending essentially on the sequence $\{k_l\}_{l\in ¥_0}$.

They also defined extended Gamma function $\Gamma_p^{(k_l)}(z)$ and the extended Beta function in the form

$$\Gamma_p^{(k_l)}(z) = \int_0^{\infty} t^{z-1}\,\Theta\left(\{k_l\};-t-\frac{p}{t}\right)dt, \qquad \left(\Re(p) \ge 0,\ \Re(z) > 0,\right)$$

$$B_p^{(k_l)}(\alpha,\beta;p) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\{k_l\};-\frac{p}{t(1-t)}\right)dt,$$

$$\left(\Re(p) \ge 0,\ \min\{\Re(\alpha),\Re(\beta)\} > 0\right)$$

By introducing one additional parameter q with $\Re(q) \ge 0$, extended Beta function can be written as

$$B_{p,q}^{(k_l)}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(\{k_l\};-\frac{p}{t}-\frac{q}{(1-t)}\right)dt,$$

$$(\min\{\Re(p),\Re(q)\} > 0; \min\{\Re(\alpha),\Re(\beta)\} > 0)$$

By using $B_p^{(k_l)}(\alpha,\beta)$ they also extended Gauss hypergeometric function as follows.

Definition 1.2:[18]The Extended Gauss hypergeometric function${}_2F_1^{(k_l)}$ is defined by

$${}_2F_1^{(k_l)}\left[\begin{matrix} a, b \\ c \end{matrix}; z; p, q\right] = \sum_{n=0}^{\infty} (a)_n \frac{B_{p,q}^{\{k_l\}}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!} \qquad \ldots(1.1)$$

$$(|z| < 1; \min\{\Re(p), \Re(q)\} \geq 0; \Re(c), \Re(b) > 0)$$

If $\Theta(k_l; z) = \exp z$ one can write

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z; p, q\right] = \sum_{n=0}^{\infty} (a)_n \frac{B_{p,q}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!} \qquad \ldots(1.2)$$

Definition 1.3: The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$, together with their familiar generalizations $B_n^{\alpha}(x)$ and $E_n^{\alpha}(x)$ of (real or complex) order α are usually defined by means of the following generating functions (see [1], [15], [16] and [17] and the references cited therein)

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{\alpha} \frac{t^n}{n!} \qquad (|t| < 2\pi;\ 1^{\alpha} = 1) \qquad \ldots(1.3)$$

$$\left(\frac{2}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{\alpha} \frac{t^n}{n!} \qquad (|t| < \pi;\ 1^{\alpha} = 1) \qquad \ldots(1.4)$$

So that obviously the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are given respectively by

$$B_n(x) = B_n^1(x), \qquad E_n(x) = E_n^1(x) \qquad (n \in N)$$

For the classical Bernoulli numbers $B_n$and the classical Euler numbers E_n

$$B_n(0) = B_n^1(0) = B_n, \quad E_n(0) = E_n^1(0) = E_n \text{respectively.}$$

In particular, Luo and Srivastava [11,12] introduced the generalized Apostol-Bernoulli polynomials $B_n^{\alpha}(x; \lambda)$ of order $\alpha \in C$; and Luo [6,7,8] introduced the generalized Apostol-Euler polynomials $E_n^{\alpha}(x; \lambda)$ of order $\alpha \in C$. These polynomials are defined, respectively as follows.

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{\alpha}(x; \lambda) \frac{t^n}{n!}, \qquad \ldots(1.5)$$

$$(|t| < 2\pi, if\ \lambda = 1;\ |t| < |\log \lambda|, if\ \lambda \neq 1;\ 1^{\alpha} = 1)$$

with

$$B_n^{\alpha}(x) = B_n^{\alpha}(x;1) \text{ and } B_n^{\alpha}(\lambda) = B_n^{\alpha}(0;\lambda),$$

where$B_n^{\alpha}(\lambda)$ denotes the so called Apostol- Bernoulli numbers of order α.

For α = 1, we can write (1.3) as

$$\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x;\lambda)\frac{t^n}{n!} \qquad (|t + \log\lambda| < 2\pi) \qquad \ldots(1.6)$$

Now

$$\left(\frac{2}{\lambda e^t + 1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x;\lambda)\frac{t^n}{n!}, \qquad (|t| < |\log(-\lambda)| < \pi;\ 1^{\alpha} = 1) \quad \ldots(1.7)$$

with

$$E_n^{\alpha}(x) = E_n^{\alpha}(x;1) \text{ and } E_n^{\alpha}(\lambda) = E_n^{\alpha}(0;\lambda),$$

where$E_n^{\alpha}(\lambda)$ denotes the so called Apostol- Euler numbers of order α.

2. EXTENDED APOSTOL-BERNOULLI POLYNOMIALS AND EXTENDEDGAUSSIAN HYPERGEOMETRIC FUNCTIONS

Extension of Apostol-Bernoulli Polynomials :

We give extensions to (1.3) and (1.5) by defining as follows

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} \Theta = \sum_{n=0}^{\infty} B_{n,\alpha}^{\{k_l\}} \frac{t^n}{n!} \qquad (|t| < 2\pi;\ 1^{\alpha} = 1) \qquad \ldots(2.1)$$

$$\frac{t}{\lambda e^t - 1}\Theta = \sum_{n=0}^{\infty} B_n^{\{k_l\}}(x;\lambda)\frac{t^n}{n!} \qquad (|t + \log\lambda| < 2\pi) \qquad \ldots(2.2)$$

Theorem 2.1: If n is a positive integer and $\Re(\lambda) > 0$, $\lambda \neq 1$ are complex numbers, then we have

$$B_n^{\{k_l\}}(x,\lambda) = \sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l(\lambda-1)^l x^{-n}\sum_{j=0}^{l}(-1)^j\binom{l}{j} j^l (x+j)^{n-l-1}\, {}_2F_1^{\{k_l\}}\left[l-n+1, l; l+1; \frac{j}{(x+j)}; p, q\right] \qquad \ldots(2.3)$$

Proof: We differentiate both sides of (2.2) with respect to the variable *t.* Applying Leibniz's rule yields

$$B_n^{\{k_l\}}(x,\lambda) = D_t^n\left\{\frac{t\Theta}{\lambda e^t - 1}\right\}\Bigg|_{t=0}, \qquad D_t = \frac{d}{dt}$$

$$=(\lambda-1)^{-1}\sum_{k=0}^{n}D_t^k\Theta D_t^{k-1}\left\{\left[\frac{\lambda}{\lambda-1}\left(e^t-1\right)+1\right]^{-1}\right\}\Bigg|_{t=0} \qquad ...(2.4)$$

Since binomial series expansion is

$$(1+\omega)^{-1}=\sum_{l=0}^{\infty}\left(-\omega\right)^{l} \qquad |\omega|<1$$

,

Setting $\omega=\dfrac{\lambda}{\lambda-1}\left(e^t-1\right)$ we have

$$B_n^{\{k_l\}}(x,\lambda)=(\lambda-1)^{-1}\sum_{k=0}^{n}D_t^k\Theta\sum_{l=0}^{k-1}\left(\frac{\lambda}{\lambda-1}\right)D_t^{k-1}\left\{(e^t-1)^l\right\}\Big|_{t=0} \qquad \ldots(2.5)$$

By the definition of Stirling numbers of the second kind (see [16])

$$(e^t-1)^l=l!\sum_{r=l}^{\infty}S\left(r,l\right)\frac{t^r}{r!}$$

$$B_n^{\{k_l\}}(x,\lambda)=\sum_{k=1}^{n}D_t^k\Theta\sum_{l=0}^{k-1}\left(-1\right)^l\lambda^l\left(\lambda-1\right)^{-l-1}l!S\left(k-1,l\right) \qquad \ldots(2.6)$$

We change sum order of k and l, and using the formula (see [16])

$$S\left(n,k\right)=\frac{1}{k!}\sum_{j=0}^{k}\left(-1\right)^{k-j}\binom{k}{j}j^n$$

we obtain

$$S\left(n,k\right)=\frac{1}{k!}\sum_{j=0}^{k}\left(-1\right)^{k-j}\binom{k}{j}j^n$$

$$B_n^{\{k_l\}}(x,\lambda)=\sum_{l=0}^{n-1}\lambda^l\left(\lambda-1\right)^{-l-1}x^{-l-1}\sum_{j=0}^{l}\left(-1\right)^j\binom{l}{j}j^l\sum_{k=0}^{n-l-1}\binom{n-1}{n-k-2}\left(\frac{j}{x}\right)^k D_t^k\Theta \quad \ldots(2.7)$$

Using the definition of extended Gauss hypergeometric functions, we get

$$B_n^{\{k_l\}}(x,\lambda)=\sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l\left(\lambda-1\right)^{-l-1}x^{-l-1}\sum_{j=0}^{l}\left(-1\right)^j\binom{l}{j}j^l\,{}_2F_1^{\{k_l\}}\left[l-n+1,1;l+1;-\frac{j}{x};p,q\right] \qquad ...(2.8)$$

Finally, we apply the transformation givenbelow (see [2])

$${}_2F_1^{\{k_l\}}(a,b;c;z;p,q)=(1-z)^{-a}\,{}_2F_1^{\{k_l\}}(a,c-b;b;\frac{z}{z-1};p,q) \quad , \quad (|\arg(1-z)|<\pi)\ \ldots(2.9)$$

and immediately obtain (2.3).

Remark 2.2: For $\Theta(k_l;t)$ =exp(t) in Theorem 2.1, the relation derived by Q. M. Luoin [9] for the generalized Bernoulli polynomials and hypergeometric functions is the special case of our result (2.3).

$$B_n(x,\lambda)=n\sum_{l=0}^{n-1}\binom{n-1}{l}\lambda^l(\lambda-1)^{-l-1}\sum_{j=0}^{l}(-1)^j\binom{l}{j}j^l(x+j)^{n-l-1}\,{}_2F_1\left[l-n+1,l;l+1;\frac{j}{(x+j)}\right]$$

Theorem 2.3: If n is a positive integer, we havethe following explicit formula for these extended generalized Bernoulli polynomials

$$B_{n,\alpha}^{\{k_l\}}(x)=\sum_{k=0}^{n}\binom{\alpha+k-1}{k}\frac{k!}{(2k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^{2k}(x+j)^{n-k}\,{}_2F_1^{\{k_l\}}\left[k-n,k-\alpha;2k+1;\frac{j}{(x+j)};p,q\right] \quad ...(2.10)$$

Proof: We differentiate both sides of (2.1) with respect to the variable t. Applying Leibniz's rule yields

$$B_{n,\alpha}^{\{k_l\}}(x)=D_t^n\left\{\left(\frac{t}{e^t-1}\right)^{\alpha}\Theta\right\}\Bigg|_{t=0}, \qquad D_t=\frac{d}{dt} \quad ...(2.11)$$

Since binomial series expansionis

$$(1+\omega)^{-\alpha}=\sum_{l=0}^{\infty}\binom{\alpha+l-1}{l}(-\omega)^l, \qquad |\omega|<1$$

Setting $(1+\omega)=\left(e^t-1\right)/t$ and applying the binomial theorem, we find

$$B_{n,\alpha}^{\{k_l\}}(x)=\sum_{s=0}^{n}D_t^{n-s}\Theta\sum_{l=0}^{s}\binom{\alpha+l-1}{l}\sum_{k=0}^{l}(-1)^k\binom{l}{k}D_t^s\left\{\left(\frac{e^t-1}{t}\right)^k\right\}\Bigg|_{t=0} \quad \ldots(2.12)$$

Now using the formula (see [4])

$$(e^t-1)^k=\sum_{r=k}^{\infty}\frac{t^r}{r!}\Delta^k 0^r, \quad \ldots(2.13)$$

where, for convenience, $$\Delta^k a^r=\Delta^k x^r\Big|_{x=a}=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(a+j)^r \quad ...(2.14)$$

Δbeingthe difference operator defined by L. Comtet (see [3])

$$\Delta f(x) = f(x+1) - f(x) \qquad \ldots(2.15)$$

So that, in general,
$$\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j) \qquad \ldots(2.16)$$

$$D_t^s \left\{ \left(\frac{e^t - 1}{t} \right)^k \right\} \Bigg|_{t=0} = \frac{s!}{(s+k)!} \Delta^k 0^{s+k}$$

And upon substituting this value in (2.12), if we rearrange the resulting triple series, we have

$$B_{n,\alpha}^{\{k_l\}}(x) = \sum_{k=0}^{n} (-1)^k \binom{\alpha+k-1}{k} \sum_{s=0}^{n-k} \frac{(s+k)!}{(s+2k)!} x^{-s} \Delta^k 0^{s+2k} \sum_{l=0}^{s} \binom{\alpha+k+l-1}{l} \qquad \ldots(2.17)$$

The innermost sum can be evaluated by appealing to the elementary combinatorial identity

$$\sum_{l=0}^{s} \binom{\lambda+l-1}{l} = \binom{\lambda+s}{s}$$

And if we further substitute for $\Delta^k 0^{s+2k}$ from the definition (2.14) with a = 0, and then use the definition of extended Gauss hypergeometric functions, we obtain

$$B_{n,\alpha}^{\{k_l\}}(x) = \sum_{k=0}^{n} \binom{\alpha+k-1}{k} \frac{k!}{(2k)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{2k} \, {}_2F_1^{\{k_l\}} \left[k-n, \alpha+k+1; 2k+1; \frac{-j}{x}; p, q \right] \ldots(2.18)$$

Finally, we apply the transformation below (see [2]) in (2.18) which leads us immediately to the explicit formula (2.10).

$${}_2F_1^{\{k_l\}}(a,b;c;z;p,q) = (1-z)^{-a} \, {}_2F_1^{\{k_l\}}(a, c-b; b; \frac{z}{z-1}; p, q), \qquad \left(\left| \arg(1-z) \right| < \pi \right)$$

Remark 2.4: For $\Theta(k_l;t) = \exp(t)$ in Theorem 2.3, the result reduces to the following explicit formula for the generalized Bernoulli polynomials (see[19])

$$B_n^{\alpha}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+k-1}{k} \frac{k!}{(2k)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{2k} (x+j)^{n-k} \, {}_2F_1 \left[k-n, k-\alpha; 2k+1; \frac{j}{(x+j)} \right] \ldots(2.19)$$

Remark 2.5: H. M.Srivastava and P. G. Todorov earlier derived the formula for the classical Bernoulli polynomials [19] for α =1, which is also the special case of Theorem 2.3.

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(2k)!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^{2k} (x+j)^{n-k} \, {}_2F_1\left[k-n, k-1; 2k+1; \frac{j}{(x+j)}\right] \qquad \ldots(2.20)$$

3. EXTENDED APOSTOL-EULER POLYNOMIALS AND EXTENDED GAUSSIAN HYPERGEOMETRIC FUNCTIONS

Extension of Apostol-Euler Polynomials :

We give extension to (1.4) which is defined as follows

$$\left(\frac{2}{e^t+1}\right)^{\alpha} \Theta = \sum_{n=0}^{\infty} E_n^{\{k_l\}} \frac{t^n}{n!} \qquad (\,|t| < \pi;\ 1^{\alpha} = 1) \qquad \ldots(3.1)$$

Theorem 3.1: For n = 0,1,..., $x \in$ ¡, $\alpha \in$ £ , the following formula in terms of the extended Gaussian hypergeometric function is true

$$E_{n,\alpha}^{\{k_l\}}(x) = \sum_{k=0}^{n} \frac{1}{2^k} \binom{\alpha+k-1}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^k (x+j)^{n-k} \, {}_2F_1^{\{k_l\}}\left[k-n, k; k+1; \frac{j}{(x+j)}; p, q\right] \qquad \ldots(3.2)$$

Proof: By Taylor's expansion and Leibniz's rule, the generating relation (3.1) yields

$$E_{n,\alpha}^{\{k_l\}}(x) = D_t^n \left\{ \left(\frac{2}{e^t+1}\right)^{\alpha} \Theta \right\} \Bigg|_{t=0}, \qquad D_t = \frac{d}{dt} \qquad \ldots(3.3)$$

Since we have binomial series expansion

$$(1+\omega)^{-\alpha} = \sum_{l=0}^{\infty} \binom{\alpha+l-1}{l} (-\omega)^{l}, \qquad |\omega| < 1$$

Setting $(1+\omega) = \left(e^t+1\right)/2$ and applying the binomial theorem, we find

$$E_{n,\alpha}^{\{k_l\}}(x) = \sum_{s=0}^{n} D_t^{n-s}\, \Theta \sum_{k=0}^{s} \frac{(-1)^k}{2^k} \binom{\alpha+k-1}{k} D_t^s \left\{ \left(e^t-1\right)^k \right\} \Bigg|_{t=0} \qquad \ldots(3.4)$$

$S(r,k)=\frac{1}{k!}\Delta^k 0^r$, where $S(r, k)$ denotes the Stirling number of the second kind defined by

$$x^k=\sum_{k=0}^{r}\binom{x}{k}k!\,S(r,k)$$

From (2.13) using (2.14), (2.15) and (2.16), we obtain

$$D_t^s\left\{\left(e^t-1\right)^k\right\}\Big|_{t=0}=\Delta^k 0^{s+k}=k!S(s,k) \qquad \text{…(3.5)}$$

Substituting this value into (3.4), we get

$$E_{n,\alpha}^{\{k_l\}}(x)=\sum_{s=0}^{n}D_t^{n-s}\Theta\sum_{k=0}^{s}\frac{(-1)^k k!}{2^k}\binom{\alpha+k-1}{k}S(s,k) \qquad \text{…(3.6)}$$

or

$$E_{n,\alpha}^{\{k_l\}}(x)=\sum_{s=0}^{n}D_t^{n-s}\Theta\sum_{k=0}^{s}\frac{(-1)^k}{2^k}\binom{\alpha+k-1}{k}\Delta^k 0^k \qquad \text{…(3.7)}$$

If we rearrange the resulting double series (3.7), we obtain

$$E_{n,\alpha}^{\{k_l\}}(x)=\sum_{k=0}^{n}\frac{(-1)^k}{2^k}\binom{\alpha+k-1}{k}\sum_{s=0}^{n-k}D_t^{n-s}\,\Theta\,\Delta^k\,0^{s+k} \qquad \text{...(3.8)}$$

Further substituting for $\Delta^k 0^{s+k}$ from (2.14) with a= 0 into (3.8)

$$E_{n,\alpha}^{\{k_l\}}(x)=\sum_{k=0}^{n}\frac{1}{2^k}\binom{\alpha+k-1}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^k\,{}_2F_1^{\{k_l\}}\left[k-n,1;k+1;\frac{-j}{x};p,q\right] \qquad \text{...(3.9)}$$

Finally, we apply the transformation below (see [2]) in (3.9) which leads us immediately to the result (3.2).

$${}_2F_1^{\{k_l\}}(a,b;c;z;p,q)=(1-z)^{-a}\,{}_2F_1^{\{k_l\}}(a,c-b;b;\frac{z}{z-1};p,q) \qquad \left(\left|\arg\left(1-z\right)\right|<\pi\right)$$

Remark 3.2: For $\Theta(k_t;t)=\exp(t)$ in Theorem 3.1, the result reduces to the following explicit formula for the generalized Euler polynomials (see[10])

$$E_n^{\alpha}(x)=\sum_{k=0}^{n}\frac{1}{2^k}\binom{n}{k}\binom{\alpha+k-1}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(x+j)^{n-k}j^k\,{}_2F_1\left[k-n,k;k+1;\frac{j}{x+j}\right] \qquad \text{...(3.10)}$$

Remark 3.3: Q.M. Luo also derived the formula for the classical Euler polynomials [10] for α =1, which is also the special case of Theorem 3.1.

$$E_n(x) = \sum_{k=0}^{n} \frac{1}{2^k} \binom{n}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x+j)^{n-k} j^k \, {}_2F_1\left[k-n, k; k+1; \frac{j}{x+j} \right] \qquad \ldots(3.11)$$

4. CONCLUSION

The results of this paper hint to interesting applications of these family of polynomials. We believe that a further insight in to the relevant theory and properties of Hermite-Bernoulli and Hermite – Euler polynomials defined by Pathan et al [13] and [14] may provide a more thorough understanding of Extended Apostol-Euler and Extended Apostol -Bernoulli polynomials and Extended Gaussian Hypergeometric functions given in this paper. The points we have touched upon in this paper show that the use of (2.1) and (2.2) of this paper with concepts and formalism of suggested in this paper can be extended to Hermite-Bernoulli and Hermite – Euler polynomials defined by Pathan et al.Further considerationson their properties will be presented in the forthcoming paper.

5. ACKNOWLEDGEMENT

The author is grateful to Professor M A Pathanfor hisvaluable comments and suggestions forthe improvement of this paper.

REFERENCES

[1] Apostol T.M.; On the Lerch zeta function, Pacific J. Math., 1951, 1, 161-167.

[2] CaturvediAparna, RaiPrakriti, Some Properties of Extended Hypergeometric Functions and its Transformations, accepted in Journal of Indian Mathematical Society.

[3] Comtet L., "Advanced Combinatorics: The Art of Finite and Infinite Expansions" (Translated from the French by J. W. Nienhuys), Reidel, Dordrecht/Boston, 1974.

[4] Gould H. W., Explicit formulas for Bernoulli numbers, Amer. Math. Monthly 1972, 79, 44-51.

[5] Minjie, Luo; A Class of Extended Hypergeometric Functions and Its Applications, arXiv:1302.2307v1, [math.CA]. 2013.

[6] Luo Q.M., Fourier expansions and integral representations for the Apostol-Bernoulli and Apostol – Euler polynomials, Math. Of Comput., 2009, 78(12), 2193-2208.

[7] Luo Q.M., The multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order, Integral Transform and Special Functions, 2009, 20(5-6), 377-391.

[8] Luo Q.M., Some formulas for the Apostol- Euler polynomials associated with Hurwitz zeta function at rational arguments, Applicable Analysis and Discrete Mathematics, 2009, 3(2), 336-346.

[9] Luo Q.M., On the Apostol- Bernoulli polynomials, Central European Journal of Mathematics, 2004, 2(4), 509-515.

[10] Luo Q.M., An Explicit formula for the Euler polynomials of higher order, Appl. Math. & Information Sciences- An International Journal, 2009, 3(1), 53-58.

[11]Luo Q.M, Srivastava H.M., Some relationships between Apostol- Bernoulli and Apostol- Euler polynomials, Computers and Mathematics with Applications, 2006, 51(3-4), 631-642.

[12]Luo Q.M, Srivastava H.M., Some generalizations of the Apostol- Bernoulli and Apostol- Euler polynomials, J. Math. Anal.and Appl., 2005, 308(1), 290-302.

[13]Pathan M. A., Khan Waseem A., Some Implicit Summation Formulas and Symmetric Identities for the Generalized Hermite–Bernoulli Polynomials,Mediterranean Journal of Mathematics,2015, Volume 12, Issue 3, 679–695.

[14]Pathan M. A., Khan Waseem A.,A New Class of Generalized Polynomials Associated with Hermite and Euler Polynomials, 2016, Volume 13, Issue 3, 913–928.

[15]Sandor J., Crisci, Handbook of Number Theory, Vol. II. Kluwer Academic Publishers, Dordrecht Boston and London, 2001

[16]Srivastava H.M., Pinter A., Remarks on some relationships between the Bernoulli and Euler polynomials, Appl. Math. Lett., 2004, 17, 375-380.

[17]Srivastava H.M., Choi J., Series associated with the Zeta and related functions, Kluwer Academic Publishers, Dordrecht Boston and London, 2001

[18]Srivastava, H. M.; Parmar R. K.; And Chopra, P.; A Class of Extended Fractional Derivative Operators and Associated Generating Relations Involving Hypergeometric Functions, Axioms 2012, 1, 238-258.

[19]Srivastava H. M., Todorov P.G., “An explicit formula for the generalized Bernoulli polynomials”, J. Math. Anal. Appl., 130, 1988, 509-513.

Ganita Sandesh Vol. 30 No. (1 & 2) 2016 pp. 13 – 26 **ISSN : 0970-9169**

CHAOS CONTROL IN THE PROBLEM OF A SATELLITE

[1]AYUB KHAN, [2]L. M. SAHA AND [3]TIL PRASAD SARMA

[1]*Department of Mathematics*

Jamia Millia Isalamia, **NEW DELHI – 110 025 (INDIA)**

[2]*Department of Mathematics*

IIIMIT Shiv Nadar Unversity, **GAUTAM BUDH NAGAR – 201 314 (U P) (INDIA)**

[3]*Department of Education in Science and Mathematics*

National Council of Educational Research and Training,

Sri Aurobindo Marg, **NEW DELHI – 110 016 (INDIA)**

[1]ayubkdu@gmail.com, [2] tpsncert@gmail.com, [3]lmsaha.msf@gmail.com

ABSTRACT

Rotational motion of a satellite influenced by certain forces like solar radiation pressure, tidal force and resistance have been investigated. The equilibrium and doubled asymptotic solution of the problem has been obtained. Melnikov' Integal evaluated to establish the non-integrability of the equation of motion. The surface of section, poicare′ map, phase plot,time series have been drawn for different ranges of values of the parameter exist in the equation to show that the motion is chaotic.

Key Words: Solar radiation pressure, tidal torque, Melnikov's Integral, Chaotic Motion

INTRODUCTION

Maciejewski(1992), Singh(1973,1986) and Khan(1994,2008) Khan,et.al(2010) have studied the Elliptically orbiting planner oscillation of satellite in the solar system. All of them studied the influence of the certain perturbative forces such as solar radiation pressure, tidal force and resistance.

Here, we have considered the spin-orbit coupling problem for a satellite. The equation of motion we have considered the effect of solar radiation pressure and tidal torque. The problem we have considered is to control chaos arising in Hamiltonian system of the satellite. Here we have calculated Hamiltonian of the equation, equilibrium solution and double asymptotic solutions, Melnikov's Intregral and lastly by the equations of motions and Melnikov's intregral we established that the motion is chaotic for certain paramatric values.

PROBLEM FORMULATION

We considered the spin –orbit coupling problem with the spin axis fixed perpendicular to the orbital plane. We assume that the satellite to be a triaxial ellipsoid with principal moments of inertia $A<B<C$, where C is the moment of inertia about the spin axis. The orbit is taken to be a fixed ellipse with semi-major axis a, eccentricity e, true anamoly ϑ and the instantaneous radius r. Fig.1 shows the position of the satellite S, around the planet P. It is defined by the distance r and the angle ϑ from the direction of peripase (true anamoly). Line PS is the planet satellite centre line and IF is the inertial line is. The angular position of the long axis relative to the line IF is denoted by x.

Thus $\varphi = x - \vartheta$ measure the orientation of the satellite's long axis relative to PS.

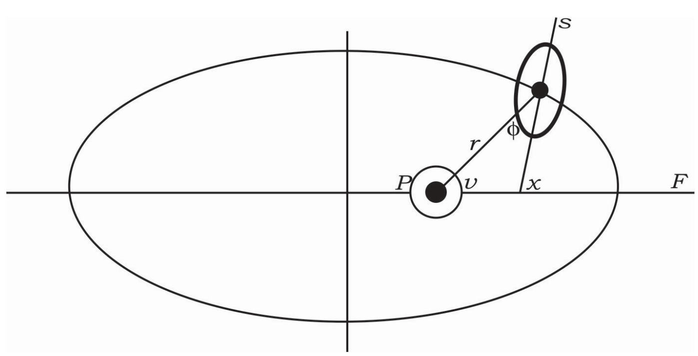


Fig. 1 . *Motion of the satellite S around the planet P*

The equation of motion together with the effect due to tidal torque due to solar radiation pressure can be written as [2, 4],

$$C\frac{d^2x}{dt^2}+\frac{3(B-A)GM}{2r^3}Sin2\phi = T-\varepsilon^1 Sin(\vartheta+\phi)\,,$$

where G is the universal gravitational constant, M is the mass of the planet and T is the tidal torque. The instantaneous tidal torque on a spinning satellite [4] is given by

$$T=\frac{-3k_2GM^2R^2}{2r^6}Sin2\delta\,,$$

where k_2 is the tidal Love (Pl. check in the thesis), *R* is the satellite's mean radius and δ is the tidal phase lag, $\delta \sim 1/Q$, which is the specific dissipation function [4]. We take tidal phase lag proportional to the frequency (tidal); the expression for the tidal torque thus becomes

$$T=-(a/r)^6[(\dot{x}-\dot{\nu})/n_1]J$$

[1], where *J* is a positive constant and n_1 is the orbital mean motion.

Then the equation of motion is given by

$$\frac{d^2\phi}{dt^2}+\frac{3(B-A)}{2cr^3}Sin2\phi=-\frac{J}{C}(a/r)^6\frac{\dot{\phi}}{n_1}-\varepsilon^1 Sin(\vartheta+\phi)$$

Taking $GM=\mu$, $3(B-A)/C=n^2$, we get

$$\frac{d^2\varphi}{dt^2}+\frac{\mu n^2}{2r^2}sin\,2\varphi=-\frac{J}{C}\left(\frac{a}{r}\right)^6\frac{\dot{\varphi}}{n_1}-\varepsilon_1\,sin(t+\varphi)$$

Let J/C= δ, where $\delta=\delta_o e$, we get after some simplification

$$\frac{d^2\varphi}{d\theta^2}+\frac{\mu n^2}{2r^2}\sin 2\varphi=\delta\left(\frac{a}{r}\right)^6\frac{1}{n_1}\frac{d\varphi}{d\theta}+\varepsilon_1\sin(\theta+\varphi)$$

This can further be written in the form

$$\frac{d^2\varphi}{d\theta^2}+\gamma\sin 2\varphi=\delta\left(\frac{a}{r}\right)^6\frac{1}{n}\frac{d\varphi}{d\theta}+\varepsilon\sin(\theta+\varphi)\,,$$

where $\gamma = \dfrac{\mu n^2}{2r^2}$.

By taking using $h^2 = \mu l$, $l = a\ (1 - e^2)$, $r^2 \vartheta = h$ and $1 / r = 1 + e\ Cos\ \vartheta$ and on changing the independent variable from θ to ϑ and taking $q = 2\ \varphi$, $\in = r^3 \varepsilon^1 / \mu$, we get an equation for q as

$$(1 + eCos\vartheta)\frac{d^2q}{d\vartheta^2} - 2eSin\frac{dq}{d\vartheta} + \frac{\beta e(1 + eCos\vartheta)^5}{(1 - e^2)^4}\frac{dq}{d\vartheta} - 4eSin\vartheta + n^2Sinq + 2\varepsilon_1 e(1 + eCos\vartheta)^{-3}Sin(q / 2 + \vartheta) = 0, \quad \ldots(1)$$

where $\beta = \alpha_0\, a^2 / n_1\, h$ and $\in = \in_1 e$

Equation (1) is equivalent to the Hamiltonian Equations

$$\frac{dq}{d\vartheta} = \frac{\partial H}{\partial p}, \frac{dp}{d\vartheta} = -\frac{\partial H}{\partial q},$$

where H is the Hamiltonian function

Multiplying the above equation by (1+ *0 Cos* ϑ), the above equation becomes

$$\frac{d}{d\vartheta}[p] + \frac{\partial H}{\partial q} = 0,$$

where $$p = (1 + eCos\vartheta)^2\frac{dq}{d\vartheta} + (1 + eCos\vartheta)^2 + \beta eq(1 + eCos\vartheta)^6$$

or $$\frac{dq}{d\vartheta} = \frac{p}{(1 + eCos\vartheta)^2} - 2 - \beta eq(1 + eCos\vartheta)^4$$

From canonical equations of motion, we have

$$\frac{\partial H}{\partial p} = \frac{\partial q}{\partial \vartheta} = \frac{p}{(1 + eCos\vartheta)^2} - 2 - \beta eq(1 + eCos\vartheta)^4$$

On integrating and neglecting higher order terms of e, we get

$$H = H_0 + H_1,$$

where $$H_0 = \frac{p^2}{2} - 2p - n^2Cosq$$

and $$H_1 = -e\left\{p^2\ Cos\vartheta + \beta pq + n^2CosqCos\vartheta + 4\varepsilon_1 Cos(q / 2 + \vartheta\right\}$$

EQUILIBRIUM SOLUTIONS

Considering $\dfrac{\partial H_0}{\partial p}=0$ and $\dfrac{\partial H_0}{\partial q}=0$, we get (π,2) is unstable solution.

Considering $\dfrac{dq}{d\vartheta}=0$ and $\dfrac{dp}{d\vartheta}=0$, we get equilibrium solution as (0,2) and (π,2).

The characteristic equation of the motion is

$$S^2+[P_\xi^0+Q_\eta{}^0]S+[P_\xi^0Q_\eta{}^0-Q_\xi{}^0P_\eta{}^0]=0$$

where

$$P_\xi^0=\frac{\partial P}{\partial \xi},\ P_\eta^0=\frac{\partial P}{\partial \eta},\ Q_\xi^0=\frac{\partial Q}{\partial \xi},\ Q_\eta^0=\frac{\partial Q}{\partial \eta},$$

Putting these values in characteristic equation , we get $S_1 = \pm$ in purely imaginary, therefore the singular points are always stable.

Similarly corresponding to equilibrium solution (π,2), we get the two real and unequal roots $\pm\, n$ which means (π,2) is a saddle point and is unstable. Thus the hyperbolic equilibrium solution at $e = 0$ is given by $q(\vartheta) = 0$, $p(\vartheta) = 2$.

Now to determine the unperturbed double asymptotic solution we have taken help of equivalent equations

$$\frac{dq}{d\vartheta}=\frac{\partial H_0}{\partial p},\quad \frac{dp}{d\vartheta}=-\frac{\partial H_0}{\partial q}$$

and

$$H_0=\frac{p^2}{2}-2p-n^2Cosq$$

we get

$$Sin(\pm(q(\vartheta))=\pm\frac{2Sinhn\vartheta}{Coh^2n\vartheta}$$

$$Cos(\pm q(\vartheta))=\pm\frac{2}{Coh^2n\vartheta}-1$$

$$p(\pm\vartheta)=\frac{2\Omega_1}{\Omega}\pm\frac{2n^2}{Coshn\vartheta}$$

$$Sin\left(\pm\frac{q(\vartheta)}{2}\right)=\pm\tanh n\vartheta$$

$$Cos\left(\pm\frac{q(\vartheta)}{2}\right)=\pm Sech\,n\vartheta$$

MELNIKOV'S INTEGRAL

The Melnikov's function is used to measure the distance between stable and unstable manifold. The system of the equations of motion will be non integrable, if Melnikov's function has simple zero for any value of the parameters. The Melnikov's function gives the idea of stable and unstable asymptotic surfaces which leads to complex structure. This method help to predict about the chaotic nature of the dynamical system. The Melnikov's integral is defined as

$$M^{\pm}(\vartheta_0) = \int_{-\infty}^{\infty} \{H_0, H_1\}\left(q^{\pm}(\vartheta-\vartheta_0), p^{\pm}(\vartheta-\vartheta_0), \vartheta\right) d\vartheta \qquad ...(2)$$

Its integrand is

$$\left\{H_0, H_1\right\}\left(q^{\pm}(\vartheta-\vartheta_0), p^{\pm}(\vartheta-\vartheta_0), \vartheta\right) = \frac{\partial H_0}{\partial q^{\pm}(\vartheta-\vartheta_0)}\frac{\partial H_1}{\partial p^{\pm}(\vartheta-\vartheta_0)} - \frac{\partial H_0}{\partial p^{\pm}(\vartheta-\vartheta_0)}\frac{\partial H_1}{\partial q^{\pm}(\vartheta-\vartheta_0)}$$

Calculating the value of integrand and substituting in (2), we get

$$M^{\pm}(\vartheta_0) = \mp 8n^2\frac{\Omega_1}{\Omega}\int_{-\infty}^{\infty}\frac{Sinhn(\vartheta-\vartheta_0)}{Cosh^2 n(\vartheta-\vartheta_0)}Cos\vartheta\, d\vartheta - 8n^4\int_{-\infty}^{\infty}\frac{Sinhn(\vartheta-\vartheta_0)}{Cosh^3 n(\vartheta-\vartheta_0)}Cos\vartheta\, d\vartheta$$

$$\mp 2\beta q n^2\int_{-\infty}^{\infty}\frac{Sinhn(\vartheta-\vartheta_0)d\vartheta}{Cosh^2 n(\vartheta-\vartheta_0)} + 4\frac{\Omega_1^{\;2}}{\Omega^2}\beta\int_{-\infty}^{\infty}d\vartheta \pm \frac{4\Omega_1}{\Omega}n^2\beta\int_{-\infty}^{\infty}\frac{d\vartheta}{Coshn(\vartheta-\vartheta_0)} \mp 4\frac{\Omega_1}{\Omega}n^2\int_{-\infty}^{\infty}\frac{Cos\vartheta Sinhn(\vartheta-\vartheta_0)d\vartheta}{Cosh^2 n(\vartheta-\vartheta_0)}$$

$$\pm 4\frac{\Omega_1}{\Omega}\varepsilon_1\int_{-\infty}^{\infty}Cos\vartheta\tanh n\left(\vartheta-\vartheta_0\right)d\vartheta \mp 4\frac{\Omega_1}{\Omega}\varepsilon_1\int_{-\infty}^{\infty}Sin\vartheta Sec\,\mathrm{h}\, n\left(\vartheta-\vartheta_0\right)d\vartheta \pm 4\frac{\Omega_1}{\Omega}n^2\beta\int_{-\infty}^{\infty}\frac{d\vartheta}{Coshn(\vartheta-\vartheta_0)}$$

$$+4n^4\beta\int_{-\infty}^{\infty}\frac{d\vartheta}{Cosh^2 n(\vartheta-\vartheta_0)} - 4n^4\int_{-\infty}^{\infty}\frac{Cos\vartheta Sinhn(\vartheta-\vartheta_0)d\vartheta}{Cosh^3 n(\vartheta-\vartheta_0)} + 4n^2\varepsilon_1\int_{-\infty}^{\infty}\frac{Sinhn(\vartheta-\vartheta_0)Cos\vartheta}{Cosh^2 n(\vartheta-\vartheta_0)}$$

$$-4n^2\varepsilon_1\int_{-\infty}^{\infty}\frac{Sin\vartheta}{Cosh^2 n(\vartheta-\vartheta_0)}d\vartheta - 4\frac{\Omega_1}{\Omega}\beta\int_{-\infty}^{\infty}d\vartheta \mp 4n^2\beta\int_{-\infty}^{\infty}\frac{d\vartheta}{Coshn(\vartheta-\vartheta_0)} \pm 4n^2\int_{-\infty}^{\infty}\frac{Sinhn(\vartheta-\vartheta_0)}{Cosh^2 n(\vartheta-\vartheta_0)}Cos\vartheta d\vartheta$$

$$\mp 4\varepsilon_1\int_{-\infty}^{\infty}Cos\vartheta\tanh n\left(\vartheta-\vartheta_0\right)d\vartheta \pm 4\varepsilon_1\int_{-\infty}^{\infty}\frac{Sin\vartheta}{Coshn(\vartheta-\vartheta_0)} \qquad \ldots(3)$$

To evaluate these integrals given in (3), one has to use residue theorems of complex integrals.

For this, let us consider the integrals

$$\int_{-\infty}^{\infty} \frac{Sinh\, n(\vartheta - \vartheta_0)}{Cosh^2\, n(\vartheta - \vartheta_0)} Cos\, \vartheta d\vartheta$$

Putting $\vartheta - \vartheta_0 = \beta$, we get

$$\int_{-\infty}^{\infty} \frac{Sinh\, n(\vartheta - \vartheta_0)}{Cosh^2\, n(\vartheta - \vartheta_0)} Cos\vartheta d\vartheta$$

becomes

$$-Sin\vartheta_0 \int_{-\infty}^{\infty} \frac{Sinh n\beta}{Cosh^2 n\beta}\, Sin\beta d\beta$$

The above integrals are evaluated by the method of residues as follows:

Consider integral

$$\int_{-\infty}^{\infty} \frac{e^{nz}\, e^{iz}}{\cosh^2 nz}\, dz \qquad \ldots(A)$$

The function $\mathrm{Cosh}^2\, nz$ has $-\pi \frac{\pi i}{2n}$ as its zero of order two. So $\frac{\pi i}{2n}$ is a pole of the order two of the function $\left[\frac{e^{nz} e^{iz}}{Cosh^2 nz}\right]$ with the help of Laurent series we find the residue of $\left[\frac{e^{nz} e^{iz}}{Cosh^2 nz}\right]$. For that, we expand the function about $z = \frac{\pi i}{2n}$ and obtain the coefficient of $\frac{1}{\left(z - \frac{\pi i}{2n}\right)}$ as the required residue.

Let $z - \frac{\pi i}{2n} = z_1$, then $z = z_1 + \frac{\pi i}{2n}$. Here the function to be expanded in a Laurent series about $z_1 = 0$ is

$$\frac{e^{n(z_1+\frac{\pi i}{2n})}e^{i(z_1+\frac{\pi i}{2n})}}{Cosh^2 n(z_1+\frac{\pi i}{2n})} = \frac{ie^{-\frac{\pi}{2n}}e^{(n+\frac{\Omega_1}{\Omega}i)z_1}}{\left(\frac{e^{(nz_1+\frac{\pi i}{2})}+e^{(-nz_1-\frac{\pi i}{2})}}{2}\right)^2} = -\frac{ie^{-\frac{\pi}{2n}[1+(n+i)z_1+\frac{(n+i)^2 z_1}{2}+---]}}{n^2 z_1^{\,2}[1+\frac{n^2 z_1^{\,2}}{6}+\frac{n^4 z_1^{\,4}}{120}+---]^2}$$

$$= -\frac{ie^{-\frac{\pi}{2n}}}{n^2 z_1}\left\{1+(n+i)z_1+\frac{(n+i)^2 z_1^{\,2}}{2}+--\right\}\times\left\{1+\frac{n^2 z_1^{\,2}}{6}+\frac{n^4 z_1^{\,4}}{120}+--\right\}^{-2}$$

In the above expression the coefficient of $1/z_1$ is

$$\frac{-ie^{-\frac{\pi}{2n}}}{n^2}(n+i)$$

So the residue is $-\frac{i(n+i)}{n^2}e^{-\frac{\pi}{2n}}$. Then by Residue Theorem, we have

$$\oint \frac{e^{nz}e^{iz}}{cosh^2 nz}dz = \frac{2\pi(n+i)}{n^2}e^{-\frac{\pi}{2n}} \qquad \ldots(B)$$

Choosing appropriate contour shown in Fig.2 for Residue integral, we have

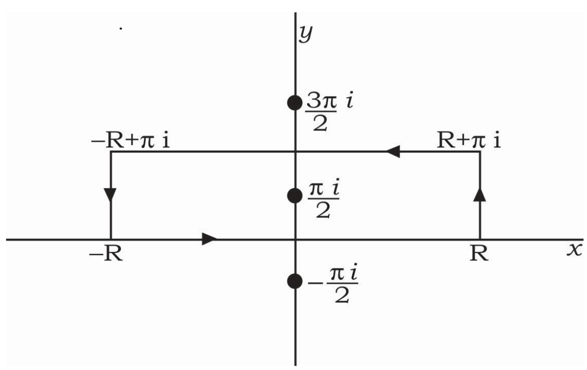


Fig. 2 *Contour for evaluation of Melnikov's function*

Now

$$\oint \frac{e^{nz}e^{iz}}{cosh^2 nz}dz = \underset{R\to\infty}{Lt}\left[\int_{-R}^{R}\frac{e^{n\beta}e^{i\beta}}{cosh^2 n\beta}d\beta + \int_{R}^{-R}\frac{e^{n(\beta+\frac{\pi}{n}i)}e^{i(\beta+\frac{\pi}{n}i)}}{cosh^2 n(\beta+\frac{\pi}{n}i)}d\beta + i\int_{0}^{\frac{\pi}{n}}\frac{e^{n(R+i\beta')}e^{i(R+i\beta')}}{cosh^2(R+i\beta')}d\beta' + i\int_{\frac{\pi}{n}}^{0}\frac{e^{n(-R+i\beta')}e^{i(-R+i\beta')}}{cosh^2(-R+i\beta')}d\beta'\right] \quad \ldots(4)$$

we can easily prove that the third and fourth integrals of (4) tend to zero.

For this we have

$$cosh^2 n(R+i\beta') = \left[\frac{e^{n(R+i\beta')}+e^{-n(R+i\beta')}}{2}\right]^2 \geq \frac{e^{2n(R+i\beta')}}{4} \text{ for the large values of } R$$

Therefore, $\left|cosh^2 n(R+i\beta')\right| \geq \left|\frac{e^{2n(R+i\beta')}}{4}\right| \geq \left|\frac{e^{2nR}}{4}\right| \geq \frac{e^{2nR}}{4}$. Now

$$\left|i\int_{0}^{\frac{\pi}{n}}\frac{e^{n(R+i\beta')}e^{i(R+i\beta')}}{cosh^2(R+i\beta')}d\beta'\right| \leq \int_{0}^{\frac{\pi}{n}}\frac{e^{n(R+i\beta')}}{cosh^2 n(R+i\beta')}d\beta' \leq 4\int_{0}^{\frac{\pi}{n}}\frac{e^{nR}}{e^{2nR}}d\beta' \leq \frac{4\pi}{n}e^{-nR} \to 0 \quad \text{as R} \to \infty$$

$$\therefore \qquad \int_{0}^{\frac{\pi}{n}}\frac{e^{n(R+i\beta')}e^{i(R+i\beta')}}{cosh^2 n(R+i\beta')}d\beta' \to 0 \qquad \text{as } R\to\infty$$

Similarly it can be shown that the fourth integral of (4) tends to zero when R is large.

From (B) and (4), we have

$$\underset{R\to\infty}{Lt}\left[\int_{-R}^{R}\frac{e^{n\beta}e^{i\beta}}{cosh^2 n\beta}d\beta - \int_{-R}^{R}\frac{e^{n(\beta+\frac{\pi}{n}i)}e^{i(\beta+\frac{\pi}{n}i)}}{cosh^2 n(\beta+\frac{\pi}{n}i)}d\beta\right] = \frac{2\pi(n+i)}{n^2}e^{-\frac{\pi}{2n}}$$

or

$$\underset{R\to\infty}{Lt}\left[\int_{-R}^{R}\frac{e^{n\beta}e^{i\beta}}{Cosh^2 n\beta}d\beta + \int_{-R}^{R}\frac{e^{n\beta}e^{i\beta}e^{-\frac{\pi}{n}}}{Cosh^2 n\beta}d\beta\right] = \frac{2\pi(n+i)}{n^2}e^{-\frac{\pi}{2n}}$$

or

$$\underset{R\to\infty}{Lt}\left[e^{\frac{\pi}{2n}} + e^{-\frac{\pi}{2n}}\right]\int_{-R}^{R}\frac{e^{n\beta}e^{i\beta}}{Cosh^2 n\beta}d\beta = \frac{2\pi(n+i)}{n^2}$$

Again $$\oint \frac{e^{-nz}e^{iz}}{Cosh^2 nz}dz = \frac{-2n(-n+i)}{n^2}$$

or $$\oint \frac{e^{-nz}e^{iz}}{Cosh^2 nz}dz = \underset{R\to\infty}{Lt}\left[\int_{-R}^{R}\frac{e^{n\beta}e^{i\beta}}{Cosh^2 n\beta}d\beta + \int_{R}^{-R}\frac{e^{-n(\beta+\frac{\pi}{n}i)}e^{i(\beta+\frac{\pi}{n}i)}}{Cosh^2 n\beta}d\beta + i\int_{0}^{\frac{\pi}{n}}\frac{e^{-n(R+i\beta')}e^{i(R+i\beta')}}{Cosh^2(R+i\beta')}d\beta' + i\int_{\frac{\pi}{n}}^{0}\frac{e^{-n(-R+i\beta')}e^{i(-R+i\beta')}}{Cosh^2(-R+i\beta')}d\beta'\right] \quad \ldots(5)$$

The imaginary integrals of (5) →0 as shown in (4). Therefore from above, we get

$$\underset{R\to\infty}{Lt}\left[\int_{-R}^{R}\frac{e^{-n\beta}e^{i\beta}}{Cosh^2 n\beta}d\beta - \int_{R}^{-R}\frac{e^{-n(\beta+\frac{\pi}{n}i)}e^{i(\beta+\frac{\pi}{n}i)}}{Cosh^2 n\beta}d\beta\right] = \frac{-2\pi(-n+i)}{n^2}$$

GRAPHICAL REPRESENTATION OF MELNIKOV'S FUNCTION

To establish the non-integrability of the equations of motion, we have drawn the graphs of Melnikov's functions $M^{\pm}(\vartheta_0)$ as shown in the following diagram.

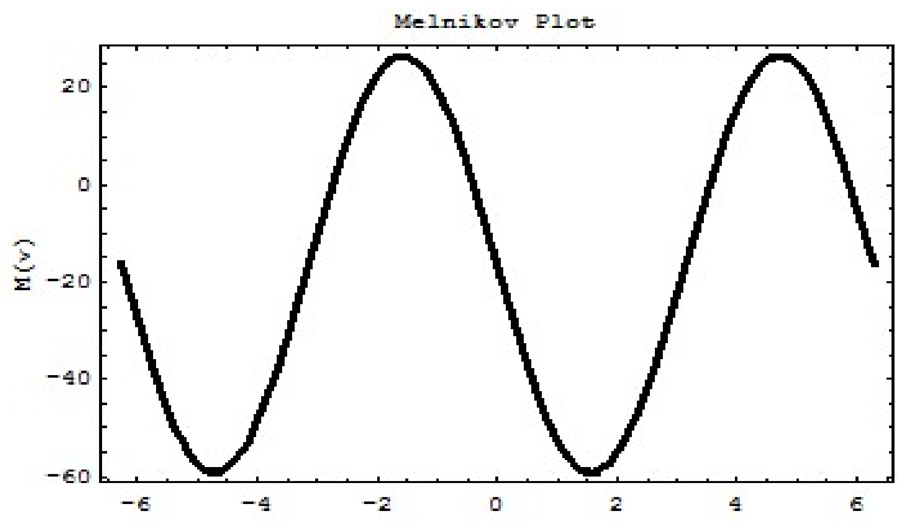


Fig.3 *Plots of Melnikov's function for* $\varepsilon = 0.5$, $\Omega_1 = 1$, $\Omega = 0.5$, $n = 1$, $\beta = 0.2$, $q = 1.5$, $e = 0.003$

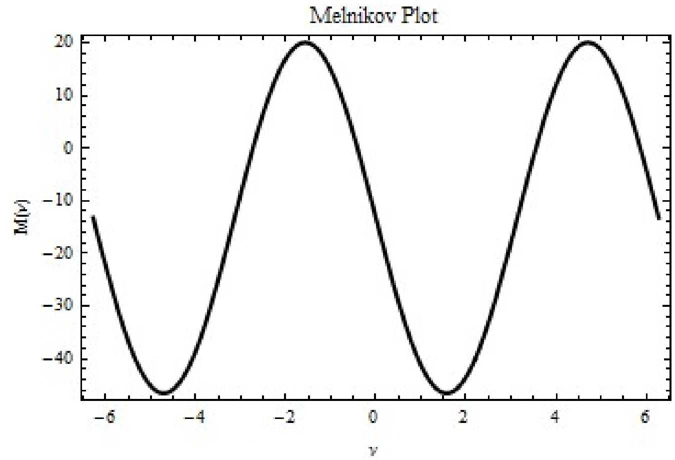


Fig. 4 *Plots of Melnikov's function for* $\varepsilon = 0.7$, $\Omega_1 = 1$, $\Omega = 0.7$,$n = 1$, $\beta = 0.2$,$q = 1.7$,$e = 0.007$

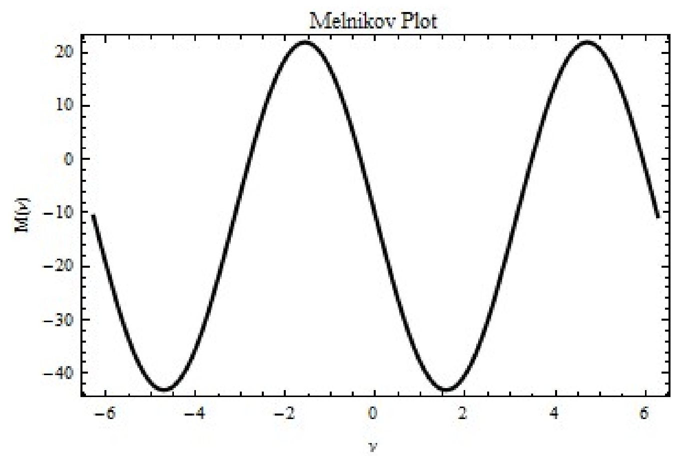


Fig.5 *Plots of Melnikov's function for* $\varepsilon = 0.4$, $\Omega_1 = 1$, $\Omega = 0.7$,$n = 1$, $\beta = 0.3$,$q = 1\ 1$,$e = 0.001$

POINCARÉ SURFACE SECTION AND POINCAÉ MAP

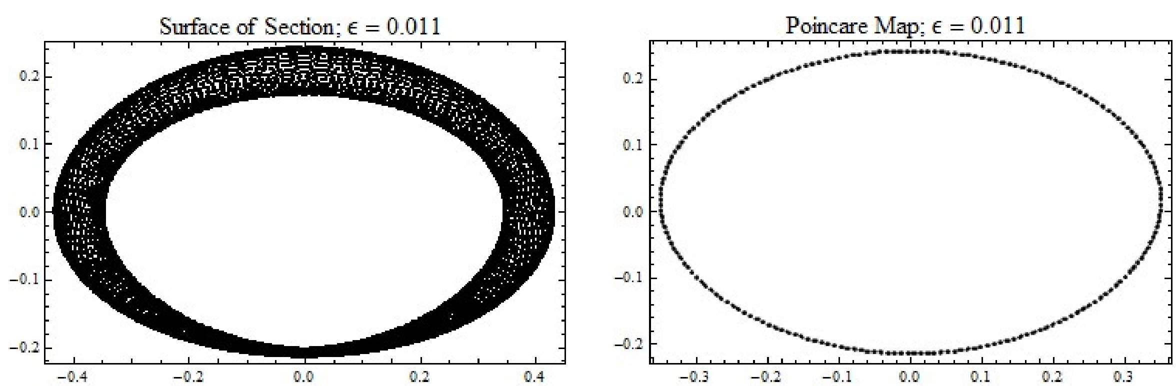


Fig.6 *Poincare surface of section & Poincare map for parameter values*
$\gamma = 0.15$, $\delta = 0.001$, $a = 0.35$, $r = 0.8$, $\varepsilon = 0.011$,$n = 0.6$.

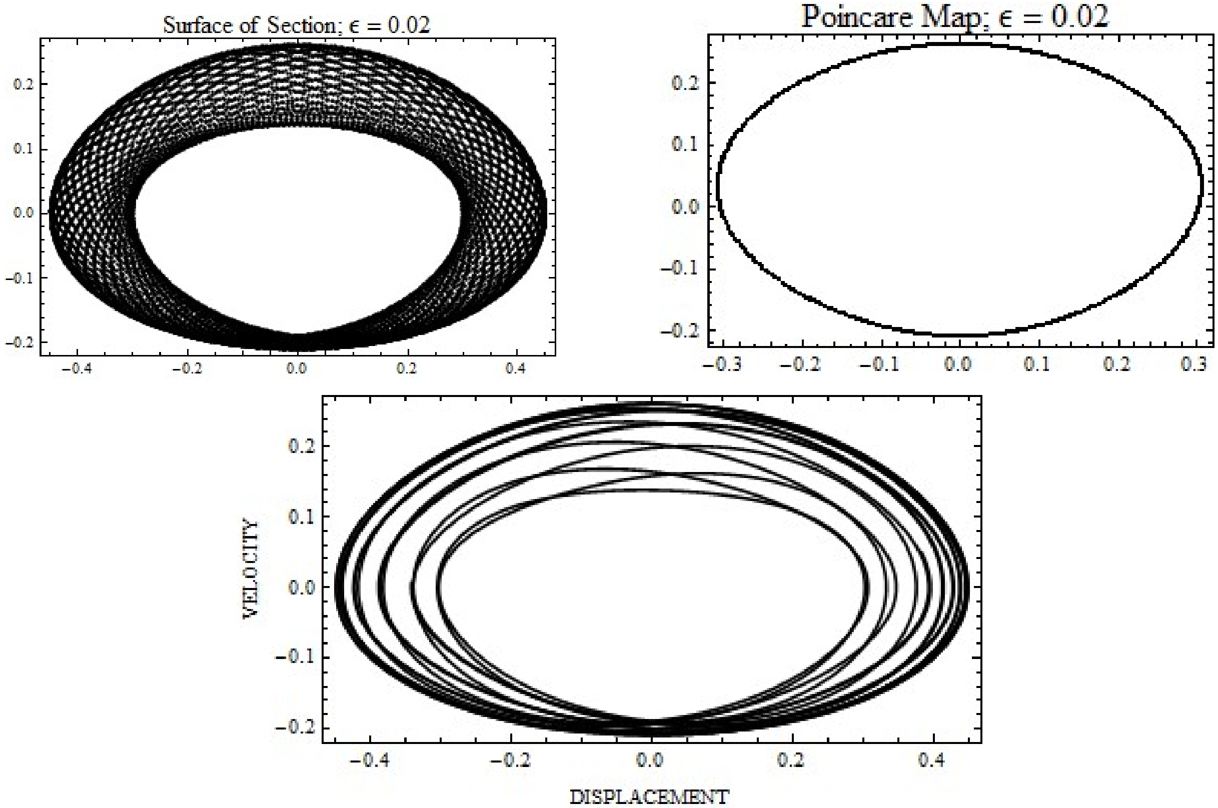
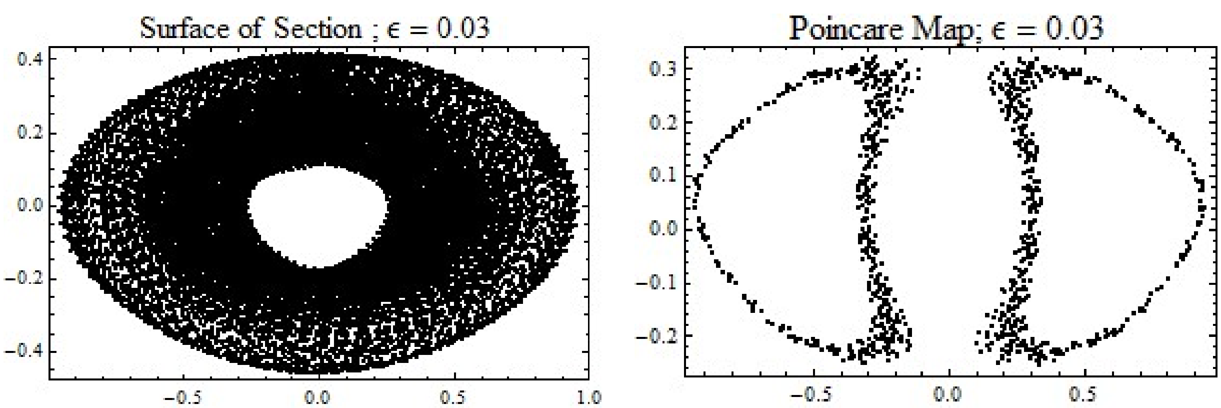


Fig.7 *Poincare surface of section, Poincare map and Phase plot for parameter values*

$\gamma = 0.15,\ \delta = 0.001,\ a = 0.35,\ r = 0.8,\ \varepsilon = 0.02, n = 0.6.$



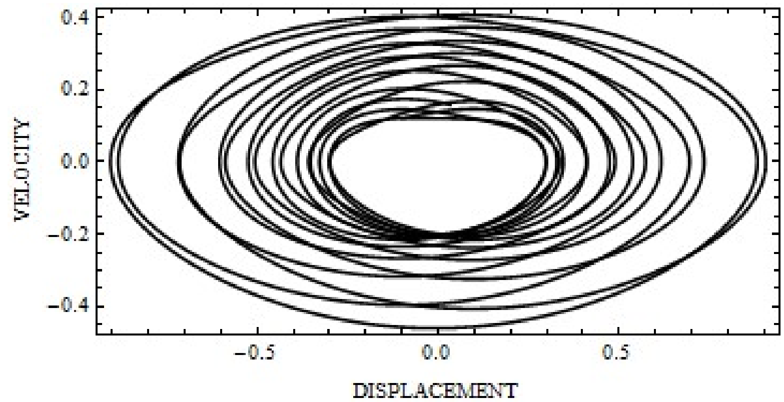


Fig.8 *Poincare surface of section, Poincare map and Phase plot for parameter values*

$\gamma = 0.15$, $\delta = 0.001$, $a = 0.35$, $r = 0.8$, $\varepsilon = 0.03$, $n = 0.6$.

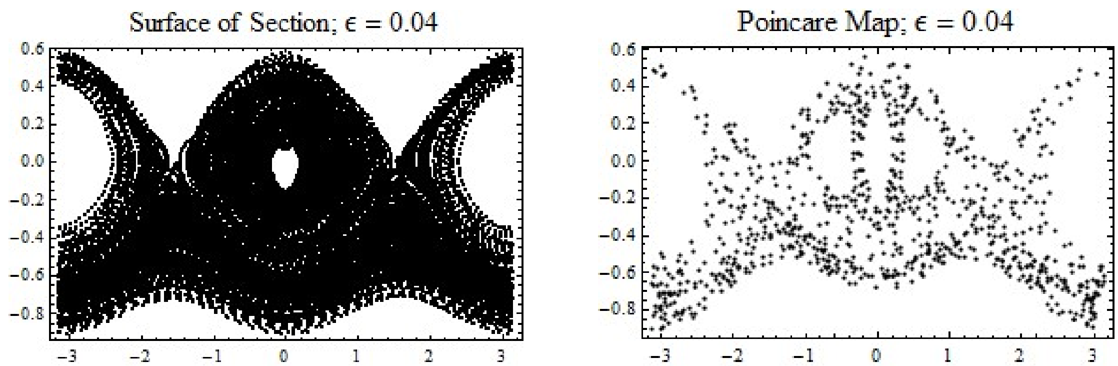


Fig.9 *Poincare surface of section and Poincare map for parameter values*

$\gamma = 0.15$, $\delta = 0.001$, a = 0.35, $r = 0.8$, $\varepsilon = 0.04$,$n = 0.6$.

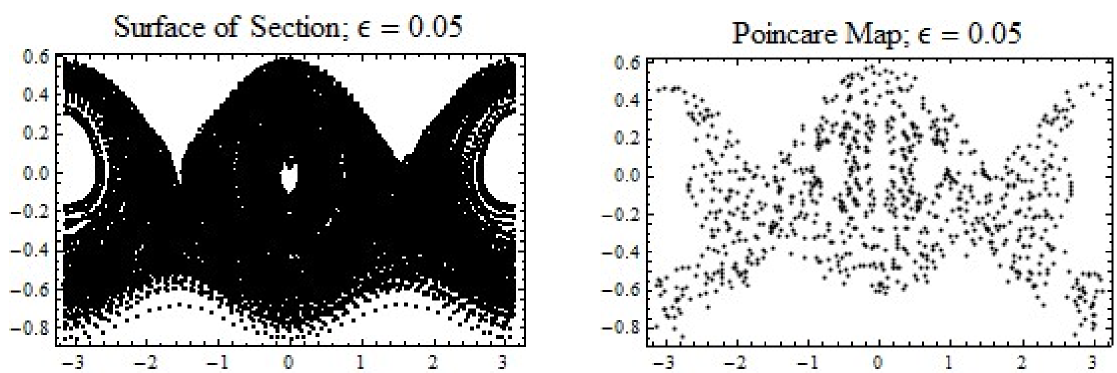


Fig.10 *Poincare surface of section and Poincare map for parameter values*

$\gamma = 0.15$, $\delta = 0.001$, $a = 0.35$, $r = 0.8$, $\varepsilon = 0.05$,$n = 0.6$.

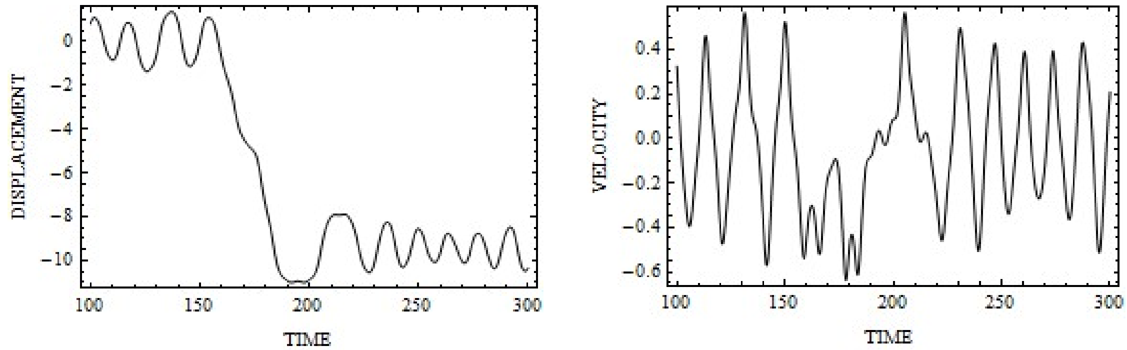


Fig.11 *Two time series plots for chaotic case, for parameter values*
$\gamma = 0.15$, $\delta = 0.001$, $a = 0.35$, $r = 0.8$, $\varepsilon = 0.05$, $n = 0.6$.

RESULT AND DISCUSSION

The graphs of Melnikov's function justifies the non-integrability of the equation of motion. One observes through successive plots, Fig.6 – Fig. 11, of surface of section, Poincare maps, time series, phase plots that the motion is regular when parameter ε is small. As ε increases from value 0.011, (as in Fig. 6), to a higher value $\varepsilon = 0.05$,(as in Fig. 9 – Fig. 11), the regular motion changes into irregular and chaotic. This suggests the fact that the parameter ε play a definite role for regular as well as for chaotic motion of the satellite. In fact here one can term this parameter as the controlling parameter of the satellite's motion.

REFERENCES

[1] Maciejewski J.A.(1992): IAU Collog. 132, Instability, Chaos and Predictability in Celestial Mechanics and Stellar Dynamics, Bhatnagar, K.B.(editor), (Commack N.Y.:Nova Sci.Publ.), p. 23

[2] Singh, R.B.(1973): Nonlinear plane oscillation of a satellite in an elliptic orbit, Cosmic Research, Vol.11 (2), p 172-187.

[3] Singh, R.B. (1986): Space Dynamics and Celestial Mechanics, in Proceeding of the International Workshop, Bhatnagar, K.B., (editor), D. Reidal Publishing Company Dordreicht, 1986 (Astrophysics and Space Science Library, Vol.127(1986), p. 480)

[4] Khan, A.(1994): Chaos in Non-Linear Planar Oscillation of a satellite in Elliptic orbit, Ph.D. thesis, University of Delhi, Delhi, 1994.

[5] Khan, A. And Goel, N.(2010): Chaos Control in the Problem of a Satellite, Applied Mathematical Sciences, Vol. 4 (25) P 1233

Ganita Sandesh Vol. 30 No. (1 & 2) 2016 pp. 27 – 38 ISSN : 0970-9169

STATISTICAL DISTRIBUTION AND PATHWAY INTEGRAL REPRESENTATION OF MULTIPARAMETER *K*-MITTAG-LEFFLER FUNCTION

[1]CHENA RAM , [2]ANITA AND [3]K.S. GEHLOT

Department of Mathematics and Statistics

Jai Narayan Vyas University **JODHPUR – 342005 (INDIA)**

[1]crc2007@rediffmail.com,[3]drksgehlot@rediffmail.com, [2]anita.neetu89@yahoo.com

ABSTRACT

The aim of this paper is to investigate the pathway fractional integral operator of the Multiparameter K-Mittag-Leffler function. We also investigate statistical distribution associated with the Multiparameter K-Mittag-Leffler function. Certain particular cases of the derived results are considered and indicated to further reduce to some known results.

Keywords:*k*- gamma function ; Multiparameter k-Mittag-Leffler function ; statistical distribution, Pathway operator.

1. INTRODUCTION AND PRELIMINARIES

The k-Pochhammer symbol was introduced by Diaz and Pariguan [3] in the form

$$(x)_{n,k}= x(x+k)(x+2k)\ldots(x+(n-1)k), \qquad \ldots(1.1)$$

where $x \in C,\ k \in R and n \in N$.

k-gamma function was defined by Diaz and Pariguan[3],as

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n!\,k^n (nk)^{\frac{z}{k}-1}}{(z)_{n,k}}, \qquad (k\in \boldsymbol{R^+} \text{ ; } z\in \mathbb{C}\backslash \boldsymbol{k}\boldsymbol{Z_0^-}). \qquad \ldots(1.2)$$

And some properties are

$$\Gamma_k(z+k) = z\,\Gamma_k(z) \text{ and } \Gamma_k(k) = 1 \text{ ;} \qquad \ldots(1.3)$$

For $R(z) > 0$ and $k \in R^+$,then $\Gamma_k(z)$ defined as the integral

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{\frac{-t^k}{k}}\, dt \qquad (k\in R^+;\ R(z) > 0) \text{ ;} \qquad \ldots (1.4)$$

The k-Pochhammer symbol $(x)_{n,k}$ defined (for $x, n \in \mathbb{C}\,; k \in R^+$)by

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)} (x \in \mathbb{C}\backslash\{0\}),$$

$$= \left[\begin{array}{ll} 1 & (n = 0), \\ x(x + k) \ldots (x + (n - 1)k) & (n \in N); \end{array}\right. \qquad \ldots (1.5)$$

From (1.4),it is easy to find the following relationship between the gamma function,Γ and the k-gamma function,Γ_k :

$$\Gamma_k(z) = k^{\frac{z}{k} - 1}\, \Gamma\left(\frac{z}{k}\right) \qquad \ldots (1.6)$$

The K-series defined by Gehlot and Ram [5] , as

$${}_pK_q^{(\beta,\eta)_m} (a_1, \ldots, a_p; b_1, \ldots, b_q, (\beta, \eta)_m; z) = {}_pK_q^{(\beta,\eta)_m}(z) \qquad \ldots (1.7)$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n z^n}{\prod_{r=1}^{q} (b_r)_n \prod_{i=1}^{m} \Gamma(\eta_i n + \beta_i)}\,, \qquad \ldots (1.8)$$

where $a_j, b_r, \beta_i \in \mathbb{C}\,;\ \eta_i \in R,\ (j = 1, \ldots, p; r = 1, \ldots, q; i = 1, \ldots, m)$.

The series is defined when none of the parameters $\boldsymbol{b_r}$(r=1,…,q), is negative integer or zero.If any parameter$\boldsymbol{a_j}$(j=1,…,p) in (1.7) is zero or negative , the series terminates in to polynomial in z, By the Ratio test,

(i) If $p < q + \sum_{i=1}^{m} \boldsymbol{\eta_i}$,then the power series on the right side of (1.8) is absolutely convergent for allz $\in C$

(ii) If $p = q + \sum_{i=1}^{m} \boldsymbol{\eta_i}$,and $|z| = 1$, then the series is absolutely convergent for all$|z| < \prod_{i=1}^{m} (|\boldsymbol{\eta}|_i)^{\boldsymbol{\eta}}$, and

$|z| = \prod_{i=1}^{m} (|\boldsymbol{\eta}|_i)^{\boldsymbol{\eta}}$,

$\mathrm{Re}\left[\sum_{r=1}^{q} (\boldsymbol{b_r}) + \sum_{i=1}^{m} (\boldsymbol{\eta_i}) - \sum_{j=1}^{p} (\boldsymbol{a_j})\right] > \frac{\boldsymbol{2+q+m-p}}{\boldsymbol{2}}$

The multiparameter K-Mittag-Leffler function defined by kuldeep singh Gehlot [6] ,as

$${}_pK_{q,k}^{(\beta,\eta)_m} (a_1, \ldots, a_p; b_1, \ldots, b_q, (\beta, \eta)_m; z) = {}_pK_{q,k}^{(\beta,\eta)_m} (z)$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{n,k} z^n}{\prod_{r=1}^{q}(b_r)_{n,k} \prod_{i=1}^{m} \Gamma_k(\eta_i n+\beta_i)}, \quad \ldots(1.9)$$

where $k \in R_+ = (0,\infty)$; $a_j, b_r, \beta_i \in C$; $\eta_i \in R$ $(j = 1,2,\ldots,p; r = 1,2,\ldots,q; i = 1,2,\ldots,m)$.

The series (1.9) is defined when none of the parameter $b_r (r = 1,2,\ldots,q) is$ negative integer or zero. If any parameter $a_j (j = 1,2,\ldots,p) in$ (1.9) is zero or negative ,the series terminates into polynomial in *z*.

Convergent conditions for series (1.9) are given by ratio test.

(i) If $p<q+\sum_{i=1}^{m}\left(\frac{\eta_i}{k}\right)$,then the power series on right of (1.9) is absolutely covergent for all $z \in C$.

(ii) If $p=q+\sum_{i=1}^{m}\left(\frac{\eta_i}{k}\right)$, then the power series on right of (1.9) is absolutely covergent for all

$$|k^{p-q-\sum_{i=1}^{m}\left(\frac{\eta_i}{k}\right)} z| < \prod_{i=1}^{m}\left(\left|\frac{\eta_i}{k}\right|\right)^{\frac{\eta_i}{k}}$$

and $|k^{p-q-\sum_{i=1}^{m}\left(\frac{\eta_i}{k}\right)} z| = \prod_{i=1}^{m}\left(\left|\frac{\eta_i}{k}\right|\right)^{\frac{\eta_i}{k}}$,Re$\left(\sum_{r=1}^{q}\left(\frac{b_r}{k}\right) + \sum_{i=1}^{m}\left(\frac{\beta_i}{k}\right) - \sum_{j=1}^{p}\left(\frac{a_j}{k}\right)\right) > \frac{2+q+m-p}{2}$.

Let $f \in L(a,b), \eta \in C$ $with$ $R(\eta), a \in R^+, and$ $\sigma < 1$ be the pathway parameter.Then the pathway fractional integration operator is defined by Nair[11]as

$$\left(P_{0+}^{(\eta,\sigma)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{a(1-\sigma)}\right]} \left[1 - \frac{a(1-\sigma)t}{x}\right]^{\frac{\eta}{1-\sigma}} f(t)dt. \quad \ldots(1.10)$$

For a real scalar σ,the pathway model for scalar random variables is represented by the following probability density function:

$$f(x) = c|x|^{\vartheta-1}[1 - a(1-\sigma)|x|^{\xi}]^{\frac{\lambda}{1-\sigma}}$$

provided that $x \in R, \vartheta, \xi \in R^+, \lambda \in R_0^+, 1 - a(1-\sigma)|x|^{\xi} > 0$. Here c is the normalizingconstant,and σ is called the pathway parameter.

For $\sigma > 1$, (1.10) can be written as follows:

$$\left(P_{0+}^{(\eta,\sigma)} f\right)(x) = x^{\eta} \int_0^{\left[\frac{x}{-a(1-\sigma)}\right]} \left[1 + \frac{a(\sigma-1)t}{x}\right]^{\frac{\eta}{-(\sigma-1)}} f(t)dt\,. \quad \ldots(1.11)$$

and

$$f(x) = c|x|^{\vartheta-1}[1 + a(\sigma-1)|x|^{\xi}]^{-\frac{\lambda}{(\sigma-1)}} \quad \ldots(1.12)$$

provided that $x \in R, \vartheta, \xi \in R^+, \lambda \in R_0^+$.

Moreover,as $\sigma \to -1$, the operator (1.10) reduces to the Laplace integral transform,and when $\sigma = 0$ and *a*=1,replacing η by $\eta - 1$,the operator (1.10) reduces to the Riemann-Liouville fractional integral operator.

2. MULTIPARAMETER K-MITTAG-LEFFLER FUNCTION AND STATISTICAL DISTRIBUTION

We investigate the density function for Multiparameter K-Mittag-Leffler function.

Theorem 1 let k,p,q,$\mu, x \in R^+$ with 0<$\mu \leq 1$ and q$\leq \mu + p$. Also , let $\eta \in$ C,

$$\mathcal{F}_x(x) = 1 - {}_pK_{q,k}^{(k,\mu)(k,\eta)_{m-1}}(-x^{\mu})$$

Then the density function f(x) of $\mathcal{F}_x(x)$is givan as follows:

$$\mathrm{f(x)} = \frac{x^{\mu-1}\prod_{j=1}^{p}(a_j)_{1,k}}{\prod_{r=1}^{q}(b_r)_{1,k}}$$

$${}_pK_{q,k}^{(\mu,\mu)(\eta+k,\eta)_{m-1}}[(a_1+k),\dots,(a_p+k);(b_1+k),$$

$$\dots,(b_q+k),(\mu,\mu),(\eta_2+k,\eta_2),\dots,(\eta_{m-1}+k,\eta_{m-1});(-x^{\mu})]. \quad \dots(2.1)$$

Proof Using (1.9) we have

$$\mathcal{F}_x(x) = 1 - \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}(-x^{\mu})^n}{\prod_{r=1}^{q}(b_r)_{n,k}\Gamma_k(\mu n+k)\prod_{i=2}^{m}\Gamma_k(\eta_i n+k)}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\prod_{j=1}^{p}(a_j)_{n,k}x^{\mu n}}{\prod_{r=1}^{q}(b_r)_{n,k}\Gamma_k(\mu n+k)\prod_{i=2}^{m}\Gamma_k(\eta_i n+k)} \quad \dots(2.2)$$

Differentiating each side of (2.2) with respect to x gives the density function

$$f(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\prod_{j=1}^{p}(a_j)_{n,k}\mu n \quad {}^{\mu n-1}}{\prod_{r=1}^{q}(b_r)_{n,k}\Gamma_k(\mu n+k)\prod_{i=2}^{m}\Gamma_k(\eta_i n+k)}.$$

Replace *n* by *n*+1, yields

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^{n+2}\prod_{j=1}^{p}(a_j)_{n+1,k}(\mu n+\mu)\ x^{\mu n+\mu-1}}{\prod_{r=1}^{q}(b_r)_{n+1,k}\Gamma_k(\mu n+\mu+\ \ )\prod_{i=2}^{m}\Gamma_k(\eta_i n+\eta_i+k)}. \quad \dots(2.3)$$

Applying the relation

$$(\gamma)_{n+j,k} = (\gamma)_{j,k}(\gamma+\rho k)_{n,k} \quad \dots(2.4)$$

and using $\Gamma_k(z+k) = z\Gamma_k(z)$

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\prod_{j=1}^{p}(a_j)_{1,k}(a_j+k)_{n,k}(\mu n+\mu)\ x^{\mu n+\mu-}}{\prod_{r=1}^{q}(b_r)_{n+1,k}\ \ (\mu n+\mu)\Gamma_k(\mu n+\mu)\prod_{i=2}^{m}\Gamma_k(\eta_i n+\eta_i+k)}$$

$$= x^{\mu-1}\prod_{j=1}^{p}(a_j)_{1,k}\sum_{n=0}^{\infty}\frac{(-1)^n\prod_{j=1}^{p}(a_j+k)_{n,k}x^{\mu n}}{\prod_{r=1}^{q}(b_r)_{n+1,k}\Gamma_k(\mu n+\mu)\prod_{i=2}^{m}\Gamma_k(\eta_i n+\eta_i+k)}$$

$$= x^{\mu-1}\prod_{j=1}^{p}(a_j)_{1,k}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j+k)_{n,k}(-x^{\mu})^n}{\prod_{r=1}^{q}(b_r)_{n+1,k}\Gamma_k(\mu n+\mu)\prod_{i=2}^{m}\Gamma_k(\eta_i n+\eta_i+k)}$$

If applying the relation (2.4) in (2.3) to both numerator and denominator,we get

$$f(x) = x^{\mu-1}\frac{\prod_{j=1}^{p}(a_j)_{1,k}}{\prod_{r=1}^{q}(b_r)_{1,k}}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j+k)_{n,k}(-x^{\mu})^n}{\prod_{r=1}^{q}(b_r+k)_{n,k}\Gamma_k(\mu n+\mu)\prod_{i=2}^{m}\Gamma_k(\eta_i n+\eta_i+k)}$$

$$=\frac{x^{\mu-1}\prod_{j=1}^{p}(a_j)_{1,k}}{\prod_{r=1}^{q}(b_r)_{1,k}}$$

$${}_pK_{q,k}^{(\mu,\mu)(\eta+k,\eta)_{m-1}}\,[(a_1+k),\dots,(a_p+k);(b_1+k),\dots,(b_q+k),(\mu,\mu),(\eta_2+k,\eta_2),\dots,(\eta_{m-1}+k,\eta_{m-1});(-x^{\mu})\,].$$

3 SPECIAL CASES

We consider some particular cases of Theorem 1.

Corollary 3.1. If we substitute k=1 in equation (2.1) ,then the following result holds

$$\mathrm{f(x)} = \frac{x^{\mu-1}\prod_{j=1}^{p}(a_j)_{1,1}}{\prod_{r=1}^{q}(b_r)_{1,1}}$$

$${}_pK_{q,1}^{(\mu,\mu)(\eta+1,\eta)_{m-1}}\,[(a_1+1),\dots,(a_p+1);(b_1+1),\dots,(b_q+1),$$
$$(\mu,\mu),(\eta_2+1,\eta_2),\dots,(\eta_{m-1}+1,\eta_{m-1});(-x^{\mu})\,]$$

$$=\frac{x^{\mu-1}\prod_{j=1}^{p}(a_j)}{\prod_{r=1}^{q}(b_r)}$$

$${}_pK_{q}^{(\mu,\mu)(\eta+1,\eta)_{m-1}}\,[(a_1+1),\dots,(a_p+1);(b_1+1),\dots,(b_q+1),$$
$$(\mu,\mu),(\eta_2+1,\eta_2),\dots,(\eta_{m-1}+1,\eta_{m-1});(-x^{\mu})\,].$$

This is new result for statistical distribution for K-Series .

Corollary 3.2 If we substitute k=1,m=1 in equation (2.1) , *then*

$$f(x) = x^{\mu-1}\frac{\prod_{j=1}^{p}(a_j)_{1,1}}{\prod_{r=1}^{q}(b_r)_{1,1}}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j+1)_{n,1}(-x^{\mu})^n}{\prod_{r=1}^{q}(b_r+1)_{n,1}\ \Gamma(\mu n+\mu)}$$

$$f(x) = x^{\mu-1}\frac{\prod_{j=1}^{p}(a_j)}{\prod_{r=1}^{q}(b_r)}\ {}_pM_q^{\mu,\mu}[(a_1+1),\dots,(a_p+1);$$

$$(b_1+1),\dots,(b_q+1),;(-x^{\mu})\ ].$$

Which is new result for statistical distribution of Generalized M-series defined by[7].

Corollary 3.3 If we substitute p=q=1,$a_1 = \rho, b_1 = k, m = 1$ in equation (2.1)$, then$

$$f(x) = x^{\mu-1}\frac{(\rho)_{1,k}}{(k)_{1,k}}\sum_{n=0}^{\infty}\frac{(\rho+k)_{n,k}(-x^{\mu})^n}{(k+k)_{n,k}\Gamma_k(\mu n+\mu)}$$

$$f(x) = \frac{\mu x^{\mu-1}}{k}\frac{(\rho)_{1,k}}{}\sum_{n=0}^{\infty}\frac{(\rho+k)_{n,k}(\frac{-x^{\mu}}{k})^n}{\Gamma_k(\mu n+\mu+k)n!}$$

$$f(x) = \frac{\mu x^{\mu-1}(\rho)_{1,k}}{k}E_{k,\mu,\mu+k}^{\rho+k}\,(-\frac{x^{\mu}}{k}).$$

This is new result for statistical distribution of generalized Mittag Leffler function.

Corollary 3.4 If we substitute p=q=m=1, $and\ a_1 = \rho$,$b_1 = 1, k = 1$ in equation (2.1) ,

$$f(x) = \mu x^{\mu-1}(\rho)\quad E_{1,\mu,\mu+1}^{\rho+1}\,(-x^{\mu})$$

$$= \mu x^{\mu-1}(\rho)\quad E_{\mu,\mu+1}^{\rho+1}\,(-x^{\mu}).$$

This result is given by C.Ram,Palu and K.S. Gehlot [2].

Corollary 3.5 If we substitute p=q=m=a_1=$b_1 = k = 1\ in\ equation$ (2.1), we have

$\mathcal{F}_x(x) = 1 - E_{\mu}$ (-x^{μ}) then

$f(x) = x^{\mu-1}E_{\mu,\mu}\,(-x^{\mu})$;0<$\mu \leq 1, x > 0$

Which is well known result given earlier by Mathai[9].

4. PATHWAY INTEGRAL REPRENSENTATION OF MULTIPARAMETER K-MITTAG-LEFFLER FUNCTION

Theorem 2 Let $\rho,\gamma,\beta\in\mathbb{C}$ with $\min\{R(\rho),R(\beta)\}>0$ and $R\left(\frac{\gamma}{1-\sigma}\right)>-1$. Also, let $k,\sigma\in R$ with $\sigma<1$ and $p,q\in R^{+}$. then,

$$(P_{0+}^{(\gamma,\sigma)}\left[t^{\frac{\beta}{k}-1}\ {}_pK_{q,k}^{(\rho,\beta)(\rho,\beta)_{m-1}}\left(\omega t^{\frac{\rho}{k}}\right)\right)(x)=\frac{x^{\gamma+\frac{\beta}{k}}k^{1+\frac{\gamma}{1-\sigma}}\Gamma\left(1+\frac{\gamma}{1-\sigma}\right)}{[a(1-\sigma)]^{\frac{\beta}{k}}}$$

$${}_pK_{q,k}^{(\rho,\beta)_m}\ [(a_1,\dots,a_p;b_1,\dots,b_q,(\rho,\beta)_{2,m},\rho,\beta+\frac{\gamma}{1-\sigma}+1;\omega(\frac{x}{a(1-\sigma)})^{\frac{\rho}{k}}]\ldots(4.1)$$

Proof Using equation (1.9) and (1.10),we have

$$(P_{0+}^{(\gamma,\sigma)}\left[t^{\frac{\beta}{k}-1}\ {}_pK_{q,k}^{(\rho,\beta)(\rho,\beta)_{m-1}}\left(\omega t^{\frac{\rho}{k}}\right)\right)(x)=x^{\gamma}\int_0^{[\frac{x}{a(1-\sigma)}]}[1-\frac{a(1-\sigma)t}{x}]^{\frac{\gamma}{1-\sigma}}\,t^{\frac{\beta}{k}-1}\ {}_pK_{q,k}^{(\rho,\beta)(\rho,\beta)_{m-1}}(\omega t^{\frac{\rho}{k}})dt\ .$$

$$=x^{\gamma}\int_0^{[\frac{x}{a(1-\sigma)}]}[1-\frac{a(1-\sigma)t}{x}]^{\frac{\gamma}{1-\sigma}}\,t^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\,(\omega t^{\frac{\rho}{k}})^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\rho_i n+\beta_i)}dt.$$

by interchanging order of summation and integration, we have,

$$=x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\rho_i n+\beta_i)}\int_0^{[\frac{x}{a(1-\sigma)}]}[1-\frac{a(1-\sigma)t}{x}]^{\frac{\gamma}{1-\sigma}}\,t^{\frac{\beta}{k}+\frac{\rho n}{k}-1}dt.$$

Put $\frac{a(1-\sigma)t}{x}=v$ and evaluate the inner integral by beta function formula, it gives

$$=x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\rho_i n+\beta_i)}\{\frac{x}{a(1-\sigma)}\}^{\frac{\beta}{k}+\frac{\rho n}{k}}\int_0^1[1-v]^{\frac{\gamma}{1-\sigma}}\,v^{\frac{\beta}{k}+\frac{\rho n}{k}-1}dv$$

$$=x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\rho_i n+\beta_i)}\{\frac{x}{a(1-\sigma)}\}^{\frac{\beta}{k}+\frac{\rho n}{k}}\frac{\Gamma\left(1+\frac{\gamma}{1-\sigma}\right)\ \Gamma(\frac{\beta}{k}+\frac{\rho n}{k})}{\Gamma(1+\frac{\gamma}{1-\sigma}+\frac{\beta}{k}+\frac{\rho n}{k})}$$

$$=x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\Gamma_k(\rho n+\beta)\prod_{i=2}^{m}\Gamma_k(\rho_i n+\beta_i)}\{\frac{x}{a(1-\sigma)}\}^{\frac{\beta}{k}+\frac{\rho n}{k}}\frac{\Gamma\left(1+\frac{\gamma}{1-\sigma}\right)\ \Gamma(\frac{\beta}{k}+\frac{\rho n}{k})}{\Gamma(1+\frac{\gamma}{1-\sigma}+\frac{\beta}{k}+\frac{\rho n}{k})}$$

$$=x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=2}^{m}\Gamma_k(\rho_i n+\beta_i)}\{\frac{x}{a(1-\sigma)}\}^{\frac{\beta}{k}+\frac{\rho n}{k}}\frac{\Gamma\left(1+\frac{\gamma}{1-\sigma}\right)\ \Gamma(\frac{\beta}{k}+\frac{\rho n}{k})}{k^{\frac{\rho n}{k}+\frac{\beta}{k}-1}\Gamma(\frac{\beta}{k}+\frac{\rho n}{k})\Gamma(1+\frac{\gamma}{1-\sigma}+\frac{\beta}{k}+\frac{\rho n}{k})}$$

$$=x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\quad\Gamma\left(1+\frac{\gamma}{1-\sigma}\right)k^{1+\frac{\gamma}{1-\sigma}}\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=2}^{m}\Gamma_k(\rho_i n+\beta_i)\,\Gamma_k(\rho n+\beta+\left(1+\frac{\gamma}{1-\sigma}\right)k)}\{\frac{x}{a(1-\sigma)}\}^{\frac{\beta}{k}+\frac{\rho n}{k}}$$

$$= \frac{x^{\gamma+\frac{\beta}{k}}\Gamma\left(1+\frac{\gamma}{1-\sigma}\right)k^{1+\frac{\gamma}{1-\sigma}}}{[a(1-\sigma)]^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}[\omega(\frac{x}{a(1-\sigma)})^{\frac{\rho}{k}}]^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=2}^{m}\Gamma_k(\rho_i n+\beta_i)\,\Gamma_k\left(\rho n+\beta+\left(1+\frac{\gamma}{1-\sigma}\right)k\right)}$$

$$= \frac{x^{\gamma+\frac{\beta}{k}}k^{1+\frac{\gamma}{1-\sigma}}\Gamma\left(1+\frac{\gamma}{1-\sigma}\right)}{[a(1-\sigma)]^{\frac{\beta}{k}}}$$

$$c\ {}_pK_{q,k}^{(\rho,\beta)_m}\ [(a_1,\dots,a_p;b_1,\dots,b_q,(\rho,\beta)_{2,m},\rho,\beta+\frac{\gamma}{1-\sigma}+1;\omega(\frac{x}{a(1-\sigma)})^{\frac{\rho}{k}}].$$

This establish result (4.1).

Corollary 4.1:

Let $\rho,\gamma,\beta \in C$ with $\min\{R(\rho),R(\beta)\} > 0\ andR\left(\frac{\gamma}{1-\sigma}\right) > -1$. Also, let$\sigma \in Rwith\sigma 1\ andp,q \in R^{+}$and k=1 then theorem 2 reduces into

$$(P_{0+}^{(\gamma,\sigma)}\left[t^{\beta-1}\ {}_pK_{q,1}^{(\rho,\beta)(\rho,\beta)_{m-1}}(\omega t^{\rho})\right)(x) = \frac{x^{\gamma+\beta}\Gamma\left(1+\frac{\gamma}{1-\sigma}\right)}{[a(1-\sigma)]^{\beta}}$$

$${}_pK_q^{(\rho,\beta)_m}\ [(a_1,\dots,a_p;b_1,\dots,b_q,(\rho,\beta)_{2,m},\rho,\beta+\frac{\gamma}{1-\sigma}+1;\omega\left(\frac{x}{a(1-\sigma)}\right)^{\rho}], \qquad \dots(4.2)$$

where ${}_pK_q^{(\rho,\beta)_m}$ [z] is K-series defined by Kuldeep Singh Gehlot [5].

Corollary 4.2:

Let $\rho,\gamma,\beta \in C$ with $\min\{R(\rho),R(\beta)\} > 0\ and\ R\left(\frac{\gamma}{1-\sigma}\right) > -1$. Also, let $\sigma \in R\ with\ \sigma < 1$ and k=1,p=q=1,$a_1=\rho, b_1=1$ then theorem 2 reduces to

$$(P_{0+}^{(\gamma,\sigma)}\left[t^{\beta-1}\ {}_1K_{1,1}^{(\rho,\beta)(\rho,\beta)_{m-1}}(\omega t^{\rho})\right)(x) = \frac{x^{\gamma+\beta}\Gamma\left(1+\frac{\gamma}{1-\sigma}\right)}{[a(1-\sigma)]^{\beta}}$$

$$E_{\rho}[(\rho_i,\beta_i)_{2,m},\rho,\beta+\frac{\gamma}{1-\sigma}+1;\omega(\frac{x}{a(1-\sigma)})^{\rho}], \qquad \dots(4.3)$$

where $E_{\rho}[(\beta,\rho)_m;z]$ is the generalized Mittag-Leffler function studied by Kilbas[1].

Corollary 4.3:

Let $\rho,\gamma,\beta \in C$ with $\min\{R(\rho),R(\beta)\} > 0\ andR\left(\frac{\gamma}{1-\sigma}\right) > -1$. Also, let$\sigma \in Rwith\sigma < 1$ and k=1,p=q=1,$a_1=\delta, b_1=1, m=1$ then theorem 2 reduces into

$$(P_{0+}^{(\gamma,\sigma)}[t^{\beta-1}\ {}_1K_{1,1}^{(\rho,\beta)}(\omega t^{\rho})](x)=\frac{x^{\gamma+\beta}\Gamma\left(1+\frac{\gamma}{1-\sigma}\right)}{[a(1-\sigma)]^{\beta}}E^{\delta}_{\rho,\beta+\frac{\gamma}{1-\sigma}+}[\omega(\frac{x}{a(1-\sigma)})^{\rho}], \qquad \ldots(4.4)$$

Which is well known result earlier studied by Nair[11] and $E^{\rho}_{\eta,\beta}[z]$ is generalized Mittag-Leffler function studied by Prabhakar[13].

Theorem 3 Let $\rho,\gamma,\beta\in\mathrm{C}$ with $\min\{\mathrm{R}(\rho),R(\beta)\}>0\ andR\left(1-\frac{\gamma}{\sigma-1}\right)>0$. Also, let$k,\sigma\in Rwith\sigma>1\ andp,q\in R^{+}$. then,

$$(P_{0+}^{(\gamma,\sigma)}\left[t^{\frac{\beta}{k}-1}\ {}_pK_{q,k}^{(\rho,\beta)(\rho,\beta)_{m-1}}\left(\omega t^{\frac{\rho}{k}}\right)\right)(x)=\frac{x^{\gamma+\frac{\beta}{k}}k^{1-\frac{\gamma}{\sigma-1}}\Gamma\left(1-\frac{\gamma}{\sigma-1}\right)}{[-a(1-\sigma)]^{\frac{\beta}{k}}}$$

$${}_pK_{q,k}^{(\rho,\beta)_m}\ [(a_1,\ldots,a_p;b_1,\ldots,b_q,(\rho,\beta)_{2,m},\rho,\beta+1-\frac{\gamma}{\sigma-1};\omega(\frac{x}{-a(1-\sigma)})^{\frac{\rho}{k}}\ ] \qquad \ldots(4.5)$$

Proof Using equation (1.9) and (1.11),we have

$$(P_{0+}^{(\gamma,\sigma)}\left[t^{\frac{\beta}{k}-1}\ {}_pK_{q,k}^{(\rho,\beta)(\rho,\beta)_{m-1}}\left(\omega t^{\frac{\rho}{k}}\right)\right)(x)=$$

$$x^{\gamma}\int_0^{[\frac{x}{-a(1-\sigma)}]}[1+\frac{a(\sigma-1)t}{x}]^{\frac{\gamma}{-(\sigma-1)}}\,t^{\frac{\beta}{k}-1}\ {}_pK_{q,k}^{(\rho,\beta)(\rho,\beta)_{m-1}}(\omega t^{\frac{\rho}{k}})dt\ .$$

$$=x^{\gamma}\int_0^{[\frac{x}{-a(1-\sigma)}]}[1+\frac{a(\sigma-1)t}{x}]^{\frac{\gamma}{-(\sigma-1)}}\,t^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\,(\omega t^{\frac{\rho}{k}})^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_{\mathrm{k}}(\rho_i n+\beta_i)}dt.$$

by interchanging order of summation and integration, we have

$$=x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_{\mathrm{k}}(\rho_i n+\beta_i)}\int_0^{[\frac{x}{-a(1-\sigma)}]}[1+\frac{a(\sigma-1)t}{x}]^{\frac{\gamma}{-(\sigma-1)}}\,t^{\frac{\beta}{k}+\frac{\rho n}{k}-1}dt.$$

Put $\frac{-a(\sigma-1)t}{x}=v$ and evaluate the inner integral by beta function formula, it gives

$$=x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_{\mathrm{k}}(\rho_i n+\beta_i)}\{\frac{-x}{a(\sigma-1)}\}^{\frac{\beta}{k}+\frac{\rho n}{k}}\int_0^1[1-v]^{\frac{\gamma}{-(\sigma-1)}}\,v^{\frac{\beta}{k}+\frac{\rho n}{k}-1}dv$$

$$= x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\,\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\rho_i n+\beta_i)}\left\{\frac{-x}{a(\sigma-1)}\right\}^{\frac{\beta}{k}+\frac{\rho n}{k}}\frac{\Gamma\left(1-\frac{\gamma}{\sigma-1}\right)\ \Gamma\left(\frac{\beta}{k}+\frac{\rho n}{k}\right)}{\Gamma\left(1-\frac{\gamma}{\sigma-1}+\frac{\beta}{k}+\frac{\rho n}{k}\right)}$$

$$= x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\,\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\Gamma_k(\rho n+\beta)\prod_{i=2}^{m}\Gamma_k(\rho_i n+\beta_i)}\left\{\frac{-x}{a(\sigma-1)}\right\}^{\frac{\beta}{k}+\frac{\rho n}{k}}\frac{\Gamma\left(1-\frac{\gamma}{\sigma-1}\right)\ \Gamma\left(\frac{\beta}{k}+\frac{\rho n}{k}\right)}{\Gamma\left(1-\frac{\gamma}{\sigma-1}+\frac{\beta}{k}+\frac{\rho n}{k}\right)}$$

$$= x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\,\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=2}^{m}\Gamma_k(\rho_i n+\beta_i)}\left\{\frac{-x}{a(\sigma-1)}\right\}^{\frac{\beta}{k}+\frac{\rho n}{k}}\frac{\Gamma\left(1-\frac{\gamma}{\sigma-1}\right)\ \Gamma\left(\frac{\beta}{k}+\frac{\rho n}{k}\right)}{k^{\frac{\rho n}{k}+\frac{\beta}{k}-1}\Gamma\left(\frac{\beta}{k}+\frac{\rho n}{k}\right)\Gamma\left(1-\frac{\gamma}{\sigma-1}+\frac{\beta}{k}+\frac{\rho n}{k}\right)}$$

$$= x^{\gamma}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\quad\Gamma\left(1-\frac{\gamma}{\sigma-1}\right)k^{1-\frac{\gamma}{1-\sigma}}\omega^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=2}^{m}\Gamma_k(\rho_i n+\beta_i)\,\Gamma_k\left(\rho n+\beta+\left(1+\frac{\gamma}{1-\sigma}\right)k\right)}\left\{\frac{-x}{a(\sigma-1)}\right\}^{\frac{\beta}{k}+\frac{\rho n}{k}}$$

$$= \frac{x^{\gamma+\frac{\beta}{k}}\Gamma\left(1-\frac{\gamma}{\sigma-1}\right)k^{1-\frac{\gamma}{1-\sigma}}}{[-a(1-\sigma)]^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n,k}\left[\omega\left(\frac{x}{-a(1-\sigma)}\right)^{\frac{\rho}{k}}\right]^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=2}^{m}\Gamma_k(\rho_i n+\beta_i)\,\Gamma_k\left(\rho n+\beta+\left(1-\frac{\gamma}{\sigma-1}\right)k\right)}$$

$$= \frac{x^{\gamma+\frac{\beta}{k}}k^{1-\frac{\gamma}{1-\sigma}}\Gamma\left(1-\frac{\gamma}{\sigma-1}\right)}{[-a(1-\sigma)]^{\frac{\beta}{k}}}$$

$${}_pK_{q,k}^{(\rho,\beta)_m}\left[(a_1,\dots,a_p;b_1,\dots,b_q,(\rho,\beta)_{2,m},\rho,\beta+1-\frac{\gamma}{\sigma-1};\omega\left(\frac{x}{-a(1-\sigma)}\right)^{\frac{\rho}{k}}\right].$$

This establish result (4.5).

Corollary 4.4:

Let $\rho,\gamma,\beta \in \mathbb{C}$ with $\min\{R(\rho),R(\beta)\}>0$ and $R\left(1-\frac{\gamma}{\sigma-1}\right)>0$. Also,

let $\sigma \in R$ with $\sigma>1$ and $p,q \in R^{+}$ and k=1 then theorem 3 reduces into

$$\left(P_{0+}^{(\gamma,\sigma)}\left[t^{\beta-1}\ {}_pK_{q,1}^{(\rho,\beta)(\rho,\beta)_{m-1}}(\omega t^{\rho})\right)(x) = \frac{x^{\gamma+\beta}\Gamma\left(1-\frac{\gamma}{\sigma-1}\right)}{[-a(1-\sigma)]^{\beta}}$$

$${}_pK_q^{(\rho,\beta)_m}\left[(a_1,\dots,a_p;b_1,\dots,b_q,(\rho,\beta)_{2,m},\rho,\beta+1-\frac{\gamma}{\sigma-1};\omega\left(\frac{x}{-a(1-\sigma)}\right)^{\rho}\right], \quad \dots(4.6)$$

where ${}_pK_q^{(\rho,\beta)_m}$ [z] is K-series defined by Kuldeep Singh Gehlot .

Corollary 4.5:

Let $\rho,\gamma,\beta \in C$ with $\min\{R(\rho),R(\beta)\} > 0$ and $R\left(1-\frac{\gamma}{\sigma-1}\right) > 0$. Also, let $\sigma \in R$ with $\sigma > 1$ and $k=1, p=q=1, a_1=\rho, b_1=1$ then theorem 3 reduces to

$$(P_{0+}^{(\gamma,\sigma)}\left[t^{\beta-1}\ {}_1K_{1,1}^{(\rho,\beta)(\rho,\beta)_{m-1}}(\omega t^{\rho})\right)(x) = \frac{x^{\gamma+\beta}\Gamma\left(1-\frac{\gamma}{\sigma-1}\right)}{[-a(1-\sigma)]^{\beta}}$$

$$E_{\rho}[(\rho_i,\beta_i)_{2,m},\rho,\beta+1-\frac{\gamma}{\sigma-1};\omega(\frac{x}{-a(1-\sigma)})^{\rho}], \quad \ldots(4.7)$$

where $E_{\rho}[(\beta,\rho)_m;z]$ is the generalized Mittag-Leffler function.

Corollary 4.6:

Let $\rho,\gamma,\beta \in C$ with $\min\{R(\rho),R(\beta)\} > 0$ and $R\left(1-\frac{\gamma}{\sigma-1}\right) > 0$. Also, let $\sigma \in R$ with $\sigma > 1$ and $k=1, p=q=1, a_1=\delta, b_1=1, m=1$ then theorem 3 reduces to

$$(P_{0+}^{(\gamma,\sigma)}[t^{\beta-1}\ {}_1K_{1,1}^{(\beta,\rho)_1}(\omega t^{\rho})](x) = \frac{x^{\gamma+\beta}\Gamma\left(1-\frac{\gamma}{\sigma-1}\right)}{[-a(1-\sigma)]^{\beta}} E_{\rho,\beta+1-\frac{\gamma}{\sigma-1}}^{\delta}[\omega(\frac{x}{-a(1-\sigma)})^{\rho}]\,. \quad \ldots(4.8)$$

Which is well known result given by Nair[11] .

REFERENCES

[1] A.A Kilbas,M.Saigo and J.J.Trijillo, On the generalized Wright-function, Frac. Cal. Appl. Anal.,5(2002),437-461.

[2] C. Ram, Palu and K.S Gehlot, Statistical distribution of K-Mittag-Leffler function, Palestine Journal of Mathematics, Vol 3(2) (2014),269-272 .

[3] Diaz, R. and Pariguan, E. , On hypergeometric functions and k-Pochhammer symbol,Div. Math. 15(2) (2007),179-192.

[4] H.M. Srivastava and R.K. Saxena, Operators of fractional integration and their applications,Appl.Math. Comut 118 (2001),1-52.

[5] Gehlot,K.S. and C. Ram, Integral representation of K-Series, Int. Trans. Math. Sci. Comp.,Vol.4 No. 2(2012),387-396.

[6] Gehlot,K.S. , Multiparameter K-Mittag-Leffler Function, Int. Math. Forum, Vol. 8 (2013),1691-1702.

[7] Gehlot,K.S. ,Certain properties of Generalized M-Series associated with Fractional Calculus,Int. J. of Math. Sci. & Engg. Appls., Vol.7(2013),143-150.

[8] Mathai,M,Haubold,HJ, Mittag-Leffler functions to pathway model to Tsallis statistics. Integral Transforms Spec. Funct. 21(11) (2010), 867-875 .

[9] Mathai,A.M,Haubold,HJ, On generalized distributions and pathways. Phys.Lett. A 372 (2008),2109-2113.

[10] Mathai,A.M,Some properties of Mittag Leffler Functions and matrix-variate analogues:A statistical perspective,Frac.Cal.App.Ana,13(2),(2010).

[11] Nair,SS, Pathway fractional integration operator. Fract.Calc.Appl. Anal. 12(3) (2009),237-252.

[12] S.G.Samko,A.A.Kilbas and O.I.Merichev, Fractional Integrals and Derivatives,Theory and Applications,Gordon and Breach,New York,(1993).

[13] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel,Yokohama Math. J.,19(1971),7-15.

Ganita Sandesh Vol. 30 No. (1 & 2) 2016 pp. 39 – 52 **ISSN : 0970-9169**

NEW f - DIVERGENCE MEASURE AND ITS INEQUALITIES

K.C.JAIN[1] AND DEVENDRA SINGH[2]

[1]*Department of Mathematics*

IIIT Kota - **JAIPUR-302017(INDIA)**

[2]*Department of Mathematics*

Malaviya National Institute of Technology,**JAIPUR-302017(INDIA)**

[1]2013rma9061@mnit.ac.in, [2]devendrasingh_maths@yahoo.com

ABSTRACT

There are several types of Divergence measures studied in information theory which measure the distance between two probability distributions and have applications in information theory and coding, signal processing, risk for binary experiment, sensor network, etc.In this paper,we derive new information divergence measures using properties of convex functions and new f-divergence measure and inter-relations among new and other well-known divergence measures are considered.

Keywords: Chi-square divergence, RelativeJ-divergence, RelativeJenson-Shannon divergence, Arithmetic-Geometric meandivergence, Harmonic mean divergence etc.

Mathematics Subject Classification 2010: 94A17,26D15

1. INTRODUCTION

Let $\Gamma_n = \left\{ P = \left(p_{1,} p_{2,} \ldots\ldots p_n\right) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2$ be the set of all complete finite discrete probability distributions. If we take $p_i \geq 0$ for some $I = 1,2,3,. \ . \ .n$, then we have to suppose that $0 \ f\left(0\right) = 0 f \left(\frac{0}{0}\right) = 0.$

Csiszar [1] introduced a generalized measure of information using *f*-divergence measure

$$C_f\left(P,Q\right) = \sum_{i=1}^{n} q_i \; f\left(\frac{p_i}{q_i}\right), \tag{1}$$

where $f:(0, \infty) \rightarrow \boldsymbol{R}$ (set of real numbers) is a convex function and $P=\left(p_1,p_2,p_3,...p_n\right)$, $Q=\left(q_1,q_2,q_3,...q_n\right)\in\Gamma_n$, where p_i ang q_i are probability mass functions. An important property of this divergence is that many known divergences can be obtained from this measure by appropriately defining the convex function *f*.There are some examples of divergence measures in the category of Csiszar's *f*- divergence measure. These measures are as follows:

i. Triangular Discrimination [9]

$$\Delta(P,Q) = \sum_{i=1}^{n} \frac{(p_i-q_i)^2}{p_i+q_i} \tag{2}$$

and
$$\Delta_m\left(P,Q\right) = \sum_{i=1}^{n} \frac{\left(p_i-q_i\right)^{2m}}{\left(p_i+q_i\right)^{2m-1}}, \quad m = 1,2,3,... \tag{3}$$

ii. Relative Information(Kullback and Leiber [7])

$$K\,(P,\,Q) = \sum_{i=1}^{n} p_i \log(\frac{p_i}{q_i}) \tag{4}$$

iii. χ^2- Divergence Measure (Pearson [2])

$$\chi^2\,(P,\,Q) = \sum_{i=1}^{n} \frac{(p_i-q_i)^2}{q_i} \tag{5}$$

iv. Relative *J*-Divergence (Dragomir,Gluscevic and Pearce [5])

$$J_R\,(P,\,Q) = \sum_{i=1}^{n}(p_i\text{-}q_i)\log(\frac{p_i+q_i}{2q_i}) \tag{6}$$

v. Relative Arithmetic-Geometric Divergence (Taneja [8])

$$G\,(Q,\,P) = \sum_{i=1}^{n}(\frac{p_i+q_i}{2}) \log(\frac{p_i+q_i}{2q_i}) \tag{7}$$

vi. Relative Jenson-Shannon Divergence (Sibson [3])

$$F\,(P,\,Q) = \sum_{i=1}^{n} p_i \log(\frac{2p_i}{p_i+q_i}) \tag{8}$$

and
$$F\,(Q,\,P) = \sum_{i=1}^{n} q_i \log(\frac{2q_i}{p_i+q_i}) \tag{9}$$

vii. Arithmetic mean divergence [4]

$$A\,(P,\,Q) = \sum_{i=1}^{n} \frac{(p_i+q_i)}{2} = 1 \tag{10}$$

viii. Jensen-Shannon divergence(Burbea,Rao and Sibson [3])

$$I(P, Q) = \frac{1}{2}[F(P,Q) + F(Q,P)]$$

$$= \frac{1}{2}[\sum_{i=1}^{n} p_i \log(\frac{2p_i}{p_i+q_i}) + \sum_{i=1}^{n} q_i \log(\frac{2q_i}{p_i+q_i}), \quad (11)$$

where $F(P, Q)$ and $F(Q, P)$ are given by (8) and (9) respectively.

ix. Arithmetic-Geometric mean Divergence (Taneja [8])

$$T(P, Q) = \frac{1}{2}[G(P, Q) + G(Q, P)]$$

$$= \sum_{i=1}^{n} \frac{p_i+q_i}{2} \log(\frac{p_i+q_i}{2\sqrt{p_i q_i}}), \quad (12)$$

where $G(Q, P)$ is given by (7)

x. Harmonic mean Divergence[4]

$$H(P, Q) = \sum_{i=1}^{n} \frac{2p_i q_i}{p_i+q_i} \quad (13)$$

2. WELL KNOWN INEQUALITIES

In this section we give some well-known inequalities which are established in literature of pure and applied mathematics. These are very useful to derive some bounds of well-known information divergence measure in literature of information theory and statistics. Using following inequalities we have derived important bounds of well-known divergence measures

$$\frac{x}{1+x} \leq \log(1+x) \leq x, x > 0 \quad (14)$$

$$x - \frac{x^2}{2} \leq \log(1+x) \leq x - \frac{x^2}{2(1+x)}, x > 0 \quad (15)$$

3. NEW *f* -DIVERGENCE MEASURE AND PROPERTIES

A new *f*- divergence measure introduced by Jain &Saraswat [11], which is given by

$$S_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i + q_i}{2q_i}\right), \quad (16)$$

where f: $R_+ \longrightarrow R_+$ is a convex function and $P, Q \in \Gamma_n$

The following results are presented by Jain &Saraswat [11]

Proposition 3.1: *Let f: [0,∞)→R be the convex function P,Q* $\in \Gamma_n$. *Then we have the following inequality*

$$S_f(P,Q) \geq f(1) \tag{17}$$

If f is normalized i.e. $f(1) = 0$, then S_f(P, Q) ≥ 0 and if f is strictly convex and equality holds iff

$$p_i = q_i \forall \ \text{i}=1, 2, 3\ldots n$$

i.e.
$$S_f(P,Q) \geq 0 \quad and \ S_f(P,Q) = 0 \ if \quad P = Q \tag{18}$$

Proposition 3.2: *If* $\boldsymbol{f_1}$ *and* $\boldsymbol{f_2}$*are two convex functions and F* $=af_1 + bf_2$*then*

$$S_F(P,Q) = a\, S_{f_1}(\text{P, Q}) + b\, S_{f_2}(\text{P, Q}),$$

where a and b are constants and P, Q$\in \ \Gamma_n$

4. DIVERGENCE MEASURE OF NEW f -DIVERGENCE MEASURE'S CLASS

In this section, we shall find out the new divergence measure with the help of following convex function. Let us consider the function

$f : (0,\infty) \to R$ such that

$$f_k(t) = \frac{(t-1)^{k+1}}{(t+1)^k} \qquad : k = 1,3,5,\cdots \tag{19}$$

Then

$$f_k'(t) = \frac{(t-1)^k (2kt + t + 1)}{(t+1)^{k+1}} \tag{20}$$

and

$$f_k''(t) = \frac{(t-1)^{k-1} 4k(k+1)t}{(t+1)^{k+2}} \tag{21}$$

The function f_k(t) is convex since $f_k''(t) \geq \ 0 \forall \ t > 0$; k = 1,3,5,… and normalized also since $f(1) = 0$.

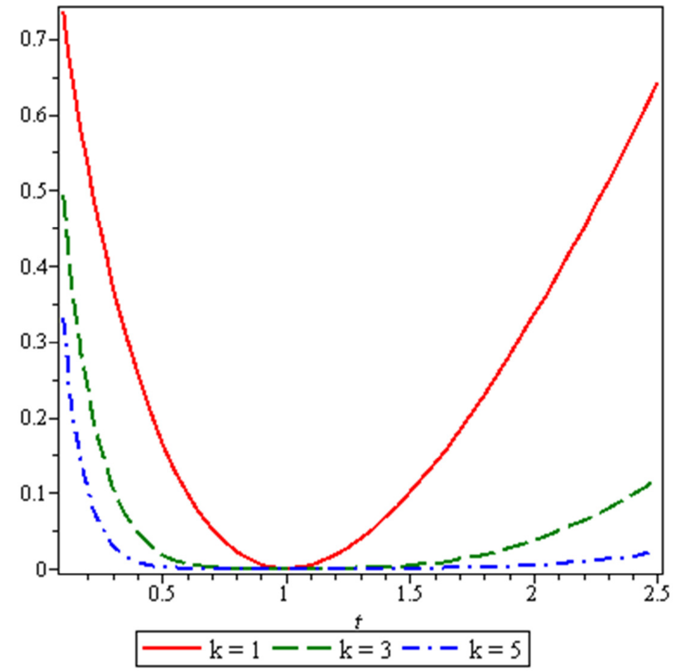


Figure 1. *Behavior of the function* $f_k(t)$

Figure 1.shows the behavior of the function $f_k(t)$ is always convex if $k=1,3,5,\ldots\ \forall\, t>0$

Now putting function (20) in (16), we obtain

$$S_f(P,Q)=M_k^c(P,Q)=\frac{1}{2}\sum_{i=1}^{n}\frac{(p_i-q_i)^{k+1}}{(p_i+3q_i)^k} \qquad ;\, k=1,3,5,\cdots \tag{22}$$

Divergence measure M_k^c (*P, Q*) is non-symmetric divergence measure since

$$M_k^c(P,Q) = M_k^c(Q,P)$$

Moreover

$$M_k^c(P,Q) \geq 0\ \forall\, P,Q\in\Gamma_n \text{ and } M_k^c(P,Q)=0 \text{ iff } P=Q$$

Now, from equation (19) for *k* = 1, 3, 5…we get the following convex functions

$$f_1(t)=\frac{(t-1)^2}{(t+1)},\ f_3(t)=\frac{(t-1)^4}{(t+1)^3},\ f_5(t)=\frac{(t-1)^6}{(t+1)^5},\ f_7(t)=\frac{(t-1)^8}{(t+1)^7},\ldots$$

We know that, sum of convex functions is also a convex function i.e. $c_1 f_1$(t) + $c_3 f_3$ + . . .is also a convex function .where c_1, c_3 , c_5 ,. . . are arbitrary positive constants and at least one of them not equal to zero.

Now,

$$F_k(\mathrm{t}) = \sum_{i=1}^{n} c_k f_k(\mathrm{t}) = c_1 f_1(t) + c_3 f_3(t) + c_5 f_5(t) + \ldots$$

$$F_k(\mathrm{t}) = c_1 \frac{(t-1)^2}{(t+1)} + c_3 \frac{(t-1)^4}{(t+1)^3} + c_5 \frac{(t-1)^6}{(t+1)^5} + \ldots \tag{23}$$

Taking

$$c_1 = 1, c_3 = {}^1/_3, c_5 = {}^1/_5 \ldots$$

$$F_1(\mathrm{t}) = 1.\frac{(t-1)^2}{(t+1)} + \frac{1}{3}\frac{(t-1)^4}{(t+1)^3} + \frac{1}{5}\frac{(t-1)^6}{(t+1)^5} + \ldots$$

$$= (\mathrm{t}-1)\left[1\left(\frac{t-1}{t+1}\right)^1 + \frac{1}{3}\left(\frac{t-1}{t+1}\right)^3 + \frac{1}{5}\left(\frac{t-1}{t+1}\right)^5 + \ldots\right]$$

$$= (\mathrm{t}-1)\frac{1}{2}\log\left\{\frac{1+\left(\frac{t-1}{t+1}\right)}{1-\left(\frac{t-1}{t+1}\right)}\right\}$$

i.e. $F_1(t) = \frac{1}{2}(\mathrm{t}-1)\log(t)$(24)

From (16), Divergence measure of *f*-divergence class for (24),

$$\mathrm{M}_1^*(\mathrm{P}, \mathrm{Q}) = \frac{1}{4}\sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) \tag{25}$$

Next , Taking,$c_1 = 0, c_3 = 1, c_5 = {}^1/_3$, $c_7 = {}^1/_5$. . . In (23), we get

$$F_3(P,Q) = \frac{1}{2}\frac{(t-1)^3}{(t+1)^2}\log(t) \tag{26}$$

Hence, Divergence measure, for (26) from (16),

$$M_3^*(P,Q) = \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i-q_i)^3}{(p_i+3q_i)^2}\log\left(\frac{p_i+q_i}{2q_i}\right) \tag{27}$$

Similarly by appropriate selection of constants, we get the following convex function

$$F_k\ (P,\ Q) = \frac{1}{2}\frac{(t-1)^k}{(t+1)^{k-1}}\log(t), \tag{28}$$

where $k = 1, 3, 5. \ . \ .$

And the corresponding series of divergence measures of *f*-divergence class

$$M_k^*(P,Q) = \frac{1}{4}\sum_{i=1}^{n}\frac{(p_i-q_i)^k}{(p_i+3q_i)^{k-1}}\log\left(\frac{p_i+q_i}{2q_i}\right) \tag{29}$$

where $k = 1, 3, 5. \ . \ .$

It may be noted that, F_k(t) in (28) satisfies $F_k(1) = 0$, so that $M_k^*(P, P) = 0$. Convexity of F_k(t) ensures that divergence measure $M_k^*(P,Q)$ is non-negative. Thus, we have

(a) $M_k^*(P,Q) \geq 0$ and $M_k^*(P,Q) = 0$ if $P = Q$

(b) $M_k^*(P, Q)$ is non-symmetric with respect to probability distribution.

5. NEW INFORMATION INEQUALITIES AND EQUALITIES

We now derive information divergence inequalities and equalities providing bounds for $M_k^*(P, Q)$ in terms of the well-known divergence measures in the following propositions

Proposition 5.1: *Let* $(P, Q) \in \Gamma_n \times \Gamma_n$ *, then we have the following new inter-relation*

$$M_1^*(P, Q) \leq [F(Q, P) + G(Q, P)], \tag{30}$$

where $M_1^*(P, Q)$,$F(Q, P)$ *and* $G(P, Q)$ *are given by* (25), (9), *and* (7) *respectively.*

Proof: From (25), we have

$$\mathrm{M}_1^*\ (P,\ Q) = \frac{1}{4}\sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$= \frac{1}{4}\sum_{i=1}^{n}(p_i + q_i - 2q_i)\log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left\{\left(\frac{p_i+q_i}{2}\right) - q_i\right\}\log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$=\frac{1}{2}\left[\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right)-\sum_{i=1}^{n}q_i\log\left(\frac{p_i+q_i}{2q_i}\right)\right]$$

$$=\frac{1}{2}\left[\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right)+\sum_{i=1}^{n}q_i\log\left(\frac{2q_i}{p_i+q_i}\right)\right]$$

$$M_1^*(\mathrm{P,\,Q})=\frac{1}{2}[G(Q,P)+F(Q,P)]$$

$$2M_1^*(\mathrm{P,\,Q})=[F(Q,P)+G(Q,P)]$$

$$M_1^*(\mathrm{P,\,Q})\ \le [F(Q,P)+G(Q,P)]$$

Hence prove the inequality.

Proposition 5.2 : *Let (P, Q)*$\in \Gamma_n \times \Gamma_n$ *, then we have the following new inter-relation*

$$M_3^*(\mathrm{P,\,Q})\le\frac{\Delta(P,Q)J_R(P,Q)}{2\,A(P,Q)} \tag{31}$$

where $M_3^*(P,Q)$, $\Delta(P,Q)$, $J_R(P,Q)$ *and* $A(P,Q)$ *are given by* (27),(2),(6) *and* (10) *respectively.*

Proof: We have

$$M_3^*(\mathrm{P,\,Q})=\frac{1}{4}\sum_{i=1}^{n}\frac{(p_i-q_i)^3}{(p_i+3q_i)^2}\log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$\Rightarrow\qquad M_3^*(\mathrm{P,\,Q})\ \le\ \sum_{i=1}^{n}\frac{(p_i-q_i)^3}{(p_i+q_i)^2}\log\left(\frac{p_i+q_i}{2q_i}\right)$$

i.e.

$$M_3^*(\mathrm{P,\,Q})\ \le\ \sum_{i=1}^{n}\frac{(p_i-q_i)^2}{(p_i+q_i)}\sum_{i=1}^{n}\frac{1}{(p_i+q_i)}\sum_{i=1}^{n}(p_i-q_i)\log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$=\ \Delta(P,Q)\,\frac{1}{2A(P,Q)}J_R(P,\,Q)$$

$$M_3^*(P,\,Q)\ \le\ \frac{\Delta(P,Q)J_R(P,Q)}{2\,A(P,Q)}$$

Hence prove the inequality.

Proposition 5.3 : *Let (P, Q)*$\in \Gamma_n \times \Gamma_n$ *, then we have the following new inter-relation*

$$4\,M_1^*(P,\,Q)=\ J_R(P,Q)=\ 2[F(Q,P)+G(Q,P)] \tag{32}$$

$$M_3^*(P, Q) \le \frac{1}{16}\chi^2(\mathrm{P, Q}) \left[\frac{1}{A(P,Q)}\right]^2 J_R(P, Q) \tag{33}$$

and
$$M_3^*(P, Q) \le \frac{1}{8}\chi^2(\mathrm{P, Q}) \left[\frac{1}{A(P,Q)}\right]^2 [F(Q, P) + G(Q, P)] \tag{34}$$

where $M_1^*(P,Q)$,$M_3^*(P,Q)$, $F(Q,P)$,$G(P,Q)$, $A(P,Q)$, $J_R(P,Q)$ *and* $\chi^2(P,Q)$ *are given by* (25),(27),(9),(7),(10),(6) *and* (5) *respectively.*

Proof: From (25), we have

$$M_1^*(P, Q) = \frac{1}{4}\sum_{i=1}^n (p_i - q_i)\, log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$\Rightarrow \qquad 4\, M_1^*(P, Q) = \sum_{i=1}^n (p_i - q_i)\, log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$\Rightarrow \qquad 4M_1^*(P, Q) = J_R(P, Q)$$

and we have

$$J_R(P, Q) = 2[F(Q, P) + G(Q, P)]$$

$$\therefore \qquad 4M_1^*(P, Q) = J_R(P, Q) = 2[F(Q, P) + G(Q, P)]$$

Hence prove the equality.

Next, from (27), we have

$$M_3^*(P, Q) = \frac{1}{4}\sum_{i=1}^n \frac{(p_i-q_i)^3}{(p_i+3q_i)^2}\log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$\le \frac{1}{4}\sum_{i=1}^n \frac{(p_i-q_i)^3}{(p_i+q_i)^2}\log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$= \frac{1}{4}\sum_{i=1}^n \frac{(p_i-q_i)^2}{q_i}\sum_{i=1}^n\left(\frac{1}{p_i+q_i}\right)^2 \left(\sum_{i=1}^n q_i\right)\sum_{i=1}^n (p_i - q_i)\log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$\therefore \qquad M_3^*(P, Q) \le \frac{1}{16}\chi^2(P, Q)\left[\frac{1}{A(P,Q)}\right]^2 J_R(P,Q)$$

Hence prove the equality

We have

$$J_R(P, Q) = 2\,[\,F(Q, P) + G(Q, P)\,]$$

So

$$M_3^*(\mathrm{P, Q}) \leq \frac{1}{8}\chi^2(P, Q) \left[\frac{1}{A(P,Q)}\right]^2 [F(Q,P) + G(Q,P)]$$

Hence prove the inequality.

Proposition 5.4: *Let* $(P, Q) \in \Gamma_n \times \Gamma_n$ *, then we have the following new inter-relation*

$$H(P, Q) + 4\, M_1^*(P,Q) - 2\, G(Q, P) \leq 2\log 2 \leq 2 + 4\, M_1^*(P,Q) - 2\, G(Q, P), \quad (35)$$

where $M_1^*(P, Q)$, $G(Q, P)$ *and* $H(P, Q)$, *are given by* (25), (7) *and* (13) *respectively.*

Proof: From (14), we have the inequality

$$\frac{x}{1+x} \leq \log(1+x) \leq x\ ; \quad x > 0$$

Taking, $x = \frac{p_i}{q_i}$, then we get

$$\frac{p_i}{p_i+q_i} \leq \log\left(\frac{p_i+q_i}{q_i}\right) \leq \frac{p_i}{q_i}$$

$$\frac{p_i}{p_i+q_i} \leq \log\left(\frac{p_i+q_i}{2q_i}.2\right) \leq \frac{p_i}{q_i}$$

$$\frac{\boldsymbol{p_i}}{\boldsymbol{p_i+q_i}} \leq \log\left(\frac{p_i+q_i}{2q_i}\right) + \log 2 \leq \frac{p_i}{q_i} \quad (36)$$

Multiplying by $2q_i$ and taking summation, we get

$$\sum_{i=1}^n \frac{2p_iq_i}{p_i+q_i} \leq 2\sum_{i=1}^n q_i \log\left(\frac{p_i+q_i}{2q_i}\right) + 2\left(\sum_{i=1}^n q_i\right)\log 2 \leq 2\sum_{i=1}^n p_i$$

$$\sum_{i=1}^n \frac{2p_iq_i}{p_i+q_i} \leq 2\sum_{i=1}^n q_i \log\left(\frac{p_i+q_i}{2q_i}\right) - \sum_{i=1}^n p_i \log\left(\frac{p_i+q_i}{2q_i}\right) + \sum_{i=1}^n p_i \log\left(\frac{p_i+q_i}{2q_i}\right) + 2\log 2 \leq 2$$

$$\sum_{i=1}^n \frac{2p_iq_i}{p_i+q_i} \leq 2\sum_{i=1}^n \left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right) - \sum_{i=1}^n (p_i-q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) + 2\log 2 \leq 2$$

$$H(P, Q) \leq 2\, G(Q, P) - 4M_1^*(P,Q) + 2\log 2 \leq 2$$

$$H(P,Q) + 4M_1^*(P,Q) - 2G(Q,P) \leq 2\log 2 \leq 2 + 4\, M_1^*(P,Q) - 2G(Q,P)$$

Hence the result

Proposition 5.5: *Let* $(P, Q) \in \Gamma_n \times \Gamma_n$ *, then we have the following new inter-relation*

$$M_1^*(P, Q) = \frac{1}{2}\mathrm{I}(P, Q) + \frac{T(P,Q)}{2\,A(P,Q)} \tag{37}$$

and

$$M_1^*(P, Q) + \mathrm{M}_1^*(\mathrm{Q}, \mathrm{P}) = \mathrm{I}(P, Q) + \mathrm{T}(P, Q), \tag{38}$$

where $M_1^*(P, Q)$, $I(P, Q)$ *and* $\mathrm{T}(P, Q)$ *are given by* (25), (11), *and* (12) *respectively.*

Proof: We have

$$M_1^*(P, Q) = \frac{1}{4}\sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$M_1^*(P, Q) = \frac{1}{4}\left[\sum_{i=1}^{n} p_i \log\left(\frac{p_i+q_i}{2q_i}\right) - \sum_{i=1}^{n} q_i \log\left(\frac{p_i+q_i}{2q_i}\right)\right]$$

$$M_1^*(P, Q) = \frac{1}{4}\left[\sum_{i=1}^{n} q_i \log\left(\frac{2q_i}{p_i+q_i}\right) - \sum_{i=1}^{n} p_i \log\left(\frac{2q_i}{p_i+q_i}\right) + \sum_{i=1}^{n} p_i \log\left(\frac{p_i+q_i}{2p_i}\right) - \sum_{i=1}^{n} p_i \log\left(\frac{p_i+q_i}{2p_i}\right)\right]$$

$$M_1^*(P, Q) = \frac{1}{4}\left[\sum_{i=1}^{n} q_i \log\left(\frac{2q_i}{p_i+q_i}\right) + \sum_{i=1}^{n} p_i \log\left(\frac{2p_i}{p_i+q_i}\right) + \sum_{i=1}^{n} p_i \log\left(\frac{p_i+q_i}{2p_i}\right) + \sum_{i=1}^{n} p_i \log\left(\frac{p_i+q_i}{2q_i}\right)\right]$$

$$M_1^*(P, Q) = \frac{1}{4}\left[\sum_{i=1}^{n} q_i \log\left(\frac{2q_i}{p_i+q_i}\right) + \sum_{i=1}^{n} p_i \log\left(\frac{2p_i}{p_i+q_i}\right)\right] + \frac{1}{4}\sum_{i=1}^{n} p_i \log\left(\frac{p_i+q_i}{2p_i}\frac{p_i+q_i}{2q_i}\right) M_1^*(P, Q)$$

$$= \frac{1}{4}[F(Q, P) + F(P, Q)] + \frac{1}{2}\sum_{i=1}^{n} p_i \log\left(\frac{p_i+q_i}{2\sqrt{p_i q_i}}\right) \tag{39}$$

But $$\mathrm{I}(P, Q) = \frac{1}{2}[F(P, Q) + F(Q, P)]$$

$$\therefore \quad M_1^*(P, Q) = \frac{1}{2}\mathrm{I}(P, Q) + \frac{1}{2}\sum_{i=1}^{n}(p_i)\sum_{i=1}^{n}\left(\frac{2}{p_i+q_i}\right)\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2\sqrt{p_i q_i}}\right)$$

$$\Rightarrow \quad M_1^*(P, Q) = \frac{1}{2}\mathrm{I}(P, Q) + \frac{1}{2\,A(P,Q)}\,T(P, Q)$$

Hence prove the required equality.

Again from (39),

$$M_1^*(P, Q) = \frac{1}{4}[F(Q, P) + F(P, Q)] + \frac{1}{2}\sum_{i=1}^{n} p_i \log\left(\frac{p_i+q_i}{2\sqrt{p_i q_i}}\right)$$

$$M_1^*(Q,P) = \frac{1}{4}[F(Q,P) + F(P,Q)] + \frac{1}{2}\sum_{i=1}^{n} q_i \log\left(\frac{p_i+q_i}{2\sqrt{p_i q_i}}\right)$$

Hence

$$M_1^*(P,Q) + M_1^*(Q,P) = \frac{1}{2}[F(Q,P) + F(P,Q)] + \sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2\sqrt{p_i q_i}}\right)$$

$$\mathrm{M}_1^*(\mathrm{P},\mathrm{Q}) + \mathrm{M}_1^*(\mathrm{Q},\mathrm{P}) = \mathrm{I}(\mathrm{P},\mathrm{Q}) + \mathrm{T}(\mathrm{P},\mathrm{Q})$$

Hence prove the equality.

Proposition 5.6: *Let* $(P, Q) \in \Gamma_n \times \Gamma_n$, then *we have the following new inter-relation*

$$K(P,Q) + F(Q,P) - 4M_1^*(P,Q) < \log 2\,, \tag{40}$$

where $M_1^*(P,Q)$, $K(P,Q)$ *and* $F(Q,P)$ *are given by* (25), (4) *and* (9) *respectively.*

Proof: We know that,

$$0 \le p_i \le 1 \;\; and \;\; 0 \le q_i \le 1$$

$$\frac{p_i}{q_i} < 1 + \frac{p_i}{q_i}$$

$$\log\left(\frac{p_i}{q_i}\right) < \log\left(1 + \frac{p_i}{q_i}\right)$$

$$\log\left(\frac{p_i}{q_i}\right) < \log\left(\frac{p_i + q_i}{2q_i}\, 2\right)$$

$$\log\left(\frac{p_i}{q_i}\right) < \log\left(\frac{p_i + q_i}{2q_i}\right) + \log 2$$

$$\sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} < \sum_{i=1}^{n} p_i \log \frac{p_i + q_i}{2q_i} + \sum_{i=1}^{n} p_i \log 2$$

$$\sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right) < \sum_{i=1}^{n} (p_i - q_i) \log\left(\frac{p_{i+} q_i}{2q_i}\right) - \sum_{i=1}^{n} q_i \log\left(\frac{2q_i}{p_i + q_i}\right) + \log 2$$

$$K(P,Q) < 4M_1^*(P,Q) - F(Q,P) + \log 2$$

$$K(P,Q) - 4M_1^*(P,Q) + F(Q,P) < \log 2$$

Hence the desired result.

6. CONCLUSIONS

In this paper, we have obtained the different information divergence measures using properties of new *f*-Divergence which are very interesting in the field of the information theory. Divergence Measures give most useful results for information theory because we can derive different information divergence measures for different value of k. We have also described different equalities and inequalities of derived *f*-divergence measures in terms of other well-known information divergence measures.

REFERENCES

[1] I. Csiszar, Information type measures of differences of probability distribution and indirect observations, Studia Math. Hungarica, 2 (1967), 299-318.

[2] K. Pearson ,On the Criterion that a given system of deviations from the probable in the case of correlated system of variables is such that it can be reasonable supposed to have arisen from random sampling, Phil. Mag., 50(1900), 157-172

[3] R.Sibson, Information radius, Z. Wahrs. Undverw.Geb, (14) (1969), 149-160.

[4] I.J.Taneja, Inequalities among logarithmic mean measures,2011.Available online: http://arxiv.org/abs/1103.2580v1.

[5] S.S.Dragomir., V.Gluscevic, and C.E.M Pearce, Approximation for the Csiszar *f*-divergence via midpoint inequalities , in inequality theory and applications - Y.J. Cho, J.K. Kim and S.S. Dragomir (Eds.), Nova Science Publishers, Inc., Huntington, New York, 1, 2001, 139-154.

[6] Jeffreys, An invariant form for the prior probability in estimation problem, Proc. Roy. Soc. Lon. Ser. A, 186(1946), 453-461.

[7] S.Kullback and R.A. Leibler., On Information and Sufficiency, Ann.Math,Statist. 22(1951), 79-86.

[8] I.J. Taneja, New developments in generalized information measures, Chapter in: Advances in Imaging and Electron Physics, Ed. P.W. Hawkes, 91(1995), 37-135.

[9] P.Kafka ,F. Osterreicher and I.Vincze, On powers of f-divergencedefining a distance,Studia Sci.Math.Hungar.,26(1991), 415-422

[10] K.C. Jain and A. Srivastava, on symmetric information divergence measures of Csiszar's *f* - divergence class, Journal of AppliedMathematics, Statistics and Informatics,3 (2007) no.1, 85-102.

[11] K.C.Jain and R.N. Saraswat ,Some new information inequalities and its Applications in information theory, International Journal of Mathematics Research, 4, no.3 (2012), 295- 307

[12] K.C.Jain and PraphullChhabra, Establishing relations among various measures by Using well known inequalities, International Journal ofModern Engineering Research, 4, no.1, (2014), 238- 246.

[13] I.J.Taneja, Generalized symmetric divergence measures and Inequalities, RGMIA Research Report Collection, http://rgmia.vu.edu.au, 7(4) (2004), Art. 9. Available online: arxiv:math.ST/0501301 v1 19 Jan 2005.

Ganita Sandesh Vol. 30 No. (1 & 2) 2016 pp. 53 –76 ISSN : 0970-9169

FINITE HORIZON EOQ MODEL FOR INSTANTANEOUS DETERIORATING ITEMS, PARTIAL BACKLOGGING AND FRACTIONAL DECREASE IN DEMAND WITH INVENTORY LEVEL DEPENDENT DEMAND RATE

[1]ANIL KUMAR SHARMA AND [2]VARSHA SHARMA

Department of Mathematics,

Raj Rishi Government Autonomous College, **ALWAR – 301 201 (INDIA)**

[1]sharma_ak002@yahoo.com, [2]varsha.sharma39@gmail.com

ABSTRACT

Paper deals with an economic order quantity (EOQ) model for instantaneous deteriorating items with partial backlogging and fractional decrease in demand for infinite time horizon incorporating inventory level dependent demand rate and deterioration begins after a certain time. In this model, shortages considered and partially backlogged. The backlogging rate depends on the waiting time for the next replenishment. The salient feature of this model is the introduction of the concept of fractional decrease in demand due to ageing of inventory. Demand at instant depends linearly on the on-hand inventory level at that instant. Deterioration of items begins after a certain time from the instant of their arrival in stock. This paper aids the retailer in minimizing the total inventory cost by calculating the optimal interval and the optimal order quantity. Numerical examples are given to prove the result. Also, the effect of changes in the different parameters on the optimal total cost is graphically presented and the implications are discussed in detail.

KEYWORDS : Partial backlogging, Deterioration, EOQ, finite horizon, fractional decrease in demand.

1. INTRODUCTION

Deterioration includes the spoilage, decay and damage of any substance with time that also cause loss of utility of the item. Deterioration plays a significant role in many inventory systems.

Over a few decades, various studies have been carried out for deterioration items such as food, medicines, volatile liquid, etc., in inventory control. Generally, inventory models deal with non-deteriorating items (i.e. items that never deteriorate) and instantaneous deteriorating items (i.e. as soon as they enter the inventory they are subject to deterioration). Most physical goods undergo decay or deterioration over time, examples being medicines, volatile liquids, blood banks, and so on. So decay or deterioration of physical goods in stock is a very realistic factor and there is a big need to consider this in inventory modeling.

Traditional inventory models were formed under the assumption of constant demand or time-dependent demand. Recently a number of inventory models are formed considering the demand to be dependent on inventory level viz. initial -level- dependent and instantaneous stock level dependent. The pioneer researcher who formed inventory models taking initial stock - level dependent demand is Gupta and Vrat [1986].

Mandal and Phaujdar [1989] corrected the flaw in Gupta and Vrat [1986] model using profit maximization rather than cost minimization as the objective. Baker and Urban [1988] developed an inventory model taking demand rate in polynomial functional form; dependent on inventory level. The same functional form was used by Datta and Pal [1990]. Datta and Pal [1990] proposed an inventory model for deteriorating items, inventory - level dependent demand and shortages. In this model the rate of deterioration is assumed to be variable. Sarkar, Mukherjee and Balan [1977] took demand to be dependent on inventory level incorporating an entirely new concept of decrease in demand (due to ageing of inventory or products reaching closer to the expiry date).

Zeng [2001] developed an inventory model using partial backordering approach and minimizing the total cost function. This model identifies the conditions for partial backordering policy to be feasible. There can be certain products, which start deteriorating after a certain period of time rather than their immediate arrival in the stock. Also there are a number of products for which the demand decreases due to ageing of these products. Jain Kumar and Advani [2008] developed a model incorporating above two realistic features and partial backlogging that is more generalized than the model given by Dye, C.Y. and Ouyang, L.Y. [2005].

Uthayakumar and Geetha [2009] formulated a replenishment policy for non-instantaneous deteriorating inventory system with partial backlogging. Chung [2009] derived a complete proof of the solution procedure for non-instantaneous deteriorating items with permissible delay in payments. Chang, Teng, and Goyal [2010] framed optimal replenishment policies for non-instantaneous deteriorating items with Stock-dependent demand. Sarkar, ghosh , Chaudhary [2012] studied an optimal inventory replenishment policy for a deteriorating item with time quadratic demand and time dependent partial backlogging with shortages in all cycles. Sarkar and Sarkar [2013] developed an improved inventory model with partial backlogging time varying deterioration and stock dependent demand. Tan and Weng [2013] developed the discrete-in-time deteriorating inventory model with time varying demand, variable deterioration rate and waiting time dependent partial backlogging.

Chauhan and Singh [2014] presented an inventory model to reflect the real situation of market for time dependent demand and deterioration with discount cash flow. Recently, Goyal et al.[2013] and Singh and Sharma [2013a, 2014] discussed inventory models for perishable items with variable demand rate. Tayal et al. [2014] proposed a two echelon supply chain model for deteriorating items with effective investment in preservation technology. Sono and joshi [2015] presented an EOQ model with backorders under linear combination of possibility and necessity measure. M.Palinivel and R.Uthayakumar [2015] developed a finite horizon EOQ model for non- instantaneous deteriorating items with price and advertisement dependent demand and partial backlogging under inflation. Goel et al.[2015] proposed supply chain model with stock dependent demand, quadratic rate of deterioration with allowable shortage.

Tripathi and Uniyal [2015] discussed the Economic order quantity model for deteriorating items with time dependent demand rate under time varying shortages. Sharma & Sharma [2015] discussed a deterministic inventory model with cubic demand and infinite time horizon with constant deterioration and salvage value. In the proceedings of the last paper, Sharma & Sharma [2015] proposed a deterministic Inventory model with selling price dependent demand rate, quadratic holding cost and quadratic time varying deteriorating rate.

Palinivel and Kumar [2016] established a finite horizon EOQ model for non- instantaneous deteriorating items with probabilistic deterioration and partial backlogging under inflation. Singh et al [2016] developed an inventory model with multivariate demands in different phases with customer returns and inflation. SR Singh [2016] developed an inventory model with multivariate demands in different phases with customer returns and inflation. He discussed the impact of customer returns on inventory system of deteriorating items under inflationary environment and partial backlogging. Kousar Jaha begum (2016) developed an EPQ model for deteriorating items with generalizes Pareto decay having selling price and time dependent demand. Sharma & Sharma [2017] proposed a deterministic inventory model with Weibull distribution deteriorating item with selling price and the time dependent demand rate.

In the present work, a deterministic inventory model for instantaneous deteriorating items, partial backlogging and fractional decrease in demand with inventory level dependent demand rate over a infinite time horizon is proposed in which no replacement or repair of deteriorated items are considered. Shortages are allowed and partially backlogged in this model. We have shown the effect due to changes in various parameters by taking suitable numerical example and sensitivity analysis.

2. ASSUMPTIONS AND NOTATIONS

To develop the mathematical model, the following assumptions are being made:

A. Assumptions

1. A single item considered over the prescribed period of planning horizon.

2. No replacement or repair of deteriorated items takes places in given cycle.

3. The lead time is Zero.

4. The replenishment takes place at an infinite rate.

The deterioration of the items begins after a time μ from the instant of their arrival in stock. Hence the deterioration of the items is assumed to be governed by the function,

$$\theta\,(t') = \theta_0\, H(\,t' - \mu) = \begin{cases} \theta_0\,, & t' > \mu \\ 0\,, & t' < \mu, \end{cases}$$

where 't' is the time measured from the instant of arrival of replenishment θ_0 $(0 < \theta_0 < 1)$ is a constant and $H(t' - \mu)$ is Heaviside's function.

5. The demand function is deterministic and is a known function of instantaneous-stock-level $q\,(t)$ is given by

$$D\,(\,t\,) = \begin{cases} \alpha + \beta\, q(t) & 0 \le t \le t_1 \\ \alpha & t_1 \le t \le T \end{cases},$$

where $\alpha > 0,\ 0 < \beta < 1$

6. Shortages are allowed and partially backlogged. During the stock- out period, the backlogging rate is variable and is dependent on the length of the waiting time for the next replenishment. So, the backlogging rate for negative inventory is , $B(t) = \dfrac{1}{[1 + \delta\,(T - t)]}$, where δ is the backlogging parameter $0 \le \delta \le 1$ and $(T - t)$ is the waiting time $(t_j \le t \le T)$; $(j = 1, 2, 3, \ldots, m)$.

7. Lot size q raises the initial inventory level at the beginning of each cycle to S after fulfilling the backorder quantity (q – S).

B. Notations

1. A is the ordering cost per order.
2. T is the fixed length of each ordering cycle.
3. $TP_f(t_1, T)$ is the profit function per unit time.
4. p is the selling price per unit.
5. C is purchasing cost price per unit per unit time.
6. i is the inventory carrying or holding cost as fraction, per unit per unit time.
7. R is the fixed opportunity cost of lost sales.
8. C_2 is the shortage cost, per unit per unit time.
9. S is the inventory level at time $t = 0$
10. μ is the length of the time in which the product has no deterioration.
11. θ is the parameter of deterioration rate of the stock.
12. $q(t)$ is the instantaneous inventory level.
13. $q_1(t)$ is the inventory level at time t, $0 \leq t \leq \mu$
14. $q_2(t)$ is the inventory level at time t, $\mu \leq t \leq t_1$
15. $q_3(t)$ is the inventory level at time t, $t_1 \leq t \leq T$
16. H is the planning horizon.
17. m is the no. f replenishment during the planning horizon, $m = \frac{H}{T}$.
18. S is the maximum inventory level.
19. BI is the maximum amount of shortages demand to be backlogged.
20. Q is the 2^{nd}, 3^{rd},..., m^{th} replenishment order size

3. FORMULATION AND THE SOLUTION OF THE MODEL

Suppose that the planning horizon H is divided into m equal parts of length $T = \frac{H}{m}$. Hence, the reorder times over the planning horizon H are $(T_j = jT)$; $(j = 0,1,2,3,\ldots,m)$.when the inventory is positive, the demand rate is dependent on the instantaneous-stock-level $q(t)$, whereas for negative inventory, the demand is partially backlogged. This model is demonstrated in figure 1.

During the time interval [0, μ], the inventory level is decreasing to the demand rate. The inventory level is decreasing to zero due to the demand and deterioration during the interval $\left[\mu, t_1\right]$. During the interval $\left[t, T\right]$, shortages occur and are accumulated until $\left(t = T_1\right)$ before they are partially backlogged. Based on the above description, during the time interval [0, μ], the inventory level reduces owing to the demand only.

Hence the differential equation representing the inventory status is given by:

$$\frac{dq_1(t)}{dt} = \alpha + \beta q(t) \qquad ;0 \le t \le \mu \qquad \ldots(1)$$

with the condition $I(0) = S$, the solution of equation (1) is

$$q_1(t) = \left(S + \tfrac{\alpha}{\beta}\right) e^{-\beta t} - \tfrac{\alpha}{\beta} \qquad ;0 \le t \le \mu \qquad \ldots(2)$$

In the second interval $\left[\mu, t_1\right]$, the inventory level decreases due to demand and deterioration. Thus the differential equation below represents the inventory status:

$$\frac{dq_2(t)}{dt} + \theta q(t) = - \ \alpha + \beta q(t) - \gamma q(t) \qquad ;\mu \le t \le t_1 \qquad \ldots(3)$$

with the condition $q_2(t_1) = 0$, we get the solution of equation (3), which is

$$q_2(t) = \frac{\alpha}{\theta_0 + \beta - \gamma} \ e^{(\theta_0 + \beta - \gamma)(t_1 - t)} - 1 \qquad ; \mu \le t \le t_1 \qquad \ldots(4)$$

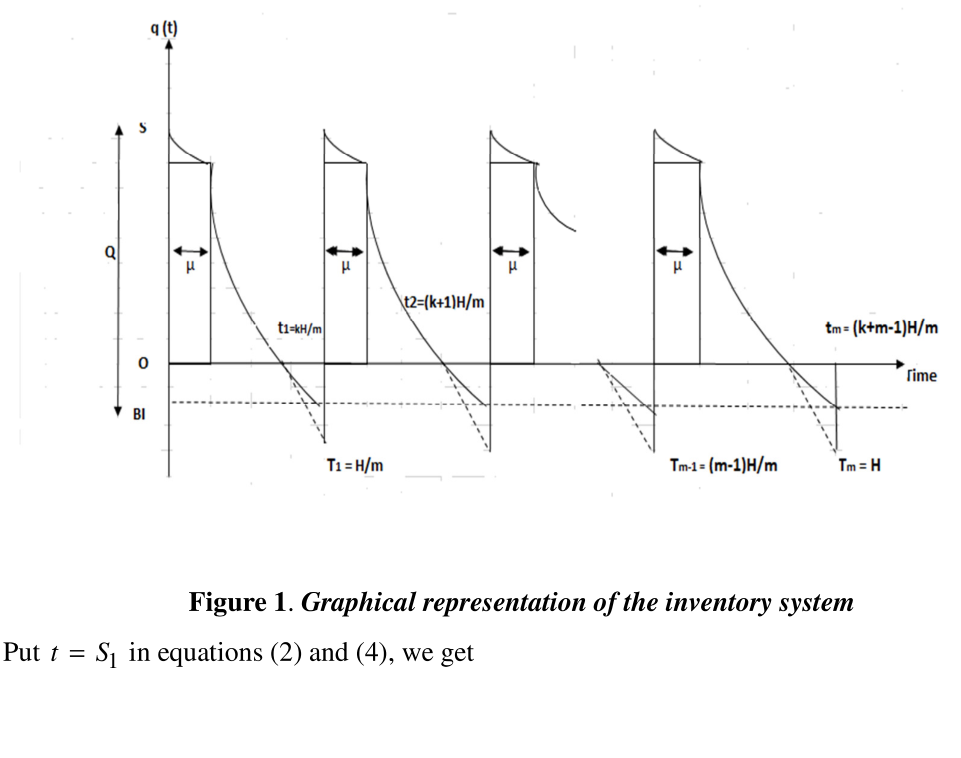


Figure 1. ***Graphical representation of the inventory system***

Put $t = S_1$ in equations (2) and (4), we get

$$S_1 = \left(S + \frac{\alpha}{\beta}\right) e^{-\beta\mu} - \frac{\alpha}{\beta} \qquad ...(5)$$

and

$$S_1 = \frac{\alpha}{\theta_0 + \beta - \gamma}\left[e^{(\theta_0+\beta-\gamma)(t_1-\mu)} - 1\right] \qquad ...(6)$$

Elimination of S_1 from equation (5) and (6), we get

$$S = \frac{\alpha e^{\beta\mu}}{\theta_0 + \beta - \gamma}\left[e^{(\theta_0+\beta-\gamma)(t_1-\mu)} - 1\right] + \frac{\alpha}{\beta}\left(e^{\beta\mu} - 1\right) \qquad ...(7)$$

Substituting equation (7) in equation (2), we get

$$q_1(t) = \left[\frac{\alpha e^{\beta\mu}}{\theta_0 + \beta - \gamma}\left[e^{(\theta_0+\beta-\gamma)(t_1-\mu)} - 1\right] + \frac{\alpha e^{\beta\mu}}{\beta}\right] e^{-\beta t} - \frac{\alpha}{\beta}$$

which on simplifies

$$q_1(t) = \left[\frac{\alpha e^{\beta\mu}}{\theta_0 + \beta - \gamma}\left[e^{(\theta_0+\beta-\gamma)(t_1-\mu)} - 1\right]\right] e^{-\beta t} - \frac{\alpha}{\beta}\left[1 - e^{-\beta(\mu-t)}\right];$$

$$0 \le t \le \mu \qquad ...(8)$$

During the third interval $[t, T]$, shortages occurred and the demand was partially backlogged. This, the inventory level at time *t* is governed by the following differential equation:

$$\frac{dq_3(t)}{dt} = \frac{-\alpha}{1 + \delta(\mathrm{T} - \mathrm{t})}; \qquad t_1 \le t \le T \qquad ...(9)$$

with the condition $q_3(t_1) = 0$, the solution of equation (9) is

$$q_3(t) = -\frac{\alpha}{\delta}\{ln[1 + \delta(\mathrm{T} - t_1)] - ln[1 + \delta(\mathrm{T} - \mathrm{t})]\};$$

$$t_1 \le t \le T \qquad ...(10)$$

Therefore the maximum inventory level and maximum amount of shortages demand to be backlogged during the first replenishment cycle are:

$$S = \frac{\alpha e^{\beta\mu}}{\theta_0 + \beta - \gamma}\left[e^{(\theta_0+\beta-\gamma)\left(\frac{kH}{m}-\mu\right)} - 1\right] + \frac{\alpha}{\beta}\left(e^{\beta\mu} - 1\right) \qquad ...(11)$$

and

$$\mathrm{B.I} = -\frac{\alpha}{\delta}\left[ln\left\{\frac{m+\delta(1-k)H}{m+\delta\mathrm{H}}\right\}\right] \quad ...(12)$$

respectively.

There are m cycles during the planning horizon. Since inventory is assumed to start and end at zero, an extra replenishment at $T_m = H$ is required to satisfy the backorders of the last cycle in the planning horizon. Therefore , there are $m + 1$ replenishment in the entire planning horizon H.

The first replenishment lot size is S.

The 2nd, 3rd,..., mth replenishment order size is

$$Q = S + BI \quad ...(13)$$

The last or $(m+1)$th replenishment lot size is BI

1. Since replenishment in each cycle is done at the start of each cycle, the present value of the ordering cost during the first cycle is

$$\mathrm{OC} = \mathrm{A} \quad ...(14)$$

2. The holding cost HC during the first replenishment cycle is

$$HC = C * i\left[\int_0^{\mu} q_1(t)dt + \int_{\mu}^{t_1} q_2(t)dt\right]$$

Using equations (2) and (4)

$$\mathrm{HC} = C * i\left\{\left(\frac{\alpha\left(e^{\beta\mu}-1\right)}{\beta^2} - 1\right) - \frac{\alpha\mu(\theta-\gamma)}{\beta(\theta+\beta-\gamma)} - \frac{\beta t_1}{\theta_0+\beta-\gamma} + \frac{\alpha}{\beta(\theta_0+\beta-\gamma)}\left[\left(e^{\beta\mu}-1\right) + \frac{\beta}{\theta_0+\beta-\gamma}\right]\right\} \quad ...(15)$$

3. The total shortages cost SC during the first replenishment cycle is given by

$$SC = -C_2\int_{t_1}^{T} q_3(t)dt$$

Using equation (10)

$$SC = \frac{C_2\alpha}{\delta}\left[(T-t_1) - \frac{1}{\delta}ln[1+\delta(T-t_1)]\right] \quad ...(16)$$

4. Opportunity Cost due to Lost sales

$$LS = \alpha R \int_{t_1}^{T} \left[1 - \frac{1}{1+\delta(T-t)}\right] dt \quad \text{which gives}$$

$$LS = \alpha R \left[(T - t_1) - \frac{1}{\delta} ln[1 + \delta(T - t_1)]\right] \qquad \dots (17)$$

5. Purchasing cost (PC) per cycle

$PC = C * S + C *$ Amount backordered $(at\ t = T))$

$$PC = C\left[\frac{\alpha e^{\beta\mu}}{\theta + \beta - \gamma}\left[e^{(\theta_0+\beta-\gamma)(t_1-\mu)} - 1\right] + \frac{\alpha}{\beta}\left(e^{\beta\mu} - 1\right)\right] + \frac{C\alpha}{\delta}\{ln[1 + \delta(T - t_1)]\} \qquad \dots (18)$$

6. Sales Revenue per cycle

$$SR = p\left\{\int_0^{\mu} demand\ in\ [0,\mu]dt + \int_{\mu}^{t_1} demand\ in\ [\mu, t_1]dt + \int_{t_1}^{T} demand\ in\ [t_1, T]\ dt\right\}$$

using equations (1), (3), (9) and (11) and solving we get

$$SR = p\left\{\frac{\alpha}{(\theta_0 + \beta - \gamma)}\left[(e^{\beta\mu} - 1) + \frac{\beta - \gamma}{\theta_0 + \beta - \gamma}\right]\left[e^{(\theta_0+\beta-\gamma)(t_1-\mu)} - 1\right] + \frac{\alpha}{\beta}\left[(e^{\beta\mu} - 1)\right] + \frac{\alpha\theta_0}{(\theta_0 + \beta - \gamma)}(t_1 - \mu) + \frac{\alpha}{\delta}\{ln[1 + \delta(T - t_1)]\}\right\} \qquad \dots (19)$$

So the total Profit function Per Unit time is given by

$$TP_f(t_1, T) = \frac{1}{T}\{sales\ revenue - ordering\ cost - holding\ cost - shortage\ cost - opportunity\ cost \quad - purchase\ cost\} \qquad \dots (20)$$

using equation (15) to (19) in equation(20), we get

$$TP_f(t_1, T) = \frac{1}{T}\left\{Xe^{(\theta_0+\beta-\gamma)(t_1-\mu)} + Z + Yt_1 - X_1(T - t_1) + Y_1 ln[1 + \delta(T - t_1)]\right\}, \qquad \dots (21)$$

where $L = \left(\frac{\alpha(e^{\beta\mu}-1)}{\beta^2}\right) - \frac{\alpha\mu(\theta-\gamma)}{\beta(\theta+\beta-\gamma)}$,

$$M = \frac{\alpha}{\beta(\theta_0+\beta-\gamma)}\left[\left(e^{\beta\mu} - 1\right) + \frac{\beta}{\theta_0+\beta-\gamma}\right],$$

$$N = \frac{\beta}{\theta_0+\beta-\gamma}$$

$$X = \frac{\alpha e^{\beta\mu}(p-C)}{(\theta_0+\beta-\gamma)} - \frac{\alpha p\theta_0}{(\theta_0+\beta-\gamma)^2} - C*i*M,$$

$$Y = \frac{\alpha p\theta_0}{\theta_0+\beta-\gamma} + C*i*M$$

$$Z = -X + \frac{\alpha}{\beta}(p-C)\left(e^{\beta\mu}-1\right) - \frac{\alpha p\theta_0\mu}{\theta_0+\beta-\gamma} - C_1 L - C_3$$

$$X_1 = \frac{C_2\alpha}{\delta} + \alpha R \qquad and \qquad Y_1 = \frac{1}{\delta}[X_1 + \alpha(p-C)]$$

4. SOLUTION PROCEDURE

For the maximization of profit we set, $\frac{\partial TP_f(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_f(t_1,T)}{\partial T} = 0$ using equation (21)

$$\frac{\partial TP_f(t_1,T)}{\partial t_1} = 0 \implies T = t_1 + \frac{Y_1}{£(t_1)} - \frac{1}{\delta} \qquad \text{... (22)}$$

where, $£(t_1) = X(\theta_0+\beta-\gamma)e^{(\theta_0+\beta-\gamma)(t_1-\mu)} + Y + X_1$

and $\quad \frac{\partial TP_f(t_1,T)}{\partial T} = 0 \implies \frac{-1}{T^2}\left\{Xe^{(\theta_0+\beta-\gamma)(t_1-\mu)} + Z + Yt_1 - X_1(T-t_1) + Y_1\ln[1+\delta(T-t_1)]\right\} +$

$$\frac{1}{T}\left\{-X_1 + \frac{Y_1\delta}{1+\delta(T-t_1)}\right\} = 0 \qquad \text{... (23)}$$

On eliminating *T* from equation (22) and (23), we get an equation in a single variable t_1 as,

$$Xe^{((\theta_0+\beta-\gamma))(t_1-\mu)}\left[1 + \frac{(\theta_0+\beta-\gamma)}{\delta} - (\theta_0+\beta-\gamma)t_1\right] + Z - Y_1 + \frac{1}{\delta}[Y+X_1] + Y_1\ln Y_1\,\delta$$

$$- Y_1\ln\left[Y + X_1 + X(\theta_0+\beta-\gamma)e^{((\theta_0+\beta-\gamma))(t_1-\mu)}\right] = 0 \qquad \text{... (24)}$$

Let t_1* be the optimal root obtained . After that we can get *T** by using equation (22). Hence t_1* and *T** jointly constitute the optimal solution provided the following conditions are satisfied,

$$\left(\frac{\partial TP_f(t_1,T)}{\partial t_1}\right)_{t=t_1^*} < 0, \quad \left(\frac{\partial TP_f(t_1,T)}{\partial T}\right)_{T=T^*} < 0,$$

and $$\left(\frac{\partial^2 TP_f(t_1,T)}{\partial {t_1}^2}\right)_{t=t_1^*}\left(\frac{\partial^2 TP_f(t_1,T)}{\partial T^2}\right)_{T=T^*} > \left(\frac{\partial^2 TP_f(t_1,T)}{\partial t_1 \partial T}\right)^2_{\substack{t=t_1^* \\ T=T^*}}$$

By using the optimal values of t_1* and T* in equation (21), optimal total profit $TP_f({t_1}^*, T^*)$ can be obtained.

Case 1: ***Complete backlogging*** **(** δ **= 0) .**

In this case, shortages are completely backlogged and profit per unit time is given as

$$TP_f(t_1,T) = \frac{1}{T}\left\{Xe^{(\theta_0+\beta-\gamma)(t_1-\mu)} + Z + Yt_1 + (p-C)(T-t_1)\alpha - \frac{C_2\alpha(T-t_1)^2}{2}\right\}$$

In this case our model is reduces to Jain & Kumar [7]

Case II : ***Complete lost sales*** **(** $\delta \to \infty$ **)**

For this case, from equation (21) we get $T^* = t_1$. In this situation optimal solution does not allow shortage. In this case profit per unit time is given as,

$$TP_f(t_1) = \frac{1}{t_1}\left\{Xe^{(\theta_0+\beta-\gamma)(t_1-\mu)} + Z + Yt_1\right\}$$

5. NUMERICAL EXAMPLE

Consider an inventory system with the following data:

$\alpha = 600, \beta = \{0.2,0.3,0.4\}$, $\mu = \{0.2,0.3,0.4\}$, $\delta = \{1,5,10,25,50\}$, $\gamma = 0.01, S = 7$,

$A = 250$, $\theta_0 = 0.05$, $C = 5$, $i = 0.35$, $C_2 = 3, p = 12$, $R = 0.2$; in appropriate units.

Using the solution procedure described above, the results are presented in table 1. From the table we see that when the backlogging parameter increases, total profit decreases.

The results obtained for constant and varying μ (and vice-versa) are shown in Tables 1 to 6. The tables also incorporate results of case 1, $\delta = 0$ (i.e. complete backlogging) and case 2, $\delta \to \infty$ (i.e. complete lost sales) as a special case.

Table-1 ***Total profit function when*** **$\mu = 0.2$ and β = 0.2**

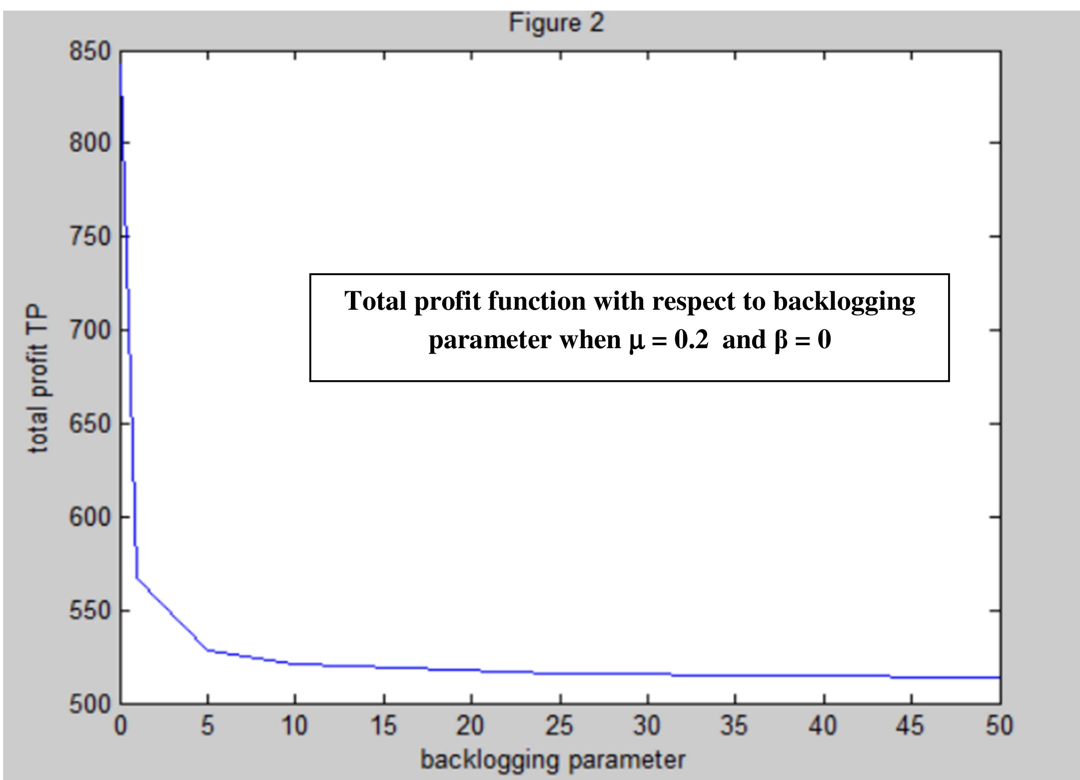
β	δ	t_1	T	TP_f
0.2	0	0.5659	0.87274	841.87641
	1	0.63867	0.75667	566.75268
	5	0.67245	0.70698	528.62619
	10	0.6791	0.69774	521.08117
	25	0.68353	0.69115	516.05884
	50	0.68568	0.68955	514.2117
	$\delta \to \infty$	0.68668	0.68668	512.47409

Table-2 ***Total profit function for*** **$\mu = 0.2$ and β = 0.3**

β	δ	t_1	T	TP_f
0.3	0	0.5659	0.89697	843.84668
	1	10.63867	0.78262	597.79521
	5	0.67245	0.73448	564.81034
	10	0.6791	0.7253	558.36507
	25	0.68353	0.71927	554.08376
	50	0.68568	0.71715	552.57988
	∞	0.68668	0.71498	551.03494

Table-3 ***Total profit function for*** **μ = 0.2 and β = 0.4**

β	δ	t_1	T	TP_f
0.4	0	0.64692	0.92804	847.2727
	1	0.71134	0.81598	631.6153
	5	0.73952	0.7696	603.7670
	10	0.74493	0.76085	598.332
	25	0.74849	0.75509	594.823
	50	0.74974	0.75308	593.575
	∞	0.75103	0.75103	592.294



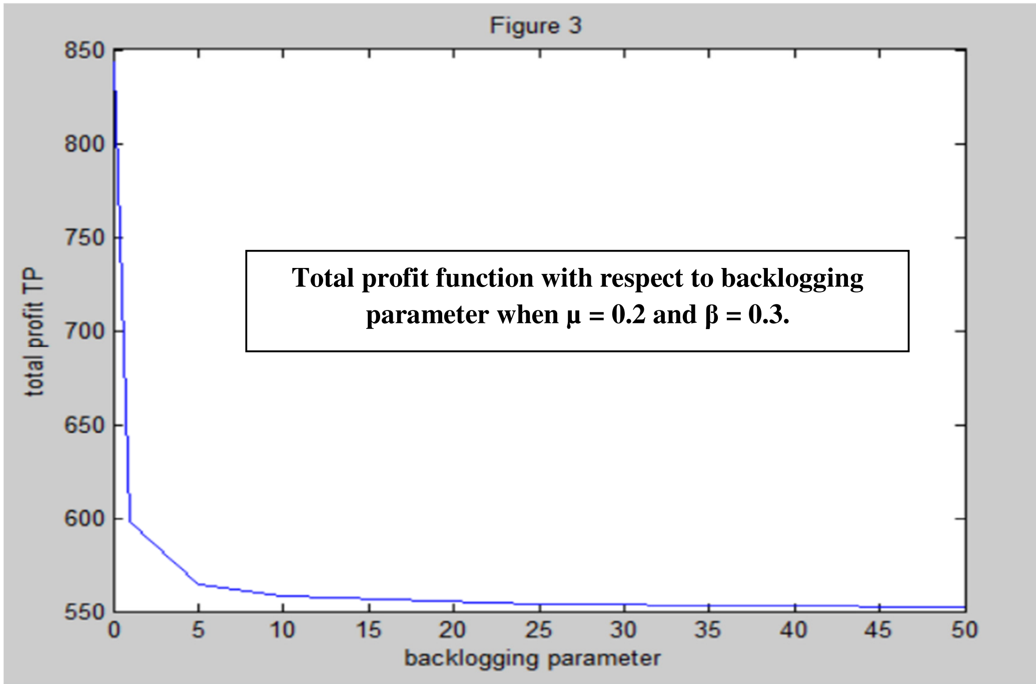


Figure 3
850
800
750
700
650
600
550
0 5 10 15 20 25 30 35 40 45 50
total profit TP
backlogging parameter
Total profit function with respect to backlogging parameter when μ = 0.2 and β = 0.3.

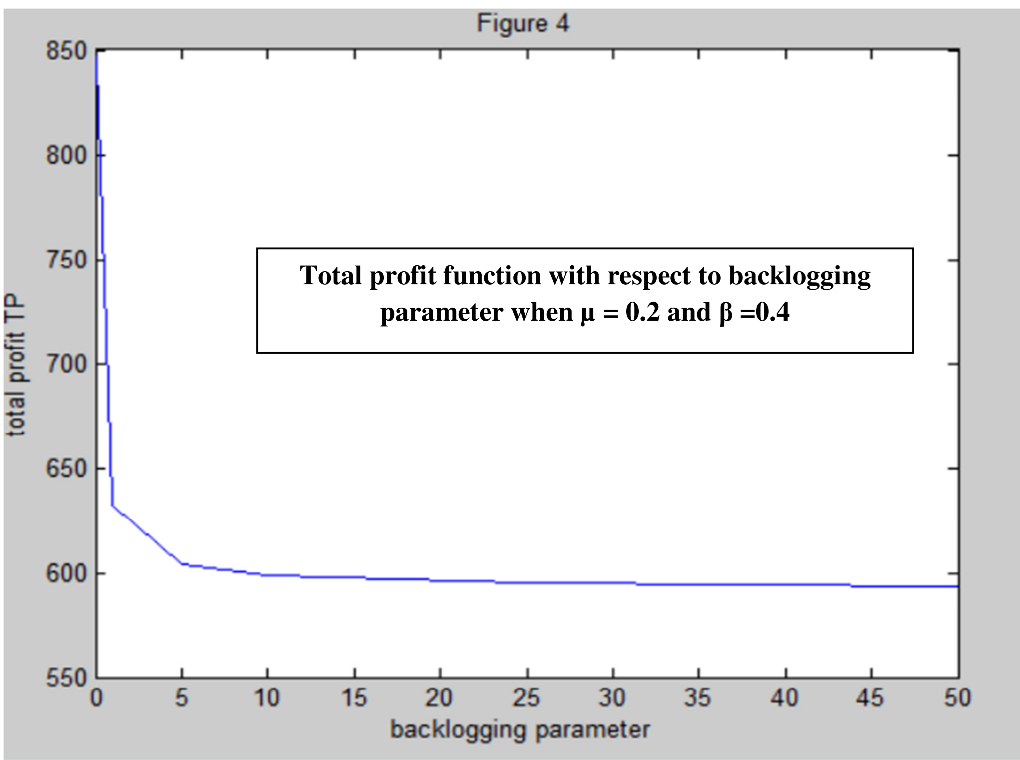


Figure 4
850
800
750
700
650
600
550
0 5 10 15 20 25 30 35 40 45 50
total profit TP
backlogging parameter
Total profit function with respect to backlogging parameter when μ = 0.2 and β =0.4

Table-4 *Total profit function for* **β = 0.2 , μ = 0.2**

μ	δ	t_1	T	TP_f
0.2	0	0.5659	0.87274	841.87641
	1	0.63867	0.75667	566.75268
	5	0.67245	0.70698	528.62619
	10	0.6791	0.69774	521.08117
	25	0.6853	0.69115	516.05884
	50	0.68568	0.68955	514.2117
	∞	0.68668	0.68668	512.47409

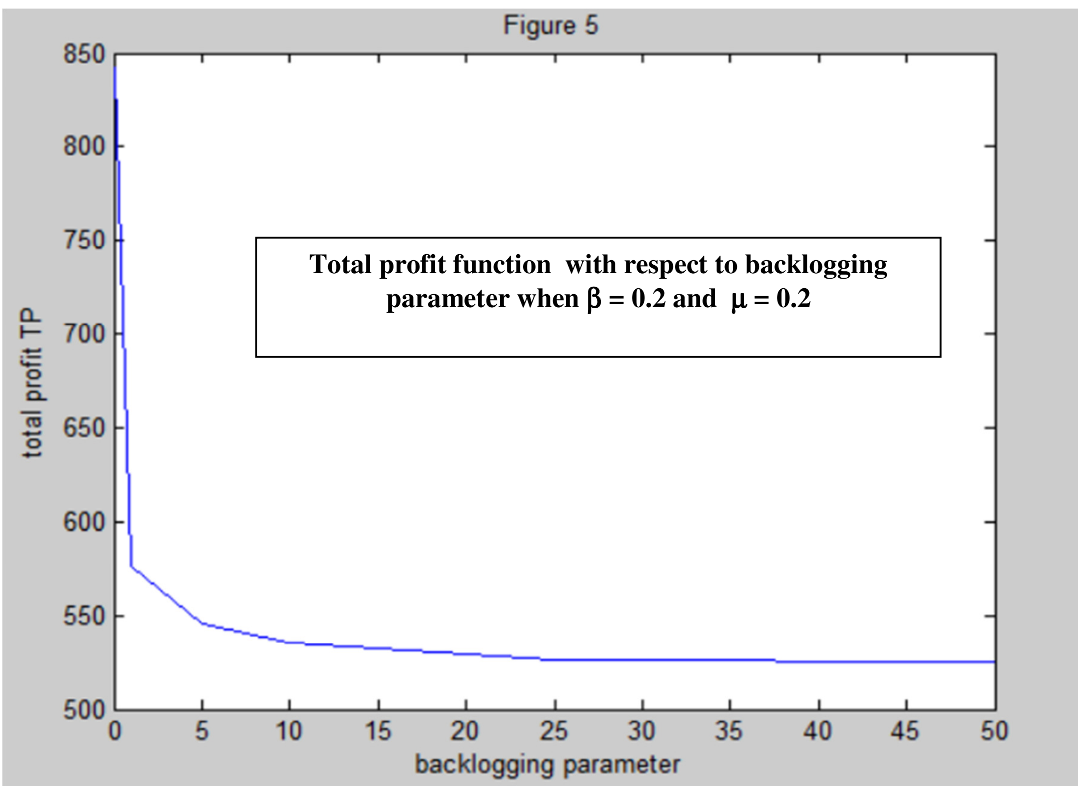


Table-5 ***Total profit function for*** **$\beta = 0.2$, $\mu = 0.3$**

μ	δ	t_1	T	$T\,P_f$
0.3	0	0.57613	0.87896	842.71407
	1	0.64637	0.76224	576.82077
	5	0.6788	0.71263	540.27405
	10	0.68518	0.70314	533.06264
	25	0.68941	0.69687	528.26645
	50	0.6909	0.69468	526.57948
	∞	0.69243	0.69243	524.84518

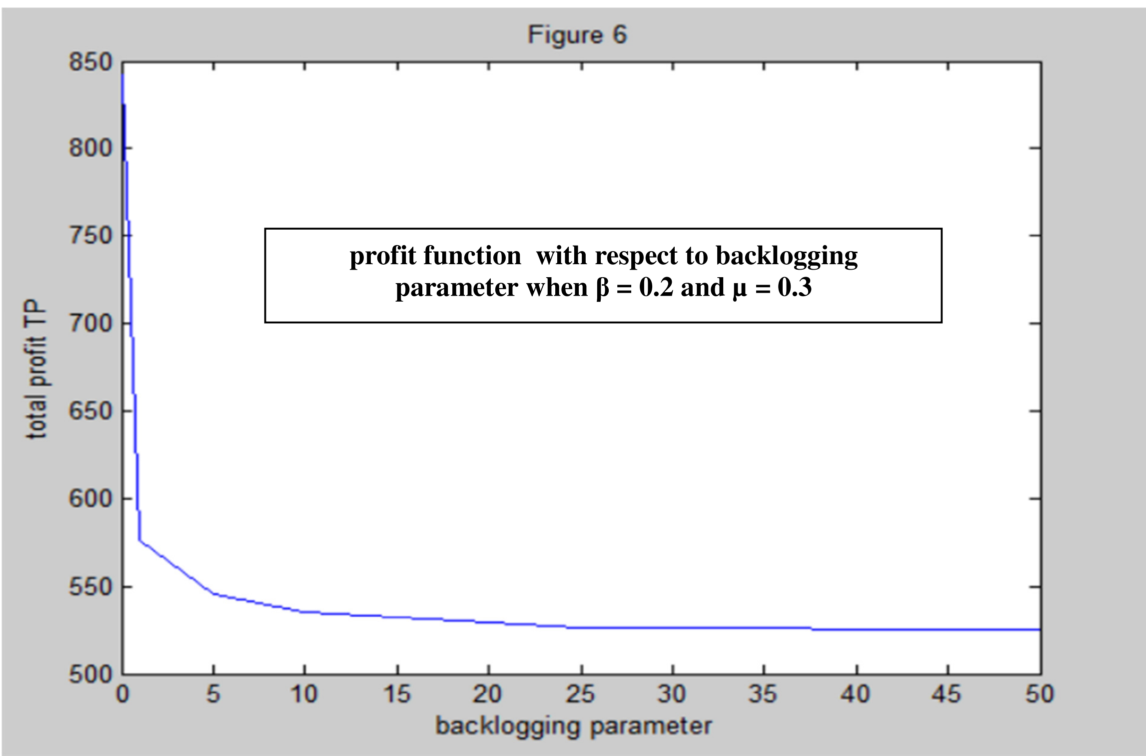
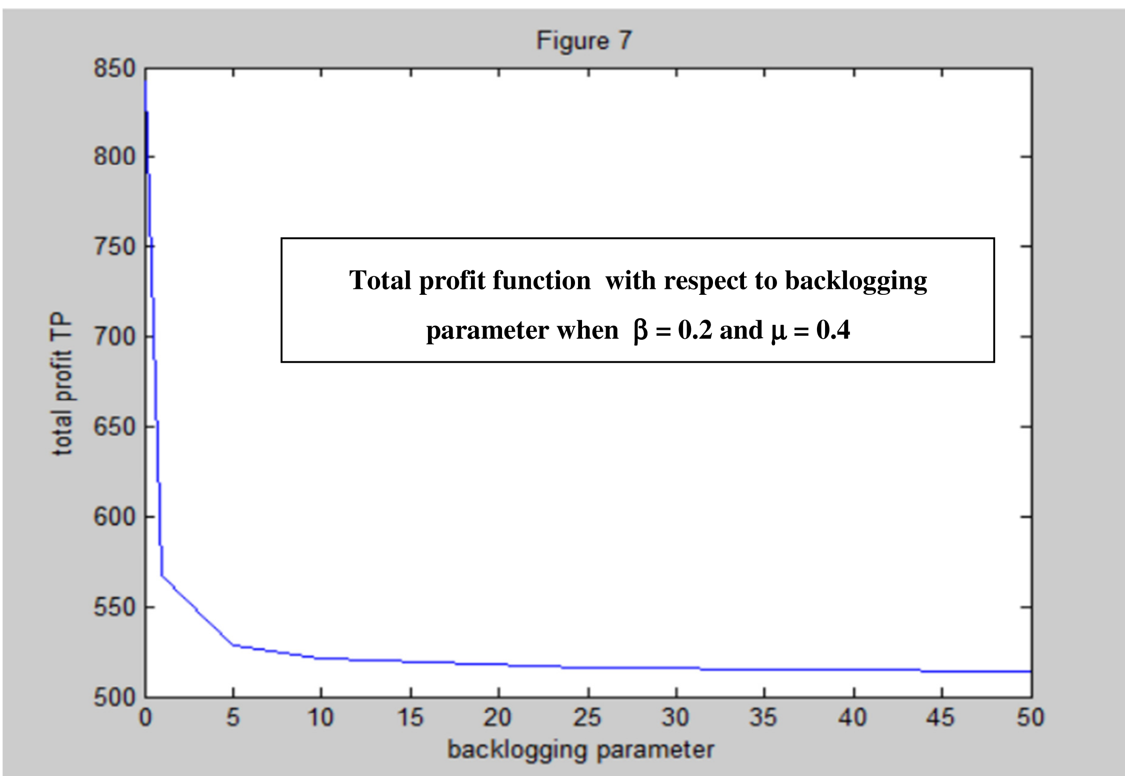


Table-6 ***Total profit function for*** **β = 0.2 and μ = 0.4**

	δ	t_1	T	TP_f
0.4	0	0.58854	0.88876	843.55173
	1	0.65669	0.77112	583.90669
	5	0.68802	0.72135	548.68534
	10	0.69416	0.71185	541.75345
	25	0.69824	0.70559	537.14667
	50	0.69967	0.70339	535.52657
	∞	0.70114	0.70114	533.86225



6. SENSITIVITY ANALYSIS

We now study the effects of changes in the value of the system parameters θ, δ, μ, p and R on the optimal replenishment policy of the example 1. We change one parameter at a time keeping the other parameters unchanged. The results are summarized in table 7. Based on the numerical results, we obtain the following managerial phenomena:

1. When the deterioration rate θ is increasing, the optimal cost is decreasing and the order quantity is increasing. Figure 8 shows that when the deterioration rate increases, the number of replenishment decreases. So, the total cost will decrease. Moreover, the minimum deterioration rate of the products will minimize the deterioration cost of the items for the retailer.

2. When the length of the fresh product time μ is decreasing, the total cost and the order quantity decreasing. Since the order quantity is reduced, automatically the holding cost of the items will also be reduced. Figure 9 shows the convexity of the total cost function with respect to changes in the parameter μ.

3. If the backlogging parameter δ is increased then the total profit and total order quantity will be decreased. In figure 10, if the backlogging parameter increases then the ordering quantity will decreases. Therefore the total profit also decreases. So, In order to minimize the cost, the retailer should increase the backlogging parameter.

4. When the discount rate of inflation R is increasing, the optimal cost is decreasing and the ordering cost is increasing. Figure 11 shows that when the discount rate increases, the number of replenishment rate decreases. So the total cost of the retailer will also be decreased.

5. When the selling price p is increasing, the total optimal cost and the order quantity are highly decreasing. but the increase of the selling price will decrease the demand as well as decrease the order quantity. So the total cost of the retailer will decrease which is illustrated in figure 12.

Table 7. ***Sensitivity analysis for various inventory parameters*.**

Parameter	**Parameter value**	***m***	***k***	***Q***	***TP***
Θ	0.1	10	0.8481	133.27	9430.6332
	0.2	8	0.3657	162.61	8788.0230
	0.3	6	0.2107	212.39	8728.7074

μ	0.2	9	0.3001	144.35	8773.3654
	0.4	8	0.3657	162.61	8788.0230
	0.6	7	0.4222	186.24	8849.1868
δ	0.05	8	0.3693	164.29	8928.9701
	0.1	8	0.3657	162.61	8788.0230
	0.15	8	0.3619	160.88	8643.5586
R	0.18	10	0.6346	132.22	9601.4801
	0.2	8	0.3654	162.61	8788.0230
	0.22	9	0.2604	213.21	8194.0680
P	15	6	0.3426	297.26	17517.8239
	20	5	0.3657	162.61	8788.0230
	25	9	0.3884	106.25	5217.3068

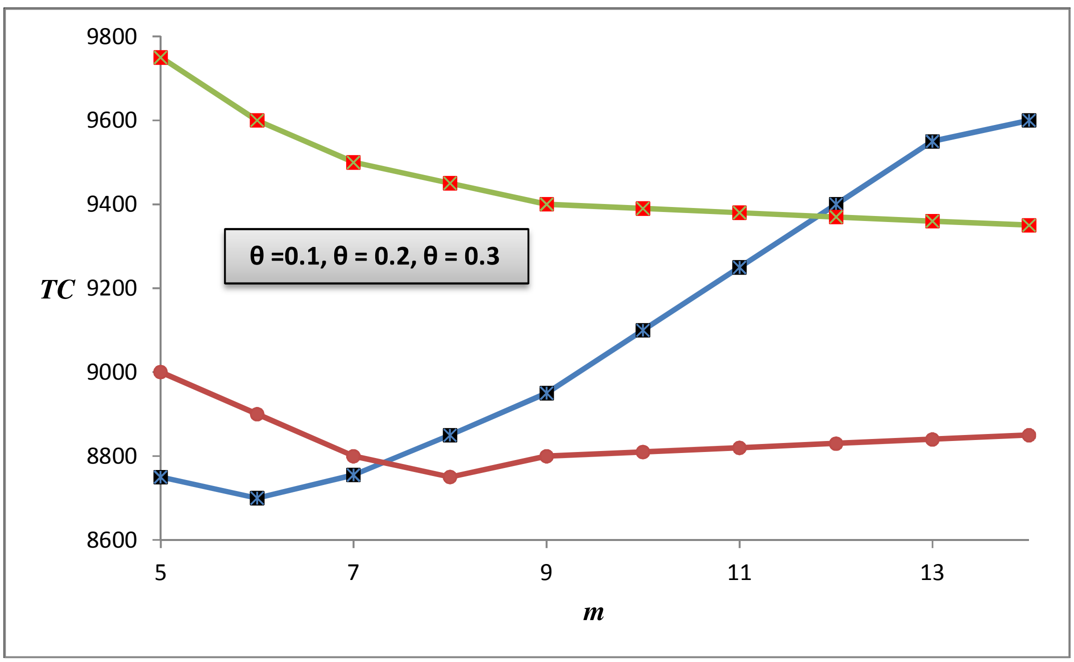


Figure 8

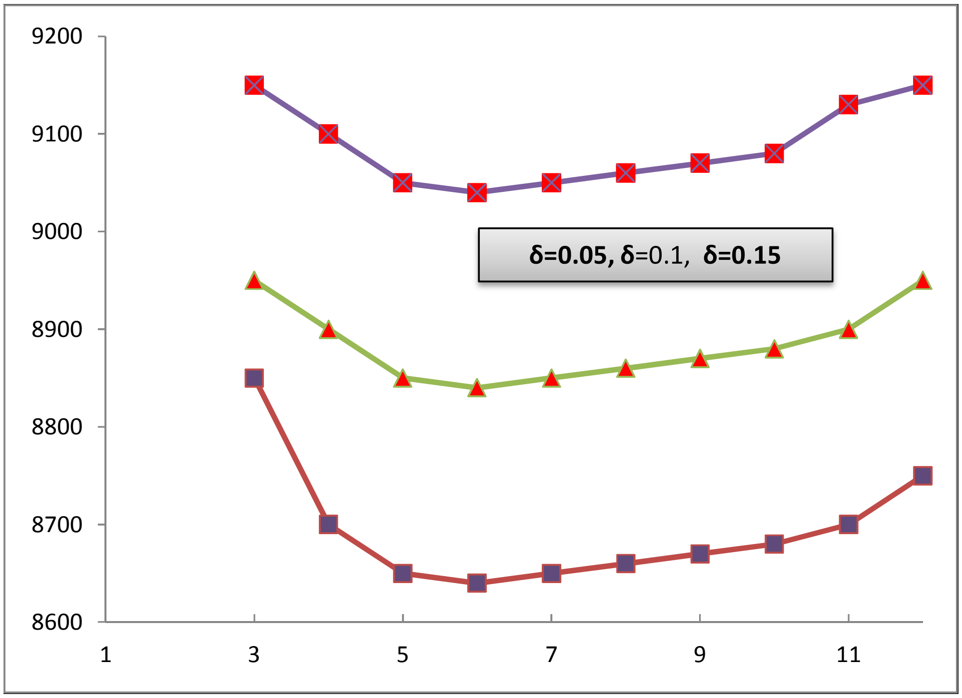


Figure 9 *Total cost vs. different values of δ*

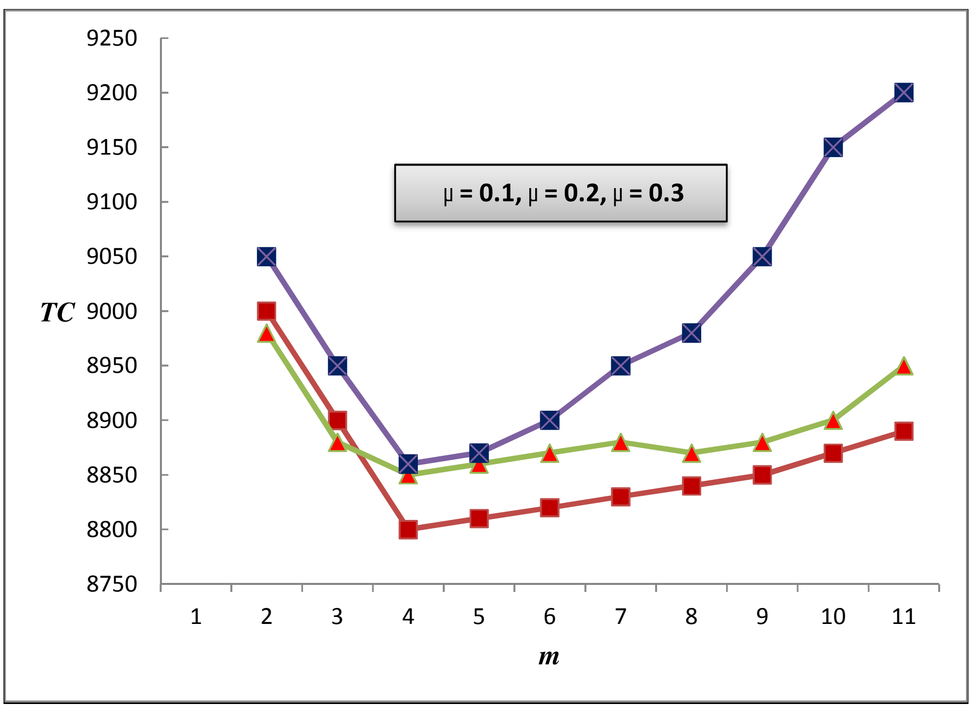


Figure 10 *Total cost vs. different values of μ*

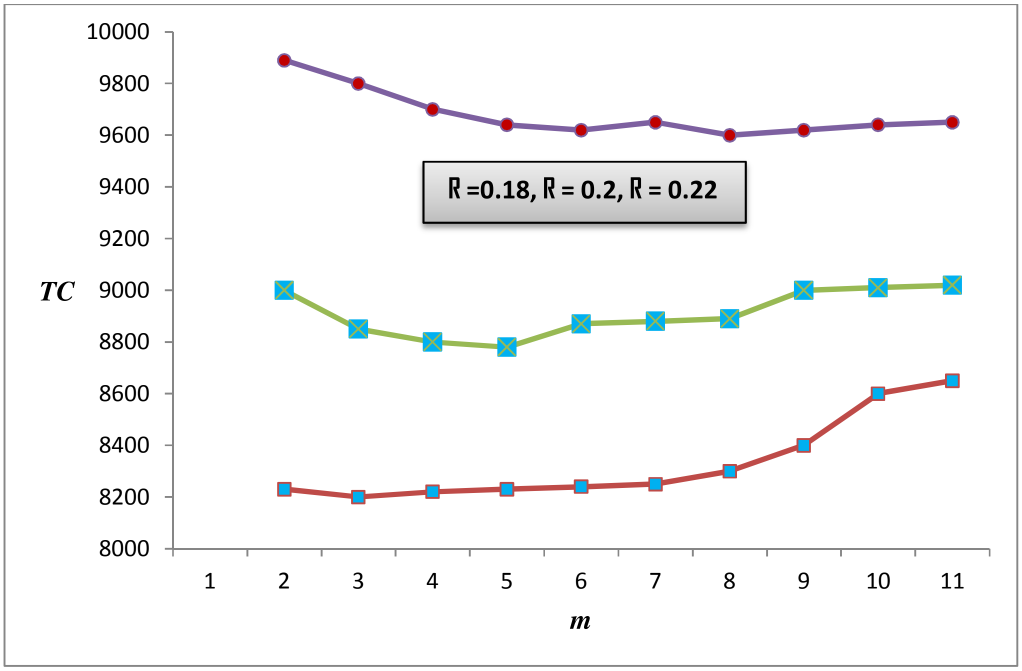


Figure 11 *Total cost vs. different values of R*

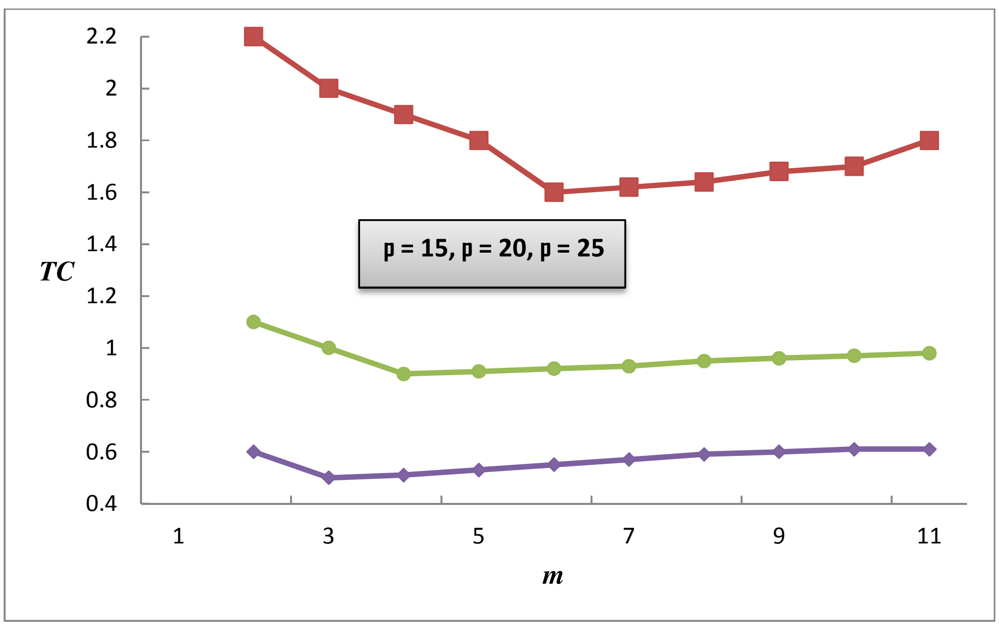


Figure 12 *Total cost vs. different values of p*

7. CONCLUSION

In this Paper, an economic order quantity (EOQ) model for instantaneous deteriorating items with partial backlogging and fractional decrease in demand for infinite time horizon incorporating inventory level dependent demand rate is discussed and deterioration begins after a certain time. In this model, shortages are allowed and partially backlogged. The backlogging rate is dependent on the waiting time for the next replenishment. The salient feature of this model is the introduction of the concept of fractional decrease in demand due to ageing of inventory. Demand at instant depends linearly on the on-hand inventory level at that instant. Deterioration of items begins after a certain time from the instant of their arrival in stock. This paper aids the retailer in minimizing the total inventory cost by finding the optimal interval and the optimal order quantity. We illustrated through a numerical example how the optimal order quantity and the optimal total cost can be derived. The results of the proposed model shows that there is a decrease in the total cost from the non-instantaneously deteriorating items compared with instantaneously deteriorating items. Also when the net discount rate of inflation and the backlogging rate increases, the optimal total cost will decrease. Furthermore, sensitivity analysis was carried out with respect to the key parameters and useful managerial insights were obtained. Graphical illustration are also given to analyze the efficiency of the model clearly.

The proposed models described some realistic features that are likely to be associated with some kinds of inventory. The model is very useful in retail businesses such as that of electronic components, fashionable clothes, domestic goods and other products.

REFERENCES

[1] Baker, R.C. and Urban, T.L. (1988). A deterministic inventory system with an inventory-level-dependent demand rate. *Journal of the Operational Research Society*, 39(9): 823-831.

[2] Begum, K.J & Devendra,P.(2016). EPQ model for deteriorating items with Generalizes Parato Decay having selling price and time dependent demand.*Vol.4(1),*1-11.

[3] Chang, C. T., Teng, J. T., and Goyal, S. K. (2010). Optimal replenishment policies for Non-Instantaneous Deteriorating items with Stock-Dependent Demand. *International Journal of Production Economics, 123,* 62-68.

[4] Chung, K. J. (2009). A complete proof on the solution procedure for Non-Instantaneous deteriorating items with permissible delay in payment, *Computers & Industries Engineering, 56:* 267-273.

[5] Datta, T.K. and Pal, A.K. (1990b). Deterministic inventory system for deteriorating item with inventory-level-dependent demand rate and shortages. *Opsearch*, 27(4): 213-224.

[6] Dye, C. Y., Ouyang, L. Y., and Hsieh, T. P.(2005). Deterministic Inventory model for deteriorating items with capacity constraint and time proportional backlogging rate, *European Journal of Operational research, 178*, 789- 807.

[7] Goyal, S. K., Singh, S. R. and Dem, H. (2013). Production policy for ameliorating/deteriorating items with ramp type demand, *International Journal of Procurement Management, Vol;.6.No.4,* 445 - 465.

[8] Goel, R. Singh, A. P., and Sharma R. (2015). Supply chain model with stock dependent demand, quadratic rate of deterioration with allowable shortage. *Int. J. of Mathematics in Operational Research,Vol.7, No.2,* 156-177.

[9] Gupta, R. and Vrat, P. (1986). Inventory model for stock-dependent consumption rate. *Opsearch*, 23(1): 19-24.

[10] Jain, S., kumar, M., and Advani P. (2008). An Inventory model with inventory level- dependent demand rate, deterioration, partial backlogging and decrease in demand. *International Journal of Operations Research Vol. 5*, No. 3, 154-159.

[11] Mandal, B.N. and Phaujdar, S. (1989a). A note on an inventory model with stock dependent consumption rate. *Opsearch*, 26(1): 43-46.

[12] Palanivel, M. and Uthayakumar, R.(2016). Finite horizon Eoq model for non deteriorating items with probabilistic deterioration and partial backlogging under inflation. *Int. J. of Mathematics in Operational Research, Vol.8, No.4,* 449-476.

[13] Sarker, B.R., Mukherjee, S., and Balan, C.V. (1997). An order level lot-size inventory model with inventory-level-dependent demand and deterioration. *International Journal of Production Economics*, 48: 227-236.

[14] Sarkar, T., Ghosh, S.K., and Chaudhuri, K.S. (2012), 'An Optimal Inventory Replenishment Policy for a Deteriorating Item with Time-Quadratic Demand and Time-Dependent Partial Backlogging with Shortages in All Cycles', *Applied Mathematics and Computation*, 218, 9147–9155.

[15] Sarkar, B., and Sarkar, S. (2013), 'An Improved Inventory Model with Partial Backlogging, Time Varying Deterioration and stock-Dependent Demand', *Economic Modeling*, 30, 924–932

[16] Sharma, Anil Kumar and Sharma, V. (2015). A deterministic Inventory model with cubic demand and infinite time horizon with constant deterioration and salvage value. *International Journal of Science and Research, Volume 5 Issue 11:*1643-1646.

[17] Sharma, Anil Kumar and Sharma, V. (2015). A deterministic Inventory model with selling price dependent demand rate, quadratic holding cost and quadratic time varying deteriorating rate. *International Journal of Science and Research, Volume 6 Issue 3:*193-198.

[18] Sharma, Anil Kumar and Sharma, V. (2017). A deterministic Inventory model with Weibull distribution deteriorating item with selling price and the time dependent demand rate. *International Academy of Physical Sciences,Vol.21 No.1*: 31 - 38.

[19] Singh, S. R. and Sharma, S. (2013a). A global optimizing policy for decaying items with ramp-type demand rate under two- level trade credit financing taking account of preservation technology, *Advance in Decision Sciences, Artical ID 126385,12:* doi :10.1155/2013/126385.

[20] Singh, S. R. and Sharma, S. (2013b). An integrated model with variable production and demand rate under inflation. *International conference on computational Intelligence: Modelling technques and applications (CIMTA), Procedia technology, Vol.10.* 381-391.

[21] Singh, S. R. and Sharma, S. (2014). Optimal trade-credit policy for perishable items deeming imperfect production and stock dependent demand. *Industrial journal of Industrial engineering computation,Vol.5,No.1;* 151-168.

[22] Singh, S.R., jain, S. and Pareek, S.(2012). A warehouse imperfect fuzzified production model with shortages under inflationary conditions , *Advance in Decision Sciences, Artical ID 638060,16:* doi :10.1155/2012/638060.

[23] Singh, S. R. and Sharma, S. (2016). Inventory model with multivariate demands in different phases with customer returns and inflation. *Int. J. of Mathematics in Operational Research, Vol.8, No.4,* 477-489.

[24] Soni. Hardik, N. ad Joshi M.(2015). EOQ model with backorders under linear combination of possibility and necessity measure. *Int. J. of Mathematics in Operational Research, Vol.7, No.2,* 194-214.

[25] Tan, Y., and Weng, M.X. (2013), A Discrete-In-Time Deteriorating inventory Model With Time-Varying Demand, Variable deterioration Rate and Waiting-Time-Dependent Partial backlogging, *International Journal of Systems Science*, 44, 1483–1493.

[26] Tayal, S., Singh, S. R. and Chauhan, A.(2014).Two echelon supply chain model for deteriorating items with effective investment in preservation technology. *Int. J. of Mathematics in Operational Research,Vol.6,No.1,* 84-105.

[27] Tripathi, R.P. and Uniyal kumar, A.(2015). Economic order quantity model for deteriorating items with time-dependent demand rate under time varying shortages. *Int. J. of Mathematics in Operational Research,Vol.7, No.6,* 706-719.

[28] Uthayakumar, R., and Geetha, K.V. (2009), 'A Replenishment policy for Non-instantaneous Deteriorating Inventory System with Partial Backlogging', *Tamsui Oxford Journal of Mathematical Sciences*, 25, 313–332.

[29] Zeng, A.M. (2001). A partial backordering approach to inventory control. *Production planning & Control, 12(7).* 660-668.

Ganita Sandesh Vol. 30 No. (1 & 2) 2016 pp. 77 – 84 **ISSN : 0970-9169**

DOUBLE DIRICHLET AVERAGE OF MAINARDI FUNCTION AND FRACTIONAL DERIVATIVE

MOHD. FARMAN ALI[1], RENU JAIN[2], MANOJ SHARMA[3]

[1, 2] *School of Mathematics and Allied Sciences,*

Jiwaji University, **GWALIOR (INDIA)**

[3] *Department of Mathematics*

RJIT, BSF Academy**, TEKANPUR (INDIA)**

[1]*mohdfarmanali@gmail.com,* [3]*manoj240674@yahoo.co.in*

ABSTRACT

The object of the preset paper is to establish a result of Double Dirichlet average of Mainardi function by using fractional derivative.

Keywords and Phrases: Dirichlet average, Mainardi function, fractional derivative and Fractional calculus operators.

1. INTRODUCTION

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions likex^t,e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n,e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process.

Recently, Gupta and Agarwal [9, 10] found that averaging process is not altogether new but directly connected with the old theory of fractional derivative. Carlson overlooked this connection whereas he has applied fractional derivative in so many cases during his entire work. Deora and Banerji [6] have found the double Dirichlet average of e^x by using fractional derivatives and they have also found the Triple Dirichlet Average of x^t by using fractional derivatives [7].

In the present paper the Dirichlet average of Mainardi function has been obtained.

2. DEFINITIONS

Some definitions which are necessary in the preparation of this paper.

2.1 Standard Simplex in $R^n, n \geq 1$:

Denote the standard simplex in R^n, $n \geq 1$ by [1, p.62].

$$E = E_n = \{S(u_{1,}u_2, u_n) \ : \ u_1 \geq 0, \ldots u_n \geq 0, \ u_1 + u_2 + \cdots + u_n \leq 1\}$$

2.2 Dirichlet Measure:

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by $E[1]$.

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \ldots u_{k-1}^{b_{k-1}-1}(1 - u_1 - \cdots - u_{k-1})b_k^{-1} du_1 \ldots du_{k-1}$$

Will be called a Dirichlet measure.

Here

$$B(b) = B(b1, \ldots bk) = \frac{\Gamma(b_1) \ldots \Gamma(b_k)}{\Gamma(b_1 + \cdots + b_k)}, C_> = \{z \in z : z \neq 0, |ph\, z| < {}^{\pi}/_{2}\},$$

Open right half plane and $C_>k$ is the k^{th} Cartesian power of $C_>$

2.3 Dirichlet Average[1, p.75]:

Let Ω be the convex set in $C_{>}$, let $z = (z_1, \ldots, z_k) \in \Omega^{\mathrm{k}}, \mathrm{k} \geq 2$ and let $u.z$ be a convex combination of $z_1, \ldots, z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define:

$$F(b, z) = \int_E f(u.z) d\, \mu_b(u) \tag{2.3}$$

F is the Dirichlet measure of f with variables $z = (z_1, \ldots, z_k)$ and parameters $b = (b_1, \ldots b_k)$.

Here

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - \cdots - u_{k-1.}$$

If $k = 1$, define $F(b, z) = f(z)$.

2.4 Mainardi Function:

The Mainardi function is

$$(z, \alpha) = \sum_{n=o}^{\infty} (-1)^n \frac{z^n}{n!\Gamma(-\alpha(n+1)+1)}, \quad \text{Here}, \alpha\, \epsilon\, C, R(\alpha) > 0. \tag{2.4}$$

Here $\alpha \in C, \boldsymbol{R}(\alpha) > 0$.

2.5 Fractional Derivative [8, p.181]:

The theory of fractional derivative with respect to an arbitrary function has been used by Erdelyi[8]. The most common definition for the fractional derivative of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^{\alpha} F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1} dt \quad , \tag{2.5}$$

where $Re(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0$.

2.6 Average of cosh x (from [4]):

let μ^b be a Dirichlet measure on the standard simplex E in R^{k-1}; k≥ 2. For every $z \in C^k$,

$$S(b,z) = \int_E M\,(u,z;\alpha)d\mu_b\ (u) \tag{2.6}$$

If $k = 1, S = (b,z) = M\,(u,z;\alpha)$.

2.7 Double Averages of Functions of One Variable (from [1, 2]):

let z be a $k \times x$ matrix with complex elements z_{ij}. Let $u = (u_1, \dots u_k) and\ v = (v_1, \dots v_k)$ be an ordered k-tuple and x-tuple of real non-negative weights $\sum u_i = 1\ and\ \sum v_j = 1$, respectively. Define:

$$u.z.v = \sum_{i=1}^{k} \sum_{j=1}^{x} u_i z_{ij} v_j \tag{2.7}$$

If z_{ij} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of $(z_{11}, \dots z_{kx})$, denote by $H(z)$.

Let $b = (b_1, \dots b_k) be$ an ordered k –tuple of complex numbers with positive real part (Re(b) > 0) and similarly for β=$(\beta_{1,} \dots \beta_x)$. Then we define $d\mu_b(u) and\ d\mu_b(v)$.

Let f be the holomorphic on a domain D in the complex plane,

If $Re(b) > 0, Re(\beta) > 0 and H(z) \subset D$, we define:

$$F(b,z,\beta) = \iint f(u,z,v)\,d\mu_b(u)d\mu_b(v) \tag{2.8}$$

Corresponding to the particular function $cosh\,x, z^t$ and e^z we define,

$$S(b,z,\beta) = \quad M\,(u,z,v;\alpha)\,d\mu_b(u)\,d\mu_b(v) \tag{2.9}$$

$$R_t(b,z,\beta) = \quad (u,z,v)^t\,d\mu_b(u)\,d\mu_b(v) \tag{2.10}$$

$$S(b,z,\beta) = \quad (e)^{u,z,v}\,d\mu_b(u)\,d\mu_b(v) \tag{2.11}$$

3. MAIN RESULTS AND PROOF

Theorem: Following equivalence relation for Double Dirichlet Average is established for ($k = x = 2$) of Mainardi Function

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho}(x-y)^{1-\rho-\rho'} D_{x-y}^{-\rho'} M\,(x,\alpha)\,(x-y)^{\rho-1} \tag{3.1}$$

Proof: Let us consider the double average for ($k = x = 2$ of $\cos h(u.z.v)$

$$S(\mu,\mu';z;\rho,\rho') = \int_0^1\int_0^1 M\,(u,z,v;\alpha)\; dm_{\mu,\mu'}(u) dm_{\rho,\rho'}(v)$$

$$= \sum_{n=o}^{\infty} \frac{(-1)^n}{\mathrm{n!}\,\Gamma(-\alpha(\mathrm{n}+1)+1)} \int_0^1\int_0^1 [u.z.v]^n dm_{\mu,\mu'}\,(u) dm_{\rho,\rho'}(v) \tag{3.2}$$

$Re(\mu) = 0, Re(\mu') = 0, Re(\rho) > 0, Re(\rho') > 0$ and

$$u.z.v = \sum_{i=1}^{2}\sum_{i=1}^{2}(u_i z_{ij} v_j) = \sum_{i=1}^{2}[u_i(z_{i1}v_1 + z_{i2}v_2)] = [u_1 z_{11} v_1 + u_1 z_{12} v_2 + u_2 z_{21} v_1 + u_2 z_{22} v_2]$$

Let $z_{11} = a, z_{12} = b, z_{21} = c, z_{22} = d$

and $\begin{cases} u_1 = u, & u_2 = 1-u \\ v_1 = v, & v_2 = 1-v \end{cases}$, thus $z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$v.z.v = uva + ub(1-v) + (1-u)cv + (1-u)d(1-v)$$
$$= uv(a-b-c+d) + u(b-d) + v(c-d) + d$$

$$dm_{\mu,\mu'}(u) = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'} u^{\mu-1}(1-u)^{\mu'-1} du$$

$$dm_{\rho,\rho'}(v) = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} v^{\rho-1}(1-v)^{\rho'-1} dv$$

Putting these values in (3.2), we have,

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} \times \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(-\alpha(n+1)+1)}$$

$$\times \int_0^1\int_0^1 [uv(a-b-c+d)+u(b-d)+v(c-d)+d]^n\, u^{\mu-1}(1-u)^{\mu'-1}v^{\rho-1}(1-v)^{\rho'-1}dudv$$

In order to obtained the fractional derivative equivalent to the above integral, we assume $a = c = x$; $b = d = y$ and get the results after solving

$$S(\mu,\mu';z;\rho,\rho') = \frac{\Gamma(\rho+\rho')}{\Gamma\rho}(x-y)^{1-\rho-\rho'} D_{x-y}^{-\rho'} \mathrm{M}(x,\alpha)\,(x-y)^{\rho-1}$$

This is complete proof of (3.1).

4. PARTICULAR CASES

(***i***) If $\rho' = v-\rho$, $y = 0$ and $\alpha = 0$ in (3.1)

$$S(\mu,\mu';z;\rho,v-\rho) = \frac{\Gamma(\rho+\rho')}{\Gamma\rho}(x)^{1-v} D_x^{\rho-v}\mathrm{M}(x)\;\;(x)^{\rho-1}$$

$$= \frac{1}{2}\left[\frac{\Gamma v}{\Gamma\rho}x^{1-v}D_x^{\rho-v}e^{-x}\,x^{\rho-1}\right]$$

$$S(\mu,\mu';z;\rho,v-\rho) = \frac{1}{2}[{}_1F_1(\rho,v;-x)]$$

(***ii***) If $\rho = -n$, $\rho' = 1+\alpha+n$, $y = 0$ and $\alpha = 0$ in (3.1) we have,

$$S(\mu,\mu';z;-n,1+\alpha+n) = \frac{1}{2}\left[\frac{\Gamma(1+\alpha)}{\Gamma(-n)}x^{-\alpha}D_x^{-n-\alpha-1}e^{-x}\,x^{-n-1}\right]$$

$$S(\mu,\mu';z;-n,1+\alpha+n) = \frac{1}{2}[{}_1F_1(-n,1+\alpha;-x)]$$

$$S(\mu,\mu';z;-n,1+\alpha+n) = \frac{1}{2}\left[\frac{L_n^{\alpha}(-x)}{L_n^{\alpha}(0)}\right]$$

where L_n^{α} is the Laguerre polynomials of degree *n*.

5. ACKNOWLEDGEMENT

Authors are grateful to referees for their valuable comments and improvement upon the paper.

REFERENCES

[1] Carlson, B. C., Special Function of Applied Mathematics, Academic Press, New York, 1977.

[2] Carlson, B. C., Appell's function F_4 as a double average, SIAM J.Math. Anal.6 (1975), 960-965.

[3] Carlson, B. C., Hidden symmetries of special functions, SIAM Rev. 12 (1970), 332-345.

[4] Carlson, B. C., Dirichlet averages of $x^t \log x$, SIAM J.Math. Anal. 18(2) (1987), 550-565.

[5] Carlson, B. C., A connection between elementary functions and higher transcendental functions, SIAM J. Appl. Math. 17 (1969), 116-140.

[6] Deora, Y. and Banerji, P. K., Double Dirichlet average of ex using fractional derivatives, J. Fractional Calculus 3 (1993), 81-86.

[7] Deora, Y. and Banerji, P. K., Double Dirichlet average and fractional derivatives, Rev.Tec.Ing.Univ. Zulia 16(2) (1993), 157-161.

[8] Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G., Tables of Integral Transforms, Vol.2 McGraw-Hill, New York, 1954.

[9] Gupta,S.C. and Agrawal, B. M., Dirichlet average and fractional derivatives, J. Indian Acad.Math. 12(1) (1990), 103-115.

[10] Gupta,S.C. and Agrawal, B. M., Double Dirichlet average of ex using fractional derivatives, Ganita Sandesh 5 (1) (1991),47-52.

[11] Mathai, A. M. and Saxena, R. K., The H-Function with Applications in Stastistics and other Disciplines, Wiley Halsted, New York, 1978.

[12] Saxena, R. K., Mathai, A. M and Haubold, H. J., Unified fractional kinetic equation and a fractional diffusion equation, J. Astrophysics and Space Science 209 (2004), 299-310.

[13] Sharma, M. and Jain, R., Double Dirichlet Average $x^t \log x$ and Fractional Derivative, J. of Indian Acad. Of Math.Vol. 28, No. 2(2006), 337-342

[14] Sharma, M. and Jain, R., Dirichlet Average of $coshx$ and Fractional Derivative, South East Asian J. Math. And Math. Sc. Vol.5, No. 2(2007), 17-22.

Estd. 1987 Reg. No. 74 / Ajmer / 1988-89, Dated 23/12/1988

राजस्थान गणित परिषद्

RAJASTHAN GAṆITA PARISHAD

Executive Committee - 2017-18

Office Bearers

President
DR. K. C. SHARMA
M.S.J.Govt. College, **BHARATPUR**

Vice President
DR. RAM NARESH SARASWAT
Manipal Universuty, **JAIPUR**

General Secretary
DR. R. P. SHARMA
Govt. Engg. College, **AJMER**

Treasurer
DR. V. C. JAIN
Govt. Engg. College, **AJMER**

Editor
DR. ANIL GOKHROO
S.P.C.Govt. College, **AJMER**

Joint Secretary
SH. R. N. MEGHWAL
Govt. College, **SUJANGARH**

MEMBERS

(3 Years)
1. PROF. G.S. RATHORE, UDAIPUR
2. DR. S.D. PUROHIT, KOTA
3. DR. A.K. MALIK, PILANI

(2 Years)
1. SH.NIKHIL JAIN, AJMER
2. SH. VIPIN KUMAR, PILANI
3. SH. HAMENT KUMAR, BHARATPUR

(1 Years)
1. DR. YASHMEEN, KOTA
2. DR. KAUR SINGH, SHRI GANGANAGAR
3. SH. LAJPAT SINGH, UDAIPUR

EX OFFICIO MEMBERS : 1. Past President, DR. V. C. JAIN, AJMER **2.** Past General Secretary, DR. P.K. MISHRA, SIKAR

CO-OPTED MEMBERS : 1. DR. D.C.GOKHROO, AJMER 2. DR. C.B. GUPTA, PILANI

Honorary Members

PROF. BANSAL, J. L., JAIPUR
DR. GOKHROO, D.C., AJMER
PROF. GUPTA, K.C., JAIPUR
DR. OM PRAKASH, JAIPUR
PROF. RAJBALI, JAIPUR,
PROF. SAXENA, R. K., JODHPUR
PROF. VERMA, G. R., KINGSTON, (USA)

Form IV (See Rule 8)

1. Place of Publication	:	AJMER (*Rajasthan*) INDIA
2. Periodicity of Publication	:	Half Yearly (*June and December*)
3. Printer's Name	:	MAHENDRA SINGH RAWAT
Nationality	:	Indian
Address	:	Bhawani Offset Printers, Topdara, AJMER (*Rajasthan*)
4. Publisher's Name	:	Dr. R. P. SHARMA
Nationality	:	Indian
Address	:	General Secretary, Rajasthan Ganita Parishad, Department of Mathematics, Govt. Engg.College, AJMER (*Rajasthan*)
5. Editor's Name	:	Dr. ANIL GOKHROO
Nationality	:	Indian
Address	:	Department of Mathematics, S. P. C. Government College, AJMER (*Rajasthan*)
6. Name and Addresses of individuals who own the newspaper and partners or shareholders holding more than 1% of the total capital.	:	Rajasthan Ganita Parishad, Regd. Head Office *Department of Mathematics,* SPC Government College, AJMER (*Rajasthan*) INDIA

I, Dr. R.P. Sharma, hereby declare that the particulars given above are true to the best of my knowledge and belief.

Dated : 23rd April, 2018 ***Dr. R. P. Sharma***

Volume 30, No. 1 and 2 (*June and Dec.* 2016)

GAṆITA SANDESH

गणित संदेश

The Sequence

Published by General Secretary, Rajasthan Ganita Parishad