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A Note on Weak Bailey Lemma Related to Basic Hypergeometric Series

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Abstract. In the present paper, we have made an attempt to establish certain summation formulae for basic hypergeometric series by making use of weak Bailey lemma and known summation formulae for truncated series.

Key words. Basic hypergeometric series, Weak Bailey lemma, truncated series, summation formulae.

1. Introduction. W. N. Bailey in 1940s elucidated the underlying structure of Rogers proof.

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_n}{(q,q)_{n-r}\,(aq,q)_{n+r}} \tag{1.1}$$

Where α_n and β_n are two sequence, then

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} a^n q^{n^2} \alpha_n \tag{1.2}$$

Under suitable condition of convergence.

(G. Andrews, R. A. Askey, R. Roy [1; Eq. (12.2.1), (12.2.2), p. 582]).

For any numbers α and q, real or complex and $|q|<1$, let

$$(a;q)_n = \begin{cases} (1-a)(1-aq)\ldots(1-aq^{n-1}) & ; n=1,2,\ldots \\ 1 & ; n=0 \end{cases} \tag{1.3}$$

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

According, we have (1.4)

Following Gasper and Rahman [2] we define a basic hypergeometric series

$${}_r\phi_s\left[a_1,\ldots,a_r;b_1,\ldots,b_s\;;q,z\right] = {}_r\phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ & ;q,z \\ b_1,b_2,\ldots,b_s \end{matrix}\right]$$

(1.5)

$$=\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n q^{\lambda n(n+1)}}{(q;q)_n(b_1;q)_n(b_2;q)_n\ldots(b_s;q)_n}z^n,\quad (|z|<1).$$

We also define a truncated series.

$${}_r\phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ & ;q,z \\ b_1,b_2,\ldots,b_s \end{matrix}\right]_N = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n q^{\lambda n(n+1)}}{(q;q)_n(b_1;q)_n(b_2;q)_n\ldots(b_s;q)_n}z^n. \tag{1.6}$$

During the process in this paper, we also make use of the following notations as

$$(a;q)_{n-r} = \frac{(q;q)_n(-1)^r q^{\binom{n}{2}-nk}}{(q^{-n};q)_r} \tag{1.7}$$

and

$$(a;q)_{n+r} = (aq;q)_n\,(aq^{n+1};q)_r \tag{1.8}$$

We shall need the following known results in our analysis,

$${}_4\phi_3\left[\begin{matrix} a,\,q\sqrt{a},\,-q\sqrt{a},\,q^{-m} \\ & ;q,-q^{-1/2+m} \\ \sqrt{a},\,-\sqrt{a},\,aq^{1+m} \end{matrix}\right] = \frac{(aq,-q^{-1/2};q)_m}{2(\sqrt{aq};q)_m(-q\sqrt{a};q)_{m-1}} + \frac{(aq,-q^{-1/2};q)_m}{2(-\sqrt{aq};q)_m(q\sqrt{a};q)_{m-1}}$$

(1.9)

(Verma and Jain [3; Eq. (4.1), p. 76])

$${}_3\phi_2\left[\begin{matrix} q, q\sqrt{a},\ q^{-m} \\ \\ \sqrt{a}, a q^{1+m}\end{matrix} ; q, -q^m\right] = \frac{\left(1+\sqrt{a}\right)\left(a q, -1; q\right)_m}{2\left(a q; q^2\right)_m} + \frac{\left(1-\sqrt{a}\right)\left(a q, -1; q\right)_m}{2\left(\sqrt{a}, -q\sqrt{a}; q\right)_m} \qquad (1.10)$$

(Verma and Jain [3; Eq. (4.2), p. 76])

$${}_2\phi_1\left[\begin{matrix} a,\ q^{-m} \\ \\ a q^{1+m}\end{matrix} ; q, -q^{\frac{1}{2}+m}\right] = \frac{\left(1+\sqrt{a}\right)\left(a q, -\sqrt{q}; q\right)_m}{2\left(-\sqrt{a q}, q\sqrt{a}; q\right)_m} + \frac{\left(1-\sqrt{a}\right)\left(a q, -\sqrt{q}; q\right)_m}{2\left(\sqrt{aq}, -q\sqrt{a}; q\right)_m} \qquad (1.11)$$

(Verma and Jain [3; Eq. (4.3), p. 76])

$${}_4\phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-m} \\ \\ \sqrt{a}, -\sqrt{a}, a q^{1+m}\end{matrix} ; q, -q^{\frac{-1}{2}+m}\right] = \frac{1}{2\sqrt{a}}\left[\frac{\left(a q, -q^{\frac{-1}{2}}; q\right)_m}{\left(\sqrt{a q}; q\right)_m\left(-q\sqrt{a}; q\right)_{m-1}} - \frac{\left(a q, -q^{\frac{-1}{2}}; q\right)_m}{\left(-\sqrt{a q}; q\right)_m\left(q\sqrt{a}; q\right)_{m-1}}\right] \qquad (1.12)$$

(Verma and Jain [3; Eq. (4.5), p. 77])

$${}_3\phi_2\left[\begin{matrix} a, q\sqrt{a},\ q^{-m} \\ \\ \sqrt{a}, a q^{1+m}\end{matrix} ; q, -q^{m+1}\right] = \frac{1+\sqrt{a}}{2\sqrt{a}}\frac{\left(a q, -1; q\right)_m}{\left(a q; q^2\right)_m} - \frac{1-\sqrt{a}}{2\sqrt{a}}\frac{\left(a q, -1; q\right)_m}{\left(\sqrt{a}, -q\sqrt{a}; q\right)_m} \qquad (1.13)$$

(Verma and Jain [3; Eq. (4.6), p. 77])

$${}_2\phi_1\left[\begin{matrix} a,\ q^{-m} \\ \\ a q^{1+m}\end{matrix} ; q, -q^{3/2+m}\right] = \frac{1+\sqrt{a}}{2\sqrt{a}}\frac{\left(a q, -\sqrt{q}; q\right)_m}{\left(-\sqrt{a q}, q\sqrt{a}; q\right)_m} - \frac{1-\sqrt{a}}{2\sqrt{a}}\frac{\left(a q, -\sqrt{q}; q\right)_m}{\left(\sqrt{a q}, -q\sqrt{a}; q\right)_m} \qquad (1.14)$$

(Verma and Jain [3; Eq. (4.7), p. 77])

$$ {}_3\phi_2\left[\begin{matrix} a, q\sqrt{a}, -q^m \\ \sqrt{a}, aq^{m+2} \end{matrix}; q, -q^{m+1}\right] = \frac{(a;q)_{m+2}\left(q\sqrt{a}, -1; q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(q, -\sqrt{aq}, \sqrt{a}, \sqrt{aq}; q\right)_{m+1}} - \frac{(a;q)_{m+2}\left(-1;q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(\sqrt{a}, -\sqrt{a}, q; q\right)_{m+1}} \tag{1.15} $$

(Verma and Jain [3; Eq. (4.8), p. 79])

$$ {}_4\phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-m} \\ \sqrt{a}, -\sqrt{a}, aq^{2+m} \end{matrix}; q, -q^{m+\frac{1}{2}}\right] = \frac{(a;q)_{m+2}\left(q\sqrt{a}, -q^{\frac{1}{2}}; q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(q, \sqrt{a} - \sqrt{a}, \sqrt{aq}; q\right)_{m+1}} \tag{1.16} $$

$$ - \frac{(a;q)_{m+2}\left(-q\sqrt{a}, -q^{-1/2}; q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(\sqrt{a}, -\sqrt{a}, -\sqrt{aq}; q\right)_{m+1}} $$

(Verma and Jain [3; Eq. (4.9), p. 79])

$$ {}_2\phi_1\left[\begin{matrix} a, q^{-m} \\ aq^{2+m} \end{matrix}; q, -q^{m+3/2}\right] = \frac{(a;q)_{m+2}\left(-\sqrt{q}; q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(q, \sqrt{a}, -\sqrt{aq}; q\right)_{m+1}} - \frac{(a;q)_{m+2}\left(-\sqrt{q}; q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(q, -\sqrt{a}, \sqrt{aq}; q\right)_{m+1}} \tag{1.17} $$

(Verma and Jain [3; Eq. (4.11), p. 79])

2. Main Results.

$$ \sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty} \frac{\left(aq, -q^{-1/2}; q\right)_m}{2\left(\sqrt{aq}; q\right)_m\left(-q\sqrt{a}; q\right)_{m-1}} + \frac{\left(aq, -q^{-1/2}; q\right)_m}{2\left(-\sqrt{aq}; q\right)_m\left(q\sqrt{a}; q\right)_{m-1}} \tag{2.1} $$

$$ \sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty} \frac{\left(1+\sqrt{a}\right)\left(aq, -1; q\right)_m}{2\left(aq; q^2\right)_m} + \frac{\left(1-\sqrt{a}\right)\left(aq, -1; q\right)_m}{2\left(\sqrt{a}, -q\sqrt{a}; q\right)_m} $$

(2.2)

$$ \sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty} \frac{\left(1+\sqrt{a}\right)\left(aq, -\sqrt{q}; q\right)_m}{2\left(-\sqrt{aq}, q\sqrt{a}; q\right)_m} + \frac{\left(1-\sqrt{a}\right)\left(aq, -\sqrt{q}; q\right)_m}{2\left(\sqrt{aq}, -q\sqrt{a}; q\right)_m} \tag{2.3} $$

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty}\frac{1}{2\sqrt{a}}\left[\frac{\left(aq,-q^{-1/2};q\right)_m}{\left(\sqrt{aq};q\right)_m\left(-q\sqrt{a};q\right)_{m-1}}-\frac{\left(aq,-q^{-1/2};q\right)_m}{\left(-\sqrt{aq};q\right)_m\left(q\sqrt{a};q\right)_{m-1}}\right]$$

(2.4)

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty}\frac{1+\sqrt{a}}{2\sqrt{a}}\frac{\left(aq,-1;q\right)_m}{\left(aq;q^2\right)_m}-\frac{1-\sqrt{a}}{2\sqrt{a}}\frac{\left(aq,-1;q\right)_m}{\left(\sqrt{a},-q\sqrt{a};q\right)_m} \tag{2.5}$$

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq;q)_\infty}\frac{1+\sqrt{a}}{2\sqrt{a}}\frac{\left(aq,-\sqrt{q};q\right)_m}{\left(-\sqrt{aq},q\sqrt{a};q\right)_m}-\frac{1-\sqrt{a}}{2\sqrt{a}}\frac{\left(aq,-\sqrt{q};q\right)_m}{\left(\sqrt{aq},-q\sqrt{a};q\right)_m} \tag{2.6}$$

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq^2;q)_\infty}\frac{(a;q)_{m+2}\left(q\sqrt{a},-1;q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(q,-\sqrt{aq},\sqrt{a},\sqrt{aq};q\right)_{m+1}}-\frac{(a;q)_{m+2}\left(-1;q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(\sqrt{a},-\sqrt{a},q;q\right)_{m+1}}$$

(2.7)

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq^2;q)_\infty}\frac{(a;q)_{m+2}\left(q\sqrt{a},-q^{1/2};q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(q,\sqrt{a}-\sqrt{a},\sqrt{aq};q\right)_{m+1}}-$$
$$\frac{(a;q)_{m+2}\left(-q\sqrt{a},-q^{-1/2};q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(\sqrt{a},-\sqrt{a},-\sqrt{aq};q\right)_{m+1}}$$

(2.8)

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \frac{1}{(aq^2;q)_\infty}\frac{(a;q)_{m+2}\left(-\sqrt{q};q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(q,\sqrt{a},-\sqrt{aq};q\right)_{m+1}}-\frac{(a;q)_{m+2}\left(-\sqrt{q};q\right)_{m+1}(q;q)_m}{2\sqrt{a}\left(q,-\sqrt{a},\sqrt{aq};q\right)_{m+1}}$$

(2.9)

3. Proof of Main Results.

1) As an illustration, we prove (2.1).

Taking $\alpha_r = \frac{q^{r(r-1)/2}\left(1-aq^{2r}\right)(a;q)_r}{(1-a)} z^r$ in (1.1), we get:

$$\beta_n = \sum_{r=0}^{n}\frac{q^{r(r-1)/2}\left(1-aq^{2r}\right)(a;q)_r\, z^r}{(1-a)(q;q)_{n-r}(aq;q)_{n+r}}$$

(3.1)

Now, substituting the resulting value of β_n and the above value of α_n in (1.2), we get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} a^n q^{n^2} \sum_{r=0}^{n} \frac{q^{r(r-1)/2} (1-aq^{2r})(a;q)_r z^r}{(1-a)(q;q)_{n-r}(aq;q)_{n+r}}$$

(3.2)

Using (1.7) and (1.8) in (3.2) after simplification, we get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (aq;q)_n} {}_4\phi_3 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-m} \\ \sqrt{a}, -\sqrt{a}, aq^{1+m} \end{matrix} ; q, -zq^m \right] \quad (3.3)$$

Talking $z = q^{-1/2}$ in (3.3) and summing the ${}_4\phi_3$ with the help of (1.9) and after simplification we get the result (2.1).

2) Proof of (2.2)

Taking $\alpha_r = \frac{q^{r(r-1)/2} (q, q\sqrt{a}; q)_r}{(q, \sqrt{a}; q)_r}$ in (1.1), we get

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r(r-1)/2} (q, q\sqrt{a}; q)_r}{(q, \sqrt{a}; q)_r (q;q)_{n-r} (aq;q)_{n+r}} \quad (3.4)$$

Now, substituting the resulting value of β_n and the above value of α_n in (1.2), using (1.7) and (1.8) in (3.4) after simplification, we get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (aq;q)_n} {}_3\phi_2 \left[\begin{matrix} q, q\sqrt{a}, q^{-m} \\ \sqrt{a}, aq^{1+m} \end{matrix} ; q, -q^m \right]$$

(3.5)

Now, summing the ${}_3\phi_2$ with the help of (1.10) and after simplification we get the result (2.2).

3) The proof of (2.3) is as follows:

Talking $\alpha_r = \frac{(a;q)_r q^{r^2/2}}{(q;q)_r}$ in (1.1), we get

$$\beta_n = \sum_{r=0}^{n} \frac{(a;q)_r q^{r^2/2}}{(q;q)_r (q;q)_{n-r} (aq;q)_{n+r}}$$

(3.6)

Now, substituting the resulting value of β_n and the above value of α_n in (1.2), using (1.7) and (1.8) in (3.6) after simplification, we get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (aq;q)_n} {}_2\phi_1 \left[\begin{matrix} a, \ q^{-m} \\ aq^{1+m} \end{matrix} ; q, -q^{\frac{1}{2}+m} \right]$$

(3.7)

Now, summing the ${}_2\phi_1$ with the help of (1.11) and after simplification we get the result (2.3).

4) The proof of (2.4) is as follows:

Talking $\alpha_r = \frac{\left(a, q\sqrt{a}, -q\sqrt{a}; q\right)_r z^r}{\left(\sqrt{a}, -\sqrt{a}; q\right)_r}$ in (1.1), we get

$$\beta_n = \sum_{r=0}^{n} \frac{\left(a, q\sqrt{a}, -q\sqrt{a}; q\right)_r z^r}{\left(\sqrt{a}, -\sqrt{a}; q\right)_r (q;q)_{n-r} (aq;q)_{n+r}}$$

(3.8)

Now, substituting the resulting value of β_n and the above value of α_n in (1.2), using (1.7) and (1.8) in (3.8) after simplification, we get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (aq;q)_n} {}_4\phi_3 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-m} \\ \sqrt{a}, -\sqrt{a}, aq^{1+m} \end{matrix} ; q, -z q^m \right] \quad (3.9)$$

Talking $z = q^{1/2}$ in (3.9) and summing the ${}_4\phi_3$ with the help of (1.12) and after simplification we get the result (2.4).

5) The proof of (2.5) is as follows:

Taking $\alpha_r = \frac{q^{r(r-1)/2} \left(a, q\sqrt{a}; q\right)_r}{\left(\sqrt{a}; q\right)_r} z^r$ in (1.1), we get

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r(r-1)/2} \left(a, q\sqrt{a}; q\right)_r}{\left(\sqrt{a}; q\right)_r (q;q)_{n-r} (aq;q)_{n+r}}$$

(3.10)

Now, substituting the resulting value of β_n and the above value of α_n in (1.2), and using (1.7) and (1.8) in (3.10) after simplification, we get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (aq;q)_n} {}_3\phi_2 \left[\begin{matrix} a, q\sqrt{a}, q^{-m} \\ \sqrt{a}, aq^{1+m} \end{matrix} ; q, -zq^m \right]$$

(3.11)

Now, talking $z = q$ in (3.11) and summing the ${}_3\phi_2$ with the help of (1.13) and after simplification we get the result (2.5).

6) The proof of (2.6) is as follows:

Talking $\alpha_r = \frac{(a;q)_r q^{r^2/2}}{(q;q)_r}$ in (1.1), we get

$$\beta_n = \sum_{r=0}^{n} \frac{(a;q)_r q^{r^2/2}}{(q;q)_r (q;q)_{n-r} (aq;q)_{n+r}}$$

(3.12)

Now, substituting the resulting value of β_n and the above value of α_n in (1.2), using (1.7) and (1.8) in (3.12) after simplification, we get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (aq;q)_n} {}_2\phi_1 \left[\begin{matrix} a, q^{-m} \\ aq^{1+m} \end{matrix} ; q, -q^{\frac{3}{2}+m} \right] \qquad (3.13)$$

Now, summing the ${}_2\phi_1$ with the help of (1.14) and after simplification we get the result (2.6).

7) Proof of (2.7)

If we replace a by aq in (1.1), we get if for $n \geq 0$

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_n}{(q,q)_{n-r} (aq^2,q)_{n+r}} \qquad (3.14)$$

Then

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} a^n q^{n^2} \sum_{r=0}^{n} \frac{\alpha_n}{(q,q)_{n-r} (aq^2,q)_{n+r}} \qquad (3.15)$$

Taking $\alpha_r = \frac{q^{r(r+1)/2} (a, q\sqrt{a}; q)_r}{(q, \sqrt{a}; q)_r} z^r$ in (3.14), we get

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r(r+1)/2} \left(a , q\sqrt{a} \,; q\right)_r z^r}{\left(q , \sqrt{a} \,; q\right)_r \left(q;q\right)_{n-r} \left(aq^2;q\right)_{n+r}}$$

(3.16)

Now, substituting the resulting value of β_n in (3.15), after simplification, we get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (aq^2;q)_n} \, {}_3\phi_2 \left[\begin{matrix} a, q\sqrt{a}, q^{-m} \\ \sqrt{a}, a q^{m+2} \end{matrix} ; q, -z q^m \right] \tag{3.17}$$

Now, talking $z = q$ in (3.17) and using (1.15) to sum the ${}_3\phi_2$, after simplification we get the result (2.7).

8) Proof of (2.8)

Taking $\alpha_r = \dfrac{q^{r(r-1)/2} \left(a , q\sqrt{a} ,-q\sqrt{a} \,; q\right)_r}{\left(q , \sqrt{a},-\sqrt{a} \,; q\right)_r} z^r$ in (3.14), we get

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r(r-1)/2} \left(a , q\sqrt{a} - q\sqrt{a}; q\right)_r z^r}{\left(q , \sqrt{a},-\sqrt{a} \,; q\right)_r \left(q;q\right)_{n-r} \left(aq^2;q\right)_{n+r}}$$

(3.18)

Now, substituting the resulting value of β_n in (3.15), after simplification, we get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (aq^2;q)_n} \, {}_4\phi_3 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, q^{-m} \\ \sqrt{a}, -\sqrt{a}, a q^{2+m} \end{matrix} ; q, -z q^m \right] \tag{3.19}$$

Now, talking $z = q^{1/2}$ in (3.19) and using (1.16) to sum the ${}_4\phi_3$, after simplification we get the result (2.8).

9) Proof of (2.9)

Taking $\alpha_r = \dfrac{q^{r(r+1)/2} \left(a \,; q\right)_r}{\left(q \,; q\right)_r} z^r$ in (3.14), we get

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r(r+1)/2} \left(a \,; q\right)_r z^r}{\left(q \,; q\right)_r \left(q;q\right)_{n-r} \left(aq^2;q\right)_{n+r}}$$

(3.20)

Now, substituting the resulting value of β_n in (3.15), after simplification, we get

$$\sum_{n=0}^{\infty} a^n q^{n^2} \beta_n = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n (aq^2;q)_n} {}_2\phi_1 \left[\begin{matrix} a, \; q^{-m} \\ \\ a\,q^{m+2} \end{matrix} ; q, -z\,q^m \right]$$

(3.21)

Now, talking $z = q^{3/2}$ in (3.21) and using (1.17) to sum the ${}_2\phi_1$, after simplification we get the result (2.9).

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Ganita Sandesh Vol. 32 (June & December, 2018) pp. 11-24 ISSN : 0970-9169

Stirling and Harmonic numbers

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Abstract: We apply the binomial coefficient polynomial to write the harmonic numbers H_n in terms of Stirling numbers of the first kind, and we employ the Melzak's identity to exhibit representations of H_n. We also deduce identities involving the harmonic numbers via their connection with the derivatives of binomial coefficients.

Keywords: Stirling numbers, Melzak's identity, Binomial coefficients, Harmonic numbers.

1. Introduction

The harmonic numbers are defined by [1-7]:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n}, \qquad n \geq 1, \tag{1}$$

and we know that the corresponding harmonic series is divergent, that is, $H_n \longrightarrow \infty$ when $n \to \infty$, this topic is considered in Sec. 2. However, it is possible to construct the finite quantity:

$$\gamma_0 \equiv lim_{\ k\to\infty}\ (H_k - Ln\ k) = 0.5772\ 1566\ 4901\ 5328\ 6060 \ldots$$
(2)

called Euler-Mascheroni's constant [5, 8-10].

In Sec. 3 the Melzak's identity [11-13] allows to deduce the relation:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+s} = \frac{n!}{(s)_{n+1}} = \frac{n!\,(s-1)!}{(s+n)!} = \frac{n!\,\Gamma(s)}{\Gamma(s+n+1)},$$
(3)

where $\Gamma(s)$ is the gamma function [5, 8, 14-17] and $(s)_m = s(s+1)(s+2)\cdots(s+m-1)$ is the Barnes-Pochhammer symbol (ascending factorial function) [5, 13, 18-21]. Thus (3) implies the following alternative expression of Euler for (1) [1, 13]:

$$H_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k}, \qquad n \geq 1.$$
(4)

If we apply $\frac{d}{ds}$ to (3) and the result is evaluated at $s = 1$, we find the property:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^2} = \frac{H_{n+1}}{n+1},$$
(5)

which is useful to prove the Coffman's formula [5, 22]:

$$\sum_{k=1}^{\infty} \frac{H_k}{k\ 2^k} = \frac{\pi^2}{12}.$$
(6)

In Sec. 4 we consider the polynomial in x of degree k :

GANITA SANDESH, Vol. 32 (June & December, 2018)

$$\binom{x}{k} = \frac{1}{k!}\, x\,(x-1)(x-2)\cdots(x-k+1) = \frac{1}{k!}\,[x]_k \,,$$

(7)

where $[x]_m$ is the descending factorial function [13, 21, 23]. Then it is possible to show the interesting connection:

$$H_k = \left[\frac{d}{dx}\binom{x}{k}\right]_{x=k} \;;$$

(8)

besides, the Stirling numbers of the first kind are defined by the relation [1, 3, 13, 15, 18, 21, 24]:

$$[x]_k = \sum_{m=0}^{k} S_k^{(m)}\, x^m \,,$$

(9)

hence (7), (8) and (9) imply the following expression for the harmonic numbers:

$$H_m = \frac{1}{m!} \sum_{j=1}^{m} j\, S_m^{(j)}\, m^{j-1} \,,$$

(10)

which is alternative to the known identity [1, 3, 13]:

$$H_m = \frac{1}{m!} \sum_{j=1}^{m} (-1)^{j+m}\, j\, S_m^{(j)} \,.$$

(11)

We remark that in (10) and (11) only participate the Stirling numbers of the first kind, but in [7] was obtained the following formula in terms of Stirling numbers of the second kind:

$$H_n = \frac{1}{n!} \sum_{k=n}^{2n-1} (-1)^{k+1} \binom{2n}{k+1} k\, S_{k-1}^{[k-n]}\,, \qquad n \geq 1. \tag{12}$$

We also exhibit that from the Spiess result [2]:

$$\sum_{k=1}^{n} \frac{H_k}{(k+1)_m} = \frac{1}{(m-1)^2} \left[\frac{1}{(m-1)!} - \frac{1+(m-1)\, H_{n+1}}{(n+2)_{m-1}} \right], \qquad m \geq 2\,, \tag{13}$$

is immediate the Cloitre's relation [20]:

$$\sum_{k=1}^{\infty} \frac{H_k}{(k+1)_m} = \frac{1}{(m-1)!\,(m-1)^2}\,, \qquad m = 2, 3, \ldots \tag{14}$$

2. Harmonic series

We have that Nicole d'Oresme (1350), Pietro Mengoli (1647) and Johann Bernoulli (1687), among others, proved that $H_n \to \infty$ if $n \to \infty$, that is, the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent. Here we shall show an alternative proof of this property, in fact, we apply the Gauss geometric series:

$$\sum_{k=1}^{\infty} z^k = \frac{z}{1-z}\,, \qquad |z| < 1\,, \tag{15}$$

for $z = e^{-\nu}$, then $\sum_{k=1}^{\infty} e^{-k\nu} = \frac{e^{-\nu}}{1-e^{-\nu}}$, whose integration with respect to ν gives the expression [25]:

$$\sum_{k=1}^{\infty} \frac{1}{k}\, e^{-k\nu} = \frac{\nu}{2} - Ln\left[2 \sinh\left(\frac{\nu}{2}\right)\right], \tag{16}$$

where we can employ $\nu \to 0$ to obtain that $H_k \to \infty$ when $k \to \infty$, which is equivalent to $\zeta(1) \to \infty$ for the Riemann zeta function [5, 26]. As an application of this result, let's remember the identity [27]:

$$2 \sum_{k=1}^{n} \frac{H_k}{k} = H_n^2 + \sum_{k=1}^{n} \frac{1}{k^2}, \qquad n \geq 1, \tag{17}$$

then is immediate that:

$$\sum_{k=1}^{n} \frac{H_k}{k} \longrightarrow \infty \quad \text{if} \quad n \longrightarrow \infty, \tag{18}$$

because $\sum_{k=1}^{\infty} \frac{1}{k^2} \equiv \zeta(2) = \frac{\pi^2}{6}$ [5].

3. Melzak's formula

Melzak [11-13, 28] deduced the interesting property:

$$f(x+s) = s \binom{s+n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(x-k)}{k+s}, \qquad n \geq 0, \qquad s \neq 0, -1, \dots - n, \tag{19}$$

for all algebraic polynomials $f(z)$ up to degree n. Thus for $f(z) = 1$ the relation (19) implies (3), which can be written in the form:

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k+s} = \frac{n!\,\Gamma(s+1) - \Gamma(s+n+1)}{s\,\Gamma(s+n+1)}, \qquad n \geq 1,$$

where the Hôpital's rule allows to consider the case $s = 0$:

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k} = lim_{s\to 0} \frac{n!\,\Gamma(s+1)\,\psi(s+1) - \Gamma(s+n+1)\,\psi(s+n+1)}{\Gamma(s+n+1)\,[1+s\,\psi(s+n+1)]} = \psi(1) - \psi(n+1), \quad (20)$$

with $\psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)}$, but we have the expression [27]:

$$\psi(z+n) - \psi(z) = \sum_{j=1}^{n} \frac{1}{j+z-1}\,, \qquad n \geq 1, \quad (21)$$

hence $\psi(n+1) - \psi(1) = H_n$ and (20) implies the representation (4) for harmonic numbers.

Now we apply $\frac{d}{ds}$ to (3):

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+s)^2} = \frac{n!\,\Gamma(s)}{\Gamma(s+n+1)}\,[\,\psi(s+n+1) - \psi(s)\,] \overset{(21)}{=} \frac{n!\,(s-1)!}{(s+n)!} \sum_{j=1}^{n+1} \frac{1}{j+s-1}\,, \quad (22)$$

which gives (5) when $s = 1$. In [6] was showed that (5) and the analytic continuation of Riemann zeta function obtained by Hasse - Sondow [29-31] allow to deduce (6) due to Coffman [22].

It is interesting to exhibit other proof of (4), in fact, from the finite geometric series:

$$\sum_{k=1}^{n} x^{k-1} = \frac{x^n-1}{x-1} = \frac{1}{x-1}\left[\sum_{k=0}^{n} \binom{n}{k} (x-1)^k - 1\right] = \sum_{k=1}^{n} \binom{n}{k} (x-1)^{k-1}\,,$$

whose integration gives the relation [1]:

$$H_n + \sum_{k=1}^{n} \binom{n}{k} \frac{(x-1)^k}{k} = \sum_{k=1}^{n} \frac{x^k}{k}, \quad (23)$$

which implies (4) for $x = 0$.

4. Stirling numbers

From (7):

$$\binom{x}{k} = \frac{\Gamma(x+1)}{k!\,\Gamma(x-k+1)}, \qquad \text{and}$$

$$\frac{d}{dx}\binom{x}{k} = \binom{x}{k}\left[\psi(x+1) - \psi(x-k+1)\right], \quad (24)$$

thus (8) is immediate. On the other hand [1, 3, 13, 15, 18, 24]:

$$\binom{x}{k} \overset{(7)}{=} \frac{1}{k!}[x]_k \overset{(9)}{=} \frac{1}{k!}\sum_{j=0}^{k} S_k^{(j)} x^j, \quad (25)$$

with the participation of Stirling numbers of the first kind [32]. If we employ (25) into (8) we obtain the representation (10), however, in the literature [1, 3, 13] we have the expression (11), which gives the following identity for Stirling numbers:

$$\sum_{k=1}^{n} k\, S_n^{(k)}\, n^{k-1} = \sum_{k=1}^{n} (-1)^{k+n}\, k\, S_n^{(k)}. \quad (26)$$

It is easy to see that:

$$\sum_{k=1}^{n} \frac{H_k}{k+1} = \sum_{j=1}^{n+1} \frac{H_j}{j} - \sum_{r=1}^{n+2} \frac{1}{r^2}, \quad (27)$$

where we can use (18) to deduce the result:

$$\sum_{k=1}^{n} \frac{H_k}{k+1} \longrightarrow \infty \qquad \text{if} \qquad n \longrightarrow \infty, \tag{28}$$

by this reason we consider the Spiess formula (13) for $m \neq 1$. It is known the very slow divergence of the harmonic series, hence:

$$\frac{1 + (m-1)\, H_{n+1}}{(n+2)_{m-1}} \longrightarrow 0 \qquad \text{when} \qquad n \longrightarrow \infty, \quad m \geq 2, \tag{29}$$

therefore (13) and (29) imply (14) due to Cloitre [20]. We remark that in [33] is the generation function:

$$\sum_{k=1}^{\infty} \frac{H_k}{k+1}\, (1-z)^{k+1} = \frac{1}{2} (Ln\, z)^2, \tag{30}$$

which also allows to deduce (28).

5. Harmonic numbers

It is well known the property [5]:

$$\frac{d}{dx}\binom{x+m}{n} = \binom{x+m}{n} \sum_{j=1}^{n} \frac{1}{j+x+m-n}\ , \tag{31}$$

in particular:

$$\left[\frac{d}{dx}\binom{x+m}{n}\right]_{x=n-m} = H_n\ , \qquad \left[\frac{d}{dx}\binom{x}{n}\right]_{x=-1} = (-1)^{n+1}\, H_n\ , \tag{32}$$

for the harmonic numbers; here we employ (31) and (32) to deduce identities involving the quantities H_n. In fact, we have the expression [13]:

$$x^n = \sum_{j=0}^{n} j! \binom{x}{j} S_n^{[j]} , \tag{33}$$

thus (32) and $[\frac{d}{dx}(33)]_{x=-1}$ imply:

$$\sum_{j=1}^{n} (-1)^j \, j! \, H_j \, S_n^{[j]} = n\,(-1)^n , \qquad n \geq 1. \tag{34}$$

We can verify (34) if we use (11):

$$\sum_{j=1}^{n} (-1)^j \, j! \, H_j \, S_n^{[j]} = \sum_{q=1}^{n} (-1)^q \, q \sum_{j=q}^{n} S_n^{[j]} \, S_j^{(q)} = (-1)^n \, n,$$

by the orthonormality of the Stirling numbers [5, 13]; hence (11) and (34) are reciprocal relations.

Lanczos [34] used the binomial expansion of Gregory-Newton to obtain the identity:

$$\sum_{k=0}^{n} \binom{x}{k}\binom{n}{k} \frac{1}{(k+1)_m} = \frac{1}{(n+1)_m} \binom{x+m+n}{n} , \tag{35}$$

where $(k+1)_m = \frac{(k+m)!}{k!}$; then (32) and $[\frac{d}{dx}(35)]_{x=-1}$ allow to deduce the formula:

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{(k+1)_m} \binom{n}{k} H_k = \frac{1}{(m-1)!\,(m+n)} (H_{m+n-1} - H_{m-1}), \qquad m \geq 1. \tag{36}$$

We have the following expression of Graham-Knuth [35]:

$$\sum_{k=0}^{n} \binom{x+k}{k} = \left(1 + \frac{n}{x+1}\right)\binom{x+n}{n}, \qquad n \geq 0, \tag{37}$$

therefore (32) and $[\frac{d}{dx}(37)]_{x=0}$ imply the property [36]:

$$\sum_{k=0}^{n} H_k = (n+1)\, H_n - n, \qquad n = 0, 1, 2, \dots, \tag{38}$$

which is a particular case of the identity [2, 3, 35-38]:

$$\sum_{k=m}^{n} \binom{k}{m} H_k = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right), \tag{39}$$

for $m = 0$.

In [13] we find the relation:

$$\sum_{k=1}^{n} (-1)^k \binom{x}{k}\, k = (-1)^n\, x \binom{x-2}{n-1}, \quad n \geq 1, \tag{40}$$

thus (32) and $[\frac{d}{dx}(40)]_{x=-1}$ generate the result [39]:

$$\sum_{k=1}^{n} k\, H_k = \binom{n+1}{2}\left(H_{n+1} - \frac{1}{2}\right), \tag{41}$$

which is deductible from [2, 13]:

$$\sum_{k=1}^{n} k^m\, H_k = \sum_{j=1}^{m} \binom{n+1}{j+1}\left(H_{n+1} - \frac{1}{j+1}\right) j!\; S_m^{[j]}, \tag{42}$$

for $m = 1$.

We know the expression:

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k\binom{x+k}{k}} = \sum_{k=1}^{n} \frac{1}{x+k} \ , \tag{43}$$

then $[\frac{d}{dx}(43)\ ]_{x=0}$ and (32) allow to obtain the identity [40]:

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k} H_k = \sum_{k=1}^{n} \frac{1}{k^2}, \tag{44}$$

which can be verified directly via the relation (4).

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On Analytic Continuation of the Appell Type Double Hurwitz – Lerch Zeta Functions

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Abstract

In this paper, we obtain some functional equations of the Appell type double zeta functions and consequently, express the series consisting of definite Appell's type double Hurwitz – Lerch Zeta functions into sum of Bernoulli polynomials. Finally, we systematically investigate their analytic continuation formulae.

Key words – Appell functions, Hurwitz –Lerch Zeta functions, functional equations, Bernoulli polynomials, analytic continuation formulae
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1. Introduction

In 2000, Choi and Srivastava [2] introduced a double zeta function

$$\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!} \frac{z^m t^n}{(a+m+n)^s}, \tag{1}$$

where, $\alpha,\beta,\beta' \in \mathbb{C}, \gamma,\gamma', a \in \mathbb{C}\setminus\mathbb{Z}_0^-, s,z,t \in \mathbb{C}$ and $\Re(s) > 0$, when $|z| < 1, |t| < 1$ and for $|z| = |t| = 1$, $\Re(\gamma + \gamma' + s - \alpha - \beta - \beta') > 0$.

Here in (1) the Pochhammer symbol, $(a)_n$, [12, p. 22] is defined as factorial function by

$$\frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n := \begin{cases} a(a+1)(a+2)\ldots(a+n-1); n \geq 1, \\ 1; n = 0, \text{for } a \neq 0. \end{cases} \tag{2}$$

Clearly setting $\gamma = \beta, \gamma' = \beta'$ in formula (1) and then on applying series rearrangement techniques [12], we get a generalized Hurwitz-Lerch type zeta function due to Goyal and Laddha [6]

$$\phi_{\alpha,\beta,\beta';\beta,\beta'}(z,t,s,a) = \sum_{M=0}^{\infty} \frac{(\alpha)_M}{M!} \frac{(z+t)^M}{(a+M)^s} = \phi^*_{\alpha}(z+t,s,a), \tag{3}$$

provided that $a \in \mathbb{C}\setminus\mathbb{Z}_0^-, s,\alpha \in \mathbb{C}$, when $|z+t| < 1$, and when $|z+t| = 1$, $(\Re(s) - \alpha) > 0, \Re(a) > 0$.

Again then on setting $\alpha = 1$ in formula (3), we find a Hurwitz-Lerch type zeta function in the form ([2], [5, p. 27], [10, Eqn. (1.4))

$$\phi_{1,\beta,\beta';\beta,\beta'}(z,t,s,a) = \sum_{M=0}^{\infty} \frac{(z+t)^M}{(a+M)^s} = \phi(z+t,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-ax}}{(1-(z+t)e^{-x})} dx, \tag{4}$$

provided that $(a \in \mathbb{C}\setminus\mathbb{Z}_0^-, s \in \mathbb{C}$, when $|z+t| < 1$, and when $|z+t| = 1, \Re(s) > 1)$.

Later on Choi and Parmar [3] introduced and studied another double zeta function in the form

$$\phi_{\alpha,\beta,\beta',\gamma}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n} m!n!} \frac{z^m t^n}{(a+m+n)^s}, \tag{5}$$

provided that $\alpha,\beta,\beta' \in \mathbb{C}, \gamma, a \in \mathbb{C}\setminus\mathbb{Z}_0^-, s,z,t \in \mathbb{C}$ and $\Re(s) > 0$, when $|z| < 1, |t| < 1$ and for $|z| = |t| = 1$, $\Re(\gamma + s - \alpha - \beta - \beta') > 0$.

When we set $z = t$ in formula (5) and then on using the formula (see, Srivastava and Manocha [12, p. 55] in it, and using Eulerian formula [10], we get following relations of Hurwitz-Lerch type zeta function

$$\phi_{\alpha,\beta,\beta',\gamma}(z,z,s,a) = \sum_{N=0}^{\infty} \frac{(\alpha)_N(\beta+\beta')_N}{(\gamma)_N N!} \frac{z^N}{(a+N)^s} = \frac{1}{\Gamma(s)}\int_0^{\infty} e^{-ax}\, x^{s-1}\, {}_2F_1\left[\begin{matrix}\alpha,\beta+\beta';\\ \gamma;\end{matrix} ze^{-x}\right]dx,$$
(6)

provided that $\alpha,\beta,\beta' \in \mathbb{C}, \gamma, a \in \mathbb{C}\setminus\mathbb{Z}_0^-, s,z,t \in \mathbb{C}$ and $\Re(s) > 0$, when $|z| < 1$, and for $|z| = 1$, $\Re(\gamma + s - \alpha - \beta - \beta') > 0$.
Here in Eqn. (6), ${}_2F_1\ [.]$ is the classical hypergeometric function due to Gauss.

Thus it would seem quite natural to investigate what is implied by the Eqns. (1), (3) - (6). In this paper we obtain some functional equations of double zeta functions of double zeta functions (1) and (5). Again, the series involving double zeta functions (1) and (5) are expressed in terms of Bernoulli polynomials and then systematically their analytic continuation formulae are discussed.

2. Certain functional equations of double zeta functions (1) and (5)

In this section, we derive functional equations of the double zeta functions (1) and (5) by obtaining some well-known zeta functions found, for instance in the literature (see for example [4], [6]-[10] and others) and some unknown zeta functions.

In the Eqn. (1), replace γ and γ' by β and β' respectively and then in the Eqn. (5), apply the formulae $\lim_{|\mu|\to\infty}(\mu)_n\left\{\frac{z}{\mu}\right\}^n = \lim_{|\lambda|\to\infty}\frac{\{\lambda z\}^n}{(\lambda)_n} = z^n$ (see in [11, p. 20]) and then making an appeal to the formula (3), we find following equivalent results

$$\lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{\alpha,\beta,\beta',\gamma}\left(\frac{\gamma z}{\beta},\frac{\gamma t}{\beta'},s,a\right) = \phi_{\alpha,\beta,\beta';\beta,\beta'}(z,t,s,a)$$
$$= \sum_{M=0}^{\infty}\frac{(\alpha)_M}{M!}\frac{(z+t)^M}{(a+M)^s} = \phi^*{}_{\alpha}(z+t,s,a),$$
(7)

provided that $a \in \mathbb{C}\setminus\mathbb{Z}_0^-, s,\alpha \in \mathbb{C}$, when $|z+t| < 1$, and when $|z+t| = 1$, $(\Re(s) - \alpha) > 0, \Re(a) > 0$.

Again in Eqn. (7), set $z = 1, t = 1, \alpha = 1$, and then making an appeal to a relation of generalized zeta function $\zeta(s,a)$ ([1, p. 249], [2, p. 89]) and the result (7), we find following connection of the formulae (1) and (5) with the generalized zeta function $\zeta(s,a)$, given by

$$\lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},s,a\right) = \phi_{1,\beta,\beta';\beta,\beta'}(1/2,1/2,s,a) = \phi^*{}_1(1,s,a)$$
$$= \sum_{M=0}^{\infty}\frac{1}{(a+M)^s} = \zeta(s,a)\ .$$
(8)

provided that $a \in \mathbb{C}\setminus\mathbb{Z}_0^-, s \in \mathbb{C}$, $(\Re(s) - 1) > 0$.
Further put $a = 1$ in Eqn. (8), and then applying the equivalence relation $\zeta(s,1) = \zeta(s) = \frac{1}{2^{s-1}}\zeta\left(s,\frac{1}{2}\right)$ in it, we get following equivalence relations

$$\lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},s,1\right) = \lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},s\right)$$
$$= \frac{1}{2^{s-1}}\lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},s,\frac{1}{2}\right),$$
(9)

$$\phi_{1,\beta,\beta';\beta,\beta'}(1/2,1/2,s,1) = \phi_{1,\beta,\beta';\beta,\beta'}(1/2,1/2,s) = \frac{1}{2^{s-1}}\phi_{1,\beta,\beta';\beta,\beta'}\left(1/2,1/2,s,\frac{1}{2}\right),$$
(10)

and

$\phi^*{}_1(1,s,1) = \phi^*{}_1(1,s) = \frac{1}{2^{s-1}}\phi^*{}_1\left(1,s,\frac{1}{2}\right).$
(11)

Since we are familiar with the formula Titchmarsh [13, pp. 21-22]
$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-1/2+s/2}\Gamma(1/2-s/2)\zeta(1-s)$, since then applying the relations given in Eqns. (9), (10) and (11), we get following functional equations

$$\pi^{-s/2}\Gamma(s/2)\lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},s,1\right)$$
$$= \pi^{-1/2+s/2}\Gamma(1/2-s/2)\lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},1-s,1\right). \qquad (12)$$

$$\pi^{-s/2}\Gamma(s/2)\phi_{1,\beta,\beta';\beta,\beta'}(1/2,1/2,s,1)$$
$$= \pi^{-1/2+s/2}\Gamma(1/2-s/2)\phi_{1,\beta,\beta';\beta,\beta'}(1/2,1/2,1-s,1).$$
(13)

$$\pi^{-s/2}\Gamma(s/2)\phi^*{}_1(1,s,1)$$
$$= \pi^{-1/2+s/2}\Gamma(1/2-s/2)\phi^*{}_1(1,1-s,1). \qquad (14)$$

3. Series of Appell's type double zeta functions into sum of Bernoulli polynomials

In this section, we will make use of a connection of zeta function and Bernoulli polynomials to get series of Appell's type double zeta functions into sum of Bernoulli polynomials. This technique may be exploited to compute the series involving the double zeta functions (1) and (5).
In Eqn. (8) applying the result due to [1, pp. 246-266],
$(-1)^n\zeta(-n,a) = \frac{1}{n+1}B_{n+1}(a)\ \forall n = 0,1,2,3,\dots;$
where, for $a \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $B_{n+1}(a)$ are the Bernoulli polynomials and $B_{n+1}(1) = B_{n+1}$, is a Bernoulli number $\forall n = 1,2,3,\dots$, we find the following series involving Bernoulli polynomials:

$$\sum_{n=0}^{\infty}\lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},-n,a\right)(-z)^n$$
$$= B_1(a) + \frac{1}{2}B_2(a)z + \frac{1}{3}B_3(a)z^2 + \frac{1}{4}B_4(a)z^3 + \frac{1}{5}B_5(a)z^4 + \cdots + \frac{1}{n+1}B_{n+1}(a)z^n + \cdots,$$
$$\Rightarrow \lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},-n,a\right)(-1)^n = \frac{1}{n+1}B_{n+1}(a)\ \forall n = 0,1,2,3,\dots.$$
(15)

$$\sum_{n=0}^{\infty}\phi_{1,\beta,\beta';\beta,\beta'}(1/2,1/2,-n,a)(-z)^n$$
$$= B_1(a) + \frac{1}{2}B_2(a)z + \frac{1}{3}B_3(a)z^2 + \frac{1}{4}B_4(a)z^3 + \frac{1}{5}B_5(a)z^4 + \cdots + \frac{1}{n+1}B_{n+1}(a)z^n + \cdots,$$
$$\Rightarrow \phi_{1,\beta,\beta';\beta,\beta'}(1/2,1/2,-n,a)(-1)^n = \frac{1}{n+1}B_{n+1}(a)\ \forall n = 0,1,2,3,\dots. \qquad (16)$$
and

$$\sum_{n=0}^{\infty}\phi^*{}_1(1,-n,a)(-z)^n$$
$$= B_1(a) + \frac{1}{2}B_2(a)z + \frac{1}{3}B_3(a)z^2 + \frac{1}{4}B_4(a)z^3 + \cdots + \frac{1}{n+1}B_{n+1}(a)z^n + \cdots,$$
$$\Rightarrow \phi^*{}_1(1,-n,a)(-1)^n = \frac{1}{n+1}B_{n+1}(a)\ \forall n = 0,1,2,3,\dots. \qquad (17)$$
Therefore using the series given in Eqns. (15), (16) and (17), we obtain the following series involving Bernoulli polynomials

$$\frac{1}{2}\sum_{n=0}^{\infty}\lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},-n,a\right)z^n$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},-n,a\right)(-z)^n$$
$$= B_1(a) + \frac{1}{3}B_3(a)z^2 + \frac{1}{5}B_5(a)z^4 + \frac{1}{7}B_7(a)z^6 + \cdots, \qquad (18)$$

$$\frac{1}{2}\sum_{n=0}^{\infty}\lim_{\gamma\to\infty,\beta\to\infty,\beta'\to\infty}\phi_{1,\beta,\beta',\gamma}\left(\frac{\gamma/2}{\beta},\frac{\gamma/2}{\beta'},-n,a\right)z^n + \frac{1}{2}\sum_{n=0}^{\infty}\phi_{1,\beta,\beta';\beta,\beta'}(1/2,1/2,-n,a)(-z)^n$$

$= B_1(a) + \frac{1}{3}B_3(a)z^2 + \frac{1}{5}B_5(a)z^4 + \frac{1}{7}B_7(a)z^6 + \cdots,$ (19)

$\frac{1}{2}\sum_{n=0}^{\infty} \phi^*{}_1(1,-n,a)\, z^n + \frac{1}{2}\sum_{n=0}^{\infty} \phi_{1,\beta,\beta';\,\beta,\beta'}(1/2\,,1/2\,,-n,a)\,(-z)^n$
$= B_1(a) + \frac{1}{3}B_3(a)z^2 + \frac{1}{5}B_5(a)z^4 + \frac{1}{7}B_7(a)z^6 + \cdots.$ (20)

By using these series (15)-(20), we may compute the series involving the double zeta functions (1) and (5).

3. Analytic continuation formulae of double zeta functions (1) and (5)

In this section, we consider the Euler integral formulae of double zeta functions (1) and (5) and then obtain their analytic continuation formulae.

Theorem A. If $\Re(s) > 0, \Re(a) > 0, \alpha, \beta, \beta', \gamma, \gamma' \in \mathbb{C}$, then the double zeta function $\phi_{\alpha,\beta,\beta';\,\gamma,\gamma'}(z,t,s,a)$ satisfies an analytic continuation formula

$$\frac{\Gamma(\alpha)\Gamma(\beta')}{\Gamma(\gamma')}\phi_{\alpha,\beta,\beta';\,\gamma,\gamma'}(z,t,s,a) = \frac{(-t^{-1})^{-\alpha}\Gamma(\alpha)\Gamma(\beta'-\alpha)}{\Gamma(\gamma'-\alpha)}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha-\gamma'+1)_n}{(\alpha-\beta'+1)_n}\phi_{\alpha,\beta,\beta',\gamma}(z,z,s,n+a+\alpha)\right\}\frac{(t)^n}{n!}$$
$$+\frac{(-t^{-1})^{-\beta'}\Gamma(\beta')\Gamma(\alpha-\beta')}{\Gamma(\gamma'-\beta')}\left\{\sum_{n=0}^{\infty}\frac{(\beta')_n(\beta'-\gamma'+1)_n}{(\beta'-\alpha+1)_n}\phi_{\alpha,\beta,\beta',\gamma}(z,z,s,n+a+\beta')\right\}\frac{(t)^n}{n!},$$

provided that $|\arg(-t^{-1})| < \pi, |z| < 1, |t^{-1}| < 1/\max\{1, |1-z|\}$.
(21)

Proof. Consider the Euler integral formula by techniques of ([2], [7], [9]) for $\Re(s) > 0, \Re(a) > 0, |z| + |t| < e^{-x} \forall x \in (0,\infty)$,

$$\phi_{\alpha,\beta,\beta';\,\gamma,\gamma'}(z,t,s,a) = \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1}e^{-ax}F_2(\alpha,\beta,\beta';\gamma,\gamma';z\,e^{-x},te^{-x})dx,$$
(22)

and make an appeal to the analytic continuation formula for the Appell function $F_2(.)$ (see, Srivastava and Karlsson [11, p. 294]) where we have replaced t by t^{-1} to get it in the form

$$\frac{\Gamma(\alpha)\Gamma(\beta')}{\Gamma(\gamma')}F_2(\alpha,\beta,\beta';\gamma,\gamma';z,t^{-1}) = \frac{(-t^{-1})^{-\alpha}\Gamma(\alpha)\Gamma(\beta'-\alpha)}{\Gamma(\gamma'-\alpha)}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha-\gamma'+1)_n}{(\alpha-\beta'+1)_n}\,{}_2F_1\left[\begin{matrix}-n,\beta;\\ \gamma;\end{matrix}z\right]\frac{t^n}{n!}$$
$$+\frac{\Gamma(\beta')\Gamma(\alpha-\beta')}{\Gamma(\gamma'-\beta')}(-t^{-1})^{-\beta'}\sum_{n=0}^{\infty}\frac{(\beta')_n(\beta'-\gamma'+1)_n}{(\beta'-\alpha+1)_n}\,{}_2F_1\left[\begin{matrix}\alpha-\beta'-n,\beta;\\ \gamma;\end{matrix}z\right]\frac{t^n}{n!},$$

$|\arg(-t^{-1})| < \pi, |z| < 1, |t^{-1}| < 1/\max\{1, |1-z|\}$.
(23)

Then under the conditions given in Eqns. (22)-(23), multiply both the sides of Eqn. (23) by $\frac{1}{\Gamma(s)}x^{s-1}e^{-ax}$ and then integrate both the sides with respect to x from 0 to ∞, we have

$$\frac{\Gamma(\alpha)\Gamma(\beta')}{\Gamma(\gamma')}\frac{1}{\Gamma(s)}\int_0^{\infty}x^{s-1}e^{-ax}\,F_2(\alpha,\beta,\beta';\gamma,\gamma';ze^{-x},t^{-1}e^{-x})dx$$
$$= \frac{(-t^{-1})^{-\alpha}\Gamma(\alpha)\Gamma(\beta'-\alpha)}{\Gamma(\gamma'-\alpha)}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_n(\alpha-\gamma'+1)_n}{(\alpha-\beta'+1)_n}\frac{1}{\Gamma(s)}\int_0^{\infty}x^{s-1}e^{-(n+a+\alpha)x}\,{}_2F_1\left[\begin{matrix}-n,\beta;\\ \gamma;\end{matrix}ze^{-x}\right]dx\right\}\frac{(t)^n}{n!}$$
$$+\frac{(-t^{-1})^{-\beta'}\Gamma(\beta')\Gamma(\alpha-\beta')}{\Gamma(\gamma'-\beta')}\left\{\sum_{n=0}^{\infty}\frac{(\beta')_n(\beta'-\gamma'+1)_n}{(\beta'-\alpha+1)_n}\frac{1}{\Gamma(s)}\int_0^{\infty}x^{s-1}e^{-(n+a+\beta')x}\,{}_2F_1\left[\begin{matrix}\alpha-\beta'-n,\beta;\\ \gamma;\end{matrix}ze^{-x}\right]dx\right\}\frac{(t)^n}{n!}.$$

(24)

Now in Eqn. (24), use the formulae (6) and (22). Under the conditions given in Eqn. (23), the general formula (21) can then be deduced.

Theorem B. If $\Re(s) > 0, \Re(a) > 0, \beta' \in \mathbb{C}, \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-$ then the double zeta function $\phi_{\alpha,1,\beta',\alpha}(z,z,s,a)$ follows an analytic continuation formula

$$\phi_{\alpha,1,\beta',\alpha}(z,z,s,a) = \phi(z,s,a) + \sum_{r=1}^{\infty} \frac{(\beta')_r}{r!} t^r \phi(z,s,a+r),$$

(25)

provided that $|z| < 1, |t| < 1$.

Proof. Under the conditions $|z| < e^{-x}, |t| < e^{-x}\ \forall x \in (0,\infty)$, consider the Euler type integral formula due to ([3], [7], [10])

$$\phi_{\alpha,1,\beta',\gamma}(z,z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax} F_1(\alpha,1,\beta';\gamma; z\,e^{-x}, te^{-x}) dx, \tag{26}$$

provided that $\Re(s) > 0, \Re(a) > 0, \alpha, \beta' \in \mathbb{C}, \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

Next, make an appeal to the relation [12, Srivastava and Manocha p. 54], we may write

$$F_1(\alpha,1,\beta';\alpha; ze^{-x}, te^{-x}) = \frac{(1-te^{-x})^{-\beta'}}{(1-ze^{-x})} = \sum_{r=0}^{\infty} \frac{(\beta')_r}{r!} t^r \frac{e^{-rx}}{(1-ze^{-x})},$$

provided that $|z| < e^{-x}, |t| < e^{-x}\ \forall x \in (0,\infty), \beta' \in \mathbb{C}, \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-$. (27)

Now under the conditions given in (26)-(27), multiply by $\frac{1}{\Gamma(s)} x^{s-1} e^{-ax}$ in both sides of Eqn. (27), and then integrate these sides with respect to x from 0 to ∞, to get

$$\frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax} F_1(\alpha,1,\beta';\alpha; ze^{-x}, te^{-x}) dx = \left\{ \sum_{r=0}^{\infty} \frac{(\beta')_r}{r!} t^r \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-(r+a)x}}{(1-ze^{-x})} dx \right\}. \tag{28}$$

Finally, use the formulae (4) and (26) in Eqn. (28), under the conditions $|z| < 1, |t| < 1$, and $\Re(s) > 0, \Re(a) > 0, \beta' \in \mathbb{C}, \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-$, to get

$$\phi_{\alpha,1,\beta',\gamma}(z,z,s,a) = \phi(z,s,a) + \sum_{r=1}^{\infty} \frac{(\beta')_r}{r!} t^r \phi(z,s,a+r). \tag{29}$$

Therefore, the **Theorem B** is followed.

4. Concluding Remarks

Analytic continuation formula of zeta function often succeeds in defining further values of a function, for example in a new region where an infinite series representation in terms of which it is initially defined becomes divergent. Due to step-wise continuation technique ,this paper work is useful in analytic number theory [1]. We have considered double zeta functions (1) and (5) and their generalizations. Various properties and similar other zeta functions have been studied in [2] and [3].On specialization of some parameters in these double zeta functions, in Section 1, we have evaluated their relations with known and unknown zeta functions. Section 2 consists of their functional equations and in Section 3, we have evaluated relations with Bernoulli polynomials. The connection of the Riemann zeta function to the Bernoulli numbers and polynomials is important from the point of view of applications. In section 4, we obtain analytic continuation formulae of double zeta functions (1) and (5).

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31 | Ganita Sandesh Vol. 32 (June & December, 2018) pp. 31-39 ISSN : 0970-9169

Some Properties of Extended τ-Gauss Hypergeometric Function and Its Transformations

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Abstract

The main objective of this paper is to introduce an extension of the τ - Gauss Hypergeometric function ${}_3R_2^{\tau}(z)$ which is obtained in terms of an extended version of the pochhammer symbol using the extended Beta function contains extra parameters and investigate its various properties such as integral representations, derivative formula, Mellin transform, and fractional calculus operators.

Keywords and phrases: Extended Gamma and extended Beta function, pochhammer symbol and its extension, τ-Gauss hypergeometric function and its extensions, Mellin transforms, fractional calculus operators

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1. Introduction

Definition 1.1([12]) Let a function $\Theta(\{k_l\}_{l\in N_0}; z)$ be analytic within the disk $|z| < R(0 < R < 1$) and let its Taylor-Maclaurin coefficients be explicitly denoted by the sequence $\{k_l\}_{l\in N_0}$. Suppose also that the function $\Theta(\{k_l\}_{l\in N_0}; z)$ can be continued analytically in the right halfplane $\Re(z) > 0$ with the asymptotic property given as follows:

$$\Theta(\{k_l\}_{l\in N_0}; z) = \begin{cases} \sum_{l=0}^{\infty} \{k_l\} \frac{z^l}{l!} & (|z| < R; 0 < R < \infty; k_0 = 1) \\ M_0 z^w \exp(z) \left[1 + O\left(\frac{1}{z}\right)\right] & (\Re(z) \to \infty; M_0 > 0; w \in \mathbb{C}) \end{cases} \quad (1)$$

for some suitable constants M_0 and w depending essentially on the sequence $\{k_l\}_{l\in N_0}$. They also defined extended Gamma function $\Gamma_p^{\{k_l\}}(z)$ and the extended Beta function.

$$\Gamma_p^{\{k_l\}}(z) = \int_0^{\infty} t^{z-1}\Theta\left(\{k_l\}; -t - \frac{p}{t}\right) dt, \quad (\Re(p) \geqslant 0, \Re(z) > 0) \quad (2)$$

and

$$B_p^{\{k_l\}}(\alpha, \beta; p) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(\{k_l\}; \frac{-p}{t(1-t)}\right) dt \quad (3)$$
$$\left(\Re(p) \geqslant 0, \mathrm{m} \quad (\Re(\alpha), \Re(\beta)) > 0\right).$$

By introducing one additional parameter q with $\Re(q) \geqslant 0$, another extension of Beta function was introduced as :

$$B_{p,q}^{\{k_l\}}(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\Theta\left(\{k_l\}; -\frac{p}{t} - \frac{q}{(1-t)}\right) dt \quad (4)$$
$$\mathrm{m} \quad (\Re(p), \Re(q)) > 0; \mathrm{m} \quad (\Re(\alpha), \Re(\beta)) > 0.$$

During the past few decades, various extensions and generalizations of well-known special functions have been studied by various researchers [3,6-8]. Chaudhry et.al. also defined a 2-parameter extension of gamma function $\Gamma(\xi)$ with the parameter (p &v) in [3] as follow :

$$\Gamma_v(\xi; p) = \begin{cases} \sqrt{\frac{2p}{\pi}} \int_0^{\infty} t^{\xi-\frac{3}{2}} e^{-t} k_{v+\frac{1}{2}}\left(\frac{p}{t}\right) dt & (\mathrm{m} \quad (\Re(p), \Re(v)) > 0; \xi \in \mathbb{C}) \\ \Gamma_p(\xi) & (v = o; \Re(\xi) > 0) \end{cases} \quad (5)$$

where $k_v(z)$ is the modified Bessel function of order v and $\Gamma_p(\xi)$ was studied in [3]. Indeed if $\mathrm{v} = 0$ in (4) and make use of the following relationship

$$k_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \quad (6)$$

then the above-extended Gamma function can be given as :

$$\Gamma_p(\xi) = \int_0^\infty t^{\xi-1} e^{-t-\frac{p}{t}} dt \qquad (\Re(p) \geqslant 0, \Re(\xi) > 0). \tag{7}$$

In the year 2012, Srivastava et.al. [11] defined the following extensions and generalization of the pochhammer symbols as follow:

$$(\xi; p,)_\mu = \begin{cases} \frac{\Gamma_p(\xi+\mu)}{\Gamma(\xi)} & (\Re(p) > 0; \xi, \mu \in \mathbb{C}), \\ (\xi)_\mu & (p = 0; \xi, \mu \in \mathbb{C} \setminus \{0\}) \end{cases} \tag{8}$$

$$(\xi; p, v)_\mu = \begin{cases} \frac{\Gamma_v(\xi+\mu;p)}{\Gamma(\xi)} & (\mathrm{m}(\Re(p), \Re(v)) > 0; \xi, \mu \in \mathbb{C}), \\ ((\xi; p)_\mu & (v = 0; \xi, \mu \in \mathbb{C} \setminus \{0\}) \end{cases} \tag{9}$$

from (4) and (8), they get

$$(\xi; p, v)_\mu = \frac{1}{\Gamma(\xi)} \sqrt{\frac{2p}{\pi}} \int_0^\infty t^{\xi+\mu-\frac{3}{2}} e^{-t} k_{v+\frac{1}{2}}\left(\frac{p}{t}\right) dt \tag{10}$$

$$(\xi; p)_\mu = \frac{1}{\Gamma(\xi)} \int_0^\infty t^{\xi+\mu-1} e^{-t-\frac{p}{t}} dt \tag{11}$$

By using (8), an extension of generalized hypergeometric function [12] ${}_pF_q$ was defined as :

$${}_pF_q\left[\begin{cases}(\rho_1; p, v), \rho_2, \rho_3 \ldots\ldots, \rho_p \\ \sigma_1, \sigma_2, \ldots\ldots, \sigma_q;\end{cases}; z\right] = \sum_{n=0}^{\infty} \frac{(\rho_1;p,v)_n (\rho_2)_n \ldots\ldots (\rho_p)_n}{(\sigma_1)_n (\sigma_2)_n \ldots\ldots (\sigma_q)_n} \frac{z^n}{n!}, \tag{12}$$

where $\rho_j \in \mathbb{C} (j = 1,2, \ldots p)$ $\sigma_j \in \mathbb{C} \setminus \mathbb{Z}_0^- (j = 1,2, \ldots q)$.

Virchenko et al.[14] studied the following τ-Gauss hypergeometric function ${}_2R_1^\tau z$ defined as:

$${}_2R_1^\tau z = {}_2R_1\{\delta_1, \delta_2; \delta_3; \tau, z\} = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)} \sum_{n=0}^{\infty} \frac{(\delta_1)_n \Gamma(\delta_2 + n\tau)}{\Gamma(\delta_3 + n\tau)} \frac{z^n}{n!}, \tag{13}$$

$\{\tau > 0; |z| < 1; \Re(\delta_3) > R(\delta_2) > 0 \text{ when } |z| = 1\}$.

They also derive the following integral representation :

$${}_2R_1(\delta_1, \delta_2; \delta_3; \tau, z) = \frac{1}{B(\delta_2, \delta_3 - \delta_2)} \int_0^\infty t^{\delta_2 - 1} (1-t)^{\delta_3 - \delta_2 - 1} (1 - zt^\tau)^{-\delta_1} dt \tag{14}$$

$(\tau > 0; |arg(1 - z)| < \pi; \Re(\delta_3) > R(\delta_2) > 0)$

in terms of classical Beta function $B(\alpha, \beta)$ defined as

$$B(\alpha,\beta)=\begin{cases}\int_0^1 t^{\alpha-1}(1-t)^{\beta=\alpha}dt & (\mathrm{m}(\Re(\alpha),\Re(\beta))>0)\\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in\mathbb{C}/\mathbb{Z}_0^-)\end{cases} \tag{15}$$

2. Main Results

Motivated mainly by investigations of the extended τ - Gauss hypergeometric function defined earlier by Srivastava et.al. [13], we introduce the extended τ-hypergeometric function. The provided result are new and have a unique property.

2.1. Extended τ-Gauss Hypergeometric Function

The extended τ-Gauss hypergeometric function as follow :

$$\begin{aligned}&{}_3R_2^{\{k_l\}}[(\delta_1;p),\delta_2,\delta_3;\delta_4,\delta_5,\tau;z,\mu_1,\mu_2]\\ &=(\delta_1,p)_n\sum_{n=0}^{\infty}\frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2+n\tau,\delta_4-\delta_2)B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_3+n\tau,\delta_5-\delta_3)}{B(\delta_2,\delta_4-\delta_2)B(\delta_3,\delta_5-\delta_3)}\frac{z^n}{n!}.\end{aligned} \tag{16}$$

$$\big(p\geqslant 0,\tau>0,|z|<1,\Re(\delta_5)>R(\delta_2),\Re(\delta_4)>R(\delta_3)>0,\mathrm{m}\quad\big(\Re(\mu_1),\Re(\mu_2)\big)>0\big)$$

3. Integral Representation and Derivative formula

In this section, we obtain integral representation and derivative formula for ${}_3R_2^{\{k_l\}}(\tau,z)$ as given in (16) :

Theorem 3.1. The following integral representation for ${}_3R_2^{\{k_l\}}(\tau,z)$ in (16) holds true:

$$\begin{aligned}&{}_3R_2^{\{k_l\}}[(\delta_1;p),\delta_2,\delta_3;\delta_4,\delta_5,\tau;z,\mu_1,\mu_2]\\ &=\frac{1}{B(\delta_2,\delta_4-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_4-\delta_2-1}\cdot\ {}_2R_1^{\{k_l\}}[(\delta_1;p),\delta_3;\delta_5,\tau;zt^{\tau},\mu_1,\mu_2]dt.\end{aligned} \tag{17}$$

$$\tau>0,\Re(p)>0,\Re(\delta_5)>R(\delta_2)>0,\Re(\delta_4)>R(\delta_3)>0 \text{ when } p=0,\mathrm{m}\quad\big(\Re(\mu_1),\Re(\mu_2)\big)>0\big)$$

Proof: Using (16) and considering the following elementary identity for the beta function

$$B(\alpha,\beta)=\begin{cases}\int_0^1 t^{\alpha-1}(1-t)^{\beta=\alpha}dt\\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\end{cases}$$

then we arrive at

$$ {}_3R_2^{\{k_l\}}[(\delta_1;p),\delta_2,\delta_3;\delta_4,\delta_5,\tau;z,\mu_1,\mu_2] $$
$$ =\frac{1}{B(\delta_2,\delta_4-\delta_2)}\frac{\Gamma(\delta_5)}{\Gamma(\delta_3)}\sum_{n=0}^{\infty}(\delta_1,p)_n\frac{\Gamma(\delta_3+n\tau)}{\Gamma(\delta_5+n\tau)}B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2+n\tau,\delta_4-\delta_2)\frac{z^n}{n!} $$
$$ =\frac{1}{B(\delta_2,\delta_4-\delta_2)}\frac{\Gamma(\delta_5)}{\Gamma(\delta_3)}\sum_{n=0}^{\infty}(\delta_1,p)n\frac{\Gamma(\delta_3+n\tau)}{\Gamma(\delta_5+n\tau)}\cdot\int_0^1 t^{\delta_2+n\tau-1}(1-t)^{\delta_4-\delta_2-1}\cdot\Theta\left(\{k_l\},-\frac{\mu_1}{t}-\frac{\mu_2}{1-t}\right)dt\frac{z^n}{n!} $$
$$ =\frac{1}{B(\delta_2,\delta_4-\delta_2)}\frac{\Gamma(\delta_5)}{\Gamma(\delta_3)}\sum_{n=0}^{\infty}(\delta_1,p)_n\frac{\Gamma(\delta_3+n\tau)}{\Gamma(\delta_5+n\tau)}\int_0^1 t^{\delta_2+n\tau-1}(1-t)^{\delta_4-\delta_2-1}\frac{z^n}{n!}dt.\Theta\left(\{k_l\},-\frac{\mu_1}{t}-\frac{\mu_2}{1-t}\right) $$

Next interchanging the order of integration and summation that is permissible

$$ =\frac{1}{B(\delta_2,\delta_4-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_4-\delta_2-1}\left\{\frac{\Gamma(\delta_5)}{\Gamma(\delta_3)}\sum_{n=0}^{\infty}(\delta_1,p)_n\frac{\Gamma(\delta_3+n\tau)}{\Gamma(\delta_5+n\tau)}\cdot\frac{(zt^\tau)^n}{n!}\Theta\left(\{k_l\},-\frac{\mu_1}{t}-\frac{\mu_2}{1-t}\right)\right\}dt $$
$$ =\frac{1}{B(\delta_2,\delta_4-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_4-\delta_2-1}\cdot\ {}_2R_1^{\{k_l\}}[(\delta_1;p),\delta_3;\delta_5,\tau;zt^\tau,\mu_1,\mu_2]dt $$

This completes the proof.

Theorem 3.2. The following derivative formula for ${}_3R_2^{\{k_l\}}(\tau,z)$ in (16) holds true:

$$ \left(\frac{d}{dz}\right)^n\left[z_3^{\delta_4-1}R_2^{\{k_l\}}[(\delta_1;p),\delta_2,\delta_3;\delta_4,\delta_5;\tau;vz^\tau,\mu_1,\mu_2]\right] $$
$$ =\frac{z^{\delta_4-n-1}\Gamma(\delta_4)}{\Gamma(\delta_4-n)}\ {}_3R_2^{\{k_l\}}[(\delta_1;p),\delta_2,\delta_3;\delta_4-n,\delta_5;\tau;vz^\tau,\mu_1,\mu_2]. \tag{18} $$

Proof: By using the series representation of ${}_3R_2^{\{k_l\}}(\tau,z)$, We introduce the extended τ hypergeometric fucntion as follow :

$$ \left[z^{\delta_4-1}\ {}_3R_2^{\{k_l\}}[(\delta_1;p),\delta_2,\delta_3;\delta_4,\delta_5;\tau;vz^\tau,\mu_1,\mu_2]\right] $$
$$ =z^{\delta_4-1}\sum_{m=0}^{\infty}\frac{(\delta_1;p)_m\Gamma(\delta_2+m\tau)\Gamma(\delta_3+m\tau)\Gamma(\delta_4)\Gamma(\delta_5)}{\Gamma(\delta_4+m\tau)\Gamma(\delta_5+m\tau)\Gamma(\delta_2)\Gamma(\delta_3)}\frac{(vz^\tau)^m}{m!} $$
$$ \frac{\Gamma(\delta_4)\Gamma(\delta_5)}{\Gamma(\delta_2)\Gamma(\delta_3)}\sum_{m=0}^{\infty}\frac{(\delta_1;p)_m\Gamma(\delta_2+m\tau)\Gamma(\delta_3+m\tau)}{\Gamma(\delta_4+m\tau)\Gamma(\delta_5+m\tau)}\frac{v^m}{m!}z^{(\delta_4+m\tau-1)}\Theta\left(\{k_l\},-\frac{\mu_1}{t}-\frac{\mu_2}{1-t}\right) $$

Now differentiating term by term under the sign of summation, we have:

$$ =\frac{\Gamma(\delta_4)\Gamma(\delta_5)}{\Gamma(\delta_2)\Gamma(\delta_3)}\sum_{m=0}^{\infty}\frac{(\delta_1;p)_m\Gamma(\delta_2+m\tau)\Gamma(\delta_3+m\tau)}{\Gamma(\delta_4+m\tau-n)\Gamma(\delta_5+m\tau)}\frac{v^m}{m!}z^{(\delta_4+m\tau-n-1)}\Theta\left(\{k_l\},-\frac{\mu_1}{t}-\frac{\mu_2}{1-t}\right) $$
$$ =z^{\delta_4-n-1}\frac{\Gamma(\delta_4)\Gamma(\delta_5)}{\Gamma(\delta_2)\Gamma(\delta_3)}\sum_{m=0}^{\infty}\frac{(\delta_1;p)_m\Gamma(\delta_2+m\tau)\Gamma(\delta_3+m\tau)}{\Gamma(\delta_4+m\tau-n)\Gamma(\delta_5+m\tau)}z^{m\tau}\frac{v^m}{m!}\Theta\left(\{k_l\},-\frac{\mu_1}{t}-\frac{\mu_2}{1-t}\right) $$
$$ =\frac{z^{\delta_4-n-1}\Gamma(\delta_4)}{\Gamma(\delta_4-n)}\ {}_3R_2^{\{k_l\}}[(\delta_1;p),\delta_2,\delta_3;\delta_4-n,\delta_5;\tau;vz^\tau,\mu_1,\mu_2] $$

This completes the proof.

4. Mellin transform of the function $_3R_2^{\{k_l\}}(\tau, z)$

The Mellin transform of a function $f(x)$ is defined by

$$M[f(x): s] = F(s) = \int_0^\infty x^{s-1} f(x) dx \quad (s \in \mathbb{C})$$

provided that the improper integral in the above equation exists.

Theorem 4.1. The Mellin transform of the extended τ-Gauss hypergeometric function $_3R_2^{\{k_l\}}(\tau, z)$ defined in (16) is given by

$$M\left[\; _3R_2^{\{k_l\}}[(\delta_1; p), \delta_2, \delta_3; \delta_4, \delta_5, \tau; z, \mu_1, \mu_2]: s\right] = \Gamma(s)(\delta_1)_s \; _3R_2^{\{k_l\}}[(\delta_1 + s), \delta_2, \delta_3; \delta_4, \delta_5, \tau; z, \mu_1, \mu_2]$$

$\big($ where $\Re(s) > 0$ and $\Re(\delta_1 + s) > 0$ when $p = 0$, m $\big(\Re(\mu_1), \Re(\mu_2)\big) > 0\big)$ (19)

Proof :

$$M\left[\; _3R_2^{\{k_l\}}[(\delta_1; p), \delta_2, \delta_3; \delta_4, \delta_5, \tau; z, \mu_1, \mu_2]: s\right]$$

$$= \int_0^\infty p^{s-1} \left[(\delta_1, p) n \sum_{n=0}^\infty \frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2 + n\tau, \delta_4 - \delta_2) B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_3 + n\tau, \delta_5 - \delta_3)}{B(\delta_2, \delta_4 - \delta_2) B(\delta_3, \delta_5 - \delta_3)} \frac{z^n}{n!} \right]$$

$$= \sum_{n=0}^\infty \frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2 + n\tau, \delta_4 - \delta_2) B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_3 + n\tau, \delta_5 - \delta_3)}{B(\delta_2, \delta_4 - \delta_2) B(\delta_3, \delta_5 - \delta_3)} \frac{z^n}{n!} \int_0^\infty p^{s-1} (\delta_1, p)_n dp$$

$$= \sum_{n=0}^\infty \frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2 + n\tau, \delta_4 - \delta_2) B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_3 + n\tau, \delta_5 - \delta_3)}{B(\delta_2, \delta_4 - \delta_2) B(\delta_3, \delta_5 - \delta_3)} \frac{z^n}{n!} \int_0^\infty p^{s-1} \frac{\Gamma_p(\delta_1 + n)}{\Gamma(\delta_1)} dp$$

Using result $\int_0^\infty p^{s-1} \Gamma_p(\delta_1 + n) dp = \Gamma(\delta_1 + s + n)\Gamma(s)$

$$= \sum_{n=0}^\infty \frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2 + n\tau, \delta_4 - \delta_2) B_{\mu_1,\mu_2}^{\{k_2\}}(\delta_3 + n\tau, \delta_5 - \delta_3)}{B(\delta_2, \delta_4 - \delta_2) B(\delta_3, \delta_5 - \delta_3)} \frac{z^n}{n!} \frac{\Gamma(\delta_1 + s + n)\Gamma(s)}{\Gamma(\delta_1)}$$

Now using $\Gamma(\delta_1 + s + n) = (\delta_1)_s (\delta_1 + s)_n \Gamma(\delta_1)$, we get

$$= \sum_{n=0}^\infty \frac{B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2 + n\tau, \delta_4 - \delta_2) B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_3 + n\tau, \delta_5 - \delta_3)}{B(\delta_2, \delta_4 - \delta_2) B(\delta_3, \delta_5 - \delta_3)} \frac{z^n}{n!} (\delta_1)_s (\delta_1 + s)_n \Gamma(s)$$

$$= (\delta_1)_s \Gamma(s) \sum_{n=0}^\infty \frac{(\delta_1 + s)_n B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2 + n\tau, \delta_4 - \delta_2) B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_3 + n\tau, \delta_5 - \delta_3)}{B(\delta_2, \delta_4 - \delta_2) B(\delta_3, \delta_5 - \delta_3)} \frac{z^n}{n!}$$

$$= (\delta_1)_s \Gamma(s) {}_3R_2^{\{k_l\}}[(\delta_1 + s), \delta_2, \delta_3; \delta_4, \delta_5, \tau; z, \mu_1, \mu_2]$$

This completes the proof.

5. Fractional Calculus Approach

The Riemann-Liouville left-sided fractional calculus operators I_{a+}^{α} and D_{a+}^{α} of order α defined by [10]

$$I_{a+}^{\alpha} f(x) = \ {}_aI_x^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad (x > a),$$

is called the Riemann-Liouville left-sided fractional integral of order α.

$$D_{a+}^{\alpha} f(x) = \ {}_aD_x^{\alpha} = \frac{1}{(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)dt}{(t-x)^{\alpha-n+1}}, \quad (n = [\alpha]+1),$$

is called the left-sided Riemann-Liouville derivative of order α. In this section, we consider the fractional differintegral operators containing extended τ Gauss hypergeometric function.

Theorem 5.1. Let $\mu \in \Re_+, \delta_1, \alpha, \delta_2, \delta_3, \delta_4, \delta_5, \mathrm{v} \in \mathbb{C}$ and $\Re(\alpha) > 0, \Re(\delta_4) > 0, min(\Re(\mu_1), \Re(\mu_2)) > 0$ and $\Re(\tau) > 0$ then for $x > \mu$ the following results holds true:

$$\begin{aligned} & I_{\mu+}^{\alpha}\left[(t-\mu)^{\delta_4-1} \ {}_3R_2^{\{k_l\}}[(\delta_1,p),\delta_2,\delta_3;\delta_4,\delta_5;\tau,v(t-\mu)^{\tau},\mu_1,\mu_2]\right](x) \\ & = \frac{(x-\mu)^{\delta_4+\alpha-1}\Gamma(\delta_4)}{\Gamma(\delta_4+\alpha)} \ {}_3R_2^{\{k_l\}}[(\delta_1,p),\delta_2,\delta_3;\delta_4+\alpha,\delta_5;\tau,v(x-\mu)^{\tau},\mu_1,\mu_2]. \end{aligned} \tag{20}$$

Proof : By using the series representation of extended τ Gauss hypergeometric function ${}_3R_2^{\{k_l\}}(\tau, z)$ as given by and interchanging the order of integration and summation we have :

$$\begin{aligned} & I_{\mu+}^{\alpha}\left[(t-\mu)^{\delta_4-1} \ {}_3R_2^{\{k_l\}}[(\delta_1,p),\delta_2,\delta_3;\delta_4,\delta_5;\tau,v(t-\mu)^{\tau};z,\mu_1,\mu_2]\right](x) \\ & = \sum_{n=0}^{\infty} \frac{(\delta_1,p)_n B_{\mu_1,\mu_2}^{\{k_l\}}(\delta_2+n\tau,\delta_4-\delta_2) B_{\mu_1,\mu_2}^{\{k_2\}}(\delta_3+n\tau,\delta_5-\delta_3)}{B(\delta_2,\delta_4-\delta_2)B(\delta_3,\delta_5-\delta_3)} \frac{v^n}{n!} I_{\mu+}^{\alpha}\left[(t-\mu)^{\delta_4+n\tau-1}\right](x) \end{aligned}$$

now for $x > \mu$, taking the power function formula into account :

$$I_{\mu+}^{\alpha}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}, \quad (\alpha,\beta \in \mathbb{C}, \Re(\alpha) > 0 \text{ and } \Re(\beta) > 0$$

then we have

$$= \frac{(x-\mu)^{\delta_4+\alpha-1}\Gamma(\delta_4)}{\Gamma(\delta_4+\alpha)} \ {}_3R_2^{\{k_l\}}[(\delta_1,p),\delta_2,\delta_3;\delta_4+\alpha,\delta_5;\tau,v(x-\mu)^{\tau};\mu_1,\mu_2]$$

This completes the proof.

Theorem 5.2. Let $\mu \in \Re_+, \delta_1, \alpha, \delta_2, \delta_3, \delta_4, \delta_5, \mathrm{v} \in \mathbb{C}$ and $\Re(\alpha) > 0, \Re(\delta_4) > 0$ and $\Re(\tau) > 0$, $\min(\Re(\mu_1), \Re(\mu_2)) > 0$ then for $x > \mu$ the following results holds true:

$$D_{\mu+}^{\alpha}\left[(t-\mu)^{\delta_4 - 1} \quad {}_3R_2^{\{k_l\}}[(\delta_1, p), \delta_2, \delta_3; \delta_4, \delta_5; \tau, v(t-\mu)^{\tau}, \mu_1, \mu_2]\right](x)$$
$$= \frac{(x-\mu)^{\delta_4-\alpha-1}\Gamma(\delta_4)}{\Gamma(\delta_4-\alpha)} \quad {}_3R_2^{\{k_l\}}[(\delta_1, p), \delta_2, \delta_3; \delta_4 - \alpha, \delta_5; \tau, v(x-\mu)^{\tau}, \mu_1, \mu_2].$$
(21)

Proof : By using series representation given by (16) and using the following relation

$$D_{\mu+}^{\alpha} f(x) = \left(\frac{d}{dx}\right)^n \left(I_{\mu+}^{n-\alpha} f\right)(x) \quad (\alpha \in \mathbb{C}; \alpha > 0, n = [\alpha] + 1)$$

then we have :

$$D_{\mu+}^{\alpha}\left[(t-\mu)^{\delta_4 - 1} \quad {}_3R_2^{\{k_l\}}[(\delta_1, p), \delta_2, \delta_3; \delta_4, \delta_5; \tau, v(t-\mu)^{\tau}, \mu_1, \mu_2]\right](x)$$

$$= \left(\frac{d}{dx}\right)^n \left[I_{\mu+}^{n-\alpha}\left((t-\mu)^{\delta_4-1} \quad {}_3R_2^{\{k_l\}}[(\delta_1, p), \delta_2, \delta_3; \delta_4, \delta_5; \tau, v(t-\mu)^{\tau}, \mu_1, \mu_2]\right)\right](x)$$
$$= \left(\frac{d}{dx}\right)^n \left(\frac{(x-\mu)^{\delta_4+n-\alpha-1}\Gamma(\delta_4)}{\Gamma(\delta_4+n-\alpha)} \quad {}_3R_2^{\{k_l\}}[(\delta_1, p), \delta_2, \delta_3; \delta_4 + n - \alpha, \delta_5; \tau, v(x-\mu)^{\tau}; \mu_1, \mu_2]\right)$$

Now by using the result given in (20), we get

$$= \frac{(x-\mu)^{\delta_4-\alpha-1}\Gamma(\delta_4)}{\Gamma(\delta_4-\alpha)} \quad {}_3R_2^{\{k_l\}}[(\delta_1, p), \delta_2, \delta_3; \delta_4 - \alpha, \delta_5; \tau, v(x-\mu)^{\tau}, \mu_1, \mu_2].$$

6. Concluding Remarks

In the present paper, we derive a new extended τ Gauss -hypergeometric function ${}_3R_2^{\{k_l\}}(\tau, z)$. Our results are motivated mainly by investigations of the τ-gauss hypergeometric function and its extension. We obtained certain integral representation, a derivative formula, Mellin transform, and fractional calculus approach of this new extended τ-hypergeometric function ${}_3R_2^{\{k_l\}}(\tau, z)$. The provided results are new and have uniqueness identity in the literature and this study can be further extended in the field of q-Calculus theory $[1,4-5]$.

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40 | Ganita Sandesh Vol. 32 (June & December, 2018) pp. 40-47 ISSN : 0970-9169

Regular and Chaotic Motion of a Satellite Influenced by Solar Radiation Pressure

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Abstract:

Regular and chaotic planar motion of an Earth satellite investigated with qualitative analysis. The model for the satellite,influenced by solar radiation pressure, described for the planar oscillations by using concepts of Beletskii. The autonomous part of motion part of the motion utilized to study qualitative behavior. Satellite's oscillations observed for cases when radiation is not influencing it as well as when it is influencing it. Emergence of chaos observed through time series, surface of sections and Poincaré maps. Lyapunov exponents calculated for different cases for clear identification of regular and chaotic motion of the satellite. Finally, a brief analysis of the investigation presented in the discussion.

1. Introduction:

Investigations on the motion of a satellite becomes interesting after publications of pioneer works, [1- 5]. These articles mostly described motion of natural satellite influenced by different forces, e.g. tidal and magnetic forces,and also motions of satellites around planets of irregular shape, e.g. Hyperion and some other satellites of Saturn. Cause of regular and chaotic motions also discussed widely. The problems of resonance capture due to various disturbing forces investigated also, [2, 6 – 7]. Use of Melnikov's integral, [8], in studying the spin-orbit-coupling problem of a satellite when tidal force is in action also evolved significant results, [9].It has been shown that tidal parameter plays an important role in changing the satellite motion from regular to irregular and vice-versa in regards to some satellite, e.g. satellitesAmalthea, 1980S27etc. Periodic variations of tidal and magnetic parameters may sometimes act like Chaos controllerdiscussed in some recent articles [10 - 11]. Influence of radiation pressure on Earth satellite is of growing interest viewing the recent researches on motion of such objects and draw significant interest to researchers of celestial mechanics. A recent work on this regard, [12], encouraged to carry study forward more investigations in this direction. The most effective tool for Identifications of chaotic and regular motion is Lyapunov characteristic exponents, (LCE), and at any stage if LCE < 0, the motion is identifies as regular and if LCE > 0 then the motion is chaotic. Actually, LCEs measure the exponential divergence of orbits that started very closely. Some of the powerful articles describing the methodology of calculation of LCEs are to be noted in this regard, [13-15].

The objective of the present investigation is to study extensively the influence of radiation pressure on the planar motion of a satellite orbiting in elliptical orbit. Steady state solutions for the autonomous part of the motion calculated and their stability criteria obtained. Regular and chaotic motions of satellite studied by taking into account no influence of radiation pressure as well as its influence on the motion. In the process of study, time series curves, surfaces of section and Poincaré maps drawn for different parameter spaces. For clear identification of regular and chaotic motions, Lyapunov characteristic exponents, LCEs, calculated numerically and represented graphically.

2. Equations of Motion:

Consider a rigid triaxial satellite moving in an elliptic orbit, of semi-major axis a and eccentricity e, around the earth under the influence of Solar radiation pressure **F** = $\boldsymbol{F_g}$ (1- q), $\boldsymbol{F_g}$ is the Solar gravitational attraction force and $0 < 1 - q \ll 1$.Principal moments of inertia are such that A < B < C, where C considered about the spin axis which is one of the principal axes. The torque to the solar radiation pressure is assumed to be perpendicular to the orbital plane. The centre of resultant radiation pressure lies on x – axis as assumed in Maciejewski [ ], Figure 1.

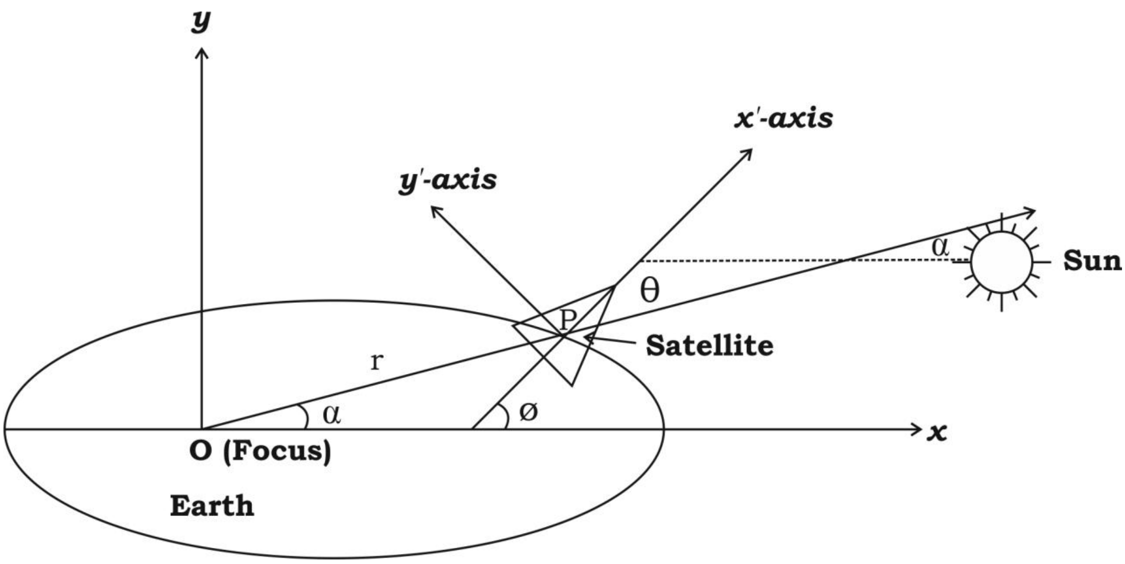


Figure 1: Motion of the Satellite Around the Earth

Let r be the instantaneous radius,α be the true anomaly and ϕbe the orientation of the satellite's long axis. Then, ϕ–α= θ, provides the measure of orientation of the satellite's long axis relative to its radius vector.The equation of motion of satellite's planar oscillation is given by

$$\frac{d^2\phi}{dt^2} = \frac{n^2}{2r^3} \sin 2(\phi-\alpha) + \epsilon \sin \phi = 0, \qquad (1)$$

where $n^2 = \frac{3\,(B-A)}{C}$ and ϵ is proportional to solar radiation torque.

Taking ϕ–α= θ,2 θ = q, $\frac{l}{r}$ = 1 + e cos α, and changing the independent variable t with α, a simplified form of the equation (1), [Bhatnagar et al], obtained as

$$(1 + e\cos\alpha)\frac{d^2q}{d\alpha^2} - 2\,e\sin\alpha\,\frac{dq}{d\alpha} + n^2\sin q$$
$$+ 2\epsilon(1 + e\cos\alpha)^{-3}\sin\left(\alpha + \frac{q}{2}\right) - 4\,e\sin\alpha = 0, \qquad (2)$$

Autonomous part of equation (2) provides

$$\frac{d^2q}{d\alpha^2} + n^2\sin q = 0. \qquad (3)$$

The Hamiltonian corresponding to motion (3) is then obtained as

$$H(q, p) = \frac{p^2}{2} - n^2\cos q$$

where, $p = \frac{dq}{d\alpha}$ is the generalized momenta.

Equilibrium points corresponding to this Hamiltonian are given by ($\pm\pi$, 0) and are of hyperbolic type, i. e. are saddle points. As these are not stable points, motion initiating near these points may not be stable. This implies motions originated in the neighborhood of these hyperbolic points are unstable and could be chaotic. Thus for chaotic oscillations of the satellite values of the system parameters as well as the initial points both are important. We study such motions in the following section.

3. Dynamics of Satellite Motion:Numerical Studies

An equivalent representation of the second order equation of motion (2) is

$$\frac{dq}{d\alpha} = y,$$
$$\frac{dy}{d\alpha} = [2\,e\sin\alpha\;y - n^2\sin q - 2\,\epsilon(1 + e\cos\alpha)^{-3}\sin\left(\alpha + \frac{q}{2}\right) + 4\,e\sin\alpha] / (1 + e\cos\alpha). \qquad (4)$$

To investigate oscillatory motion of the satellite taking into account various initial points, some are far from the hyperbolic points and some are in the neighborhood of it. Thus, for numerical calculations following initial conditions are chosen: (q_{0}, p_0) = (0, 0), (0.5,0.5), (2.8, 0.1) & (3.8, 0.5). We proceed our studies for following two cases:

Case 1: When solar radiation pressureis not taken into account, i. e. when $\epsilon = 0$:

As in this case solar radiation pressure is not influencing the motion of the satellite, the parameters influencing motion are eccentricity e of the orbit and the n, the gravitational parameter.Therefore, takingn^2 = 0.1 and e = 0.05, we have solved equations (4), numerically with above mentioned initial conditions and obtained time series plots, Figure 1,

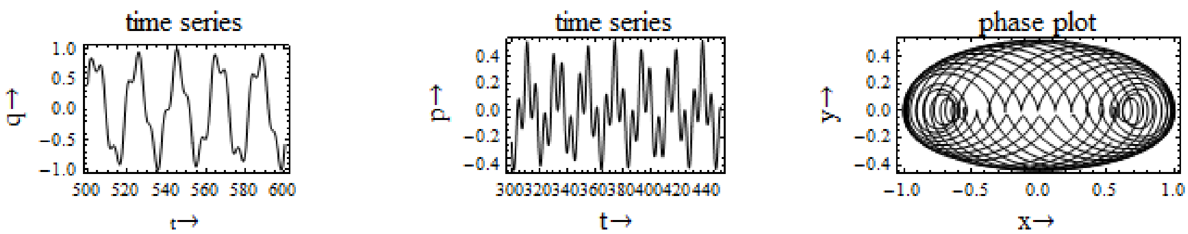
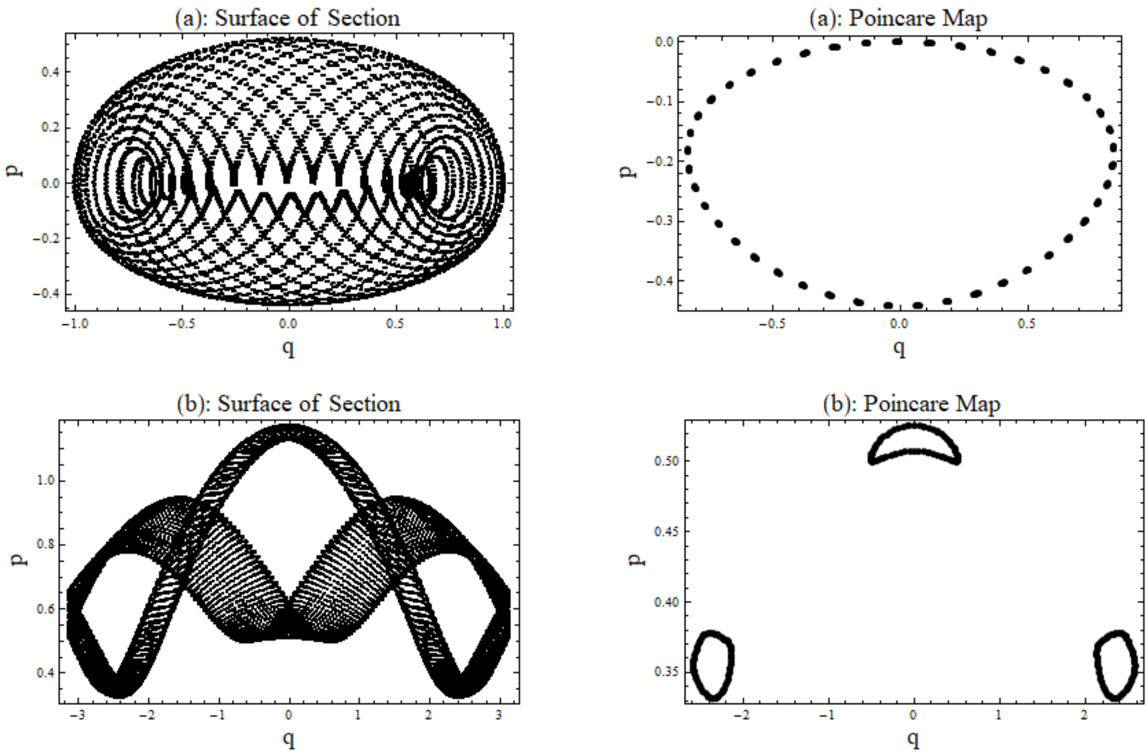


Figure 2: Time series curves (above) and phase plot for ϵ = 0, e = 0.05, n^2 = 0.1 and initial conditions (0, 0).

For ϵ = 0, e = 0.05, n^2 = 0.1 and with four different initial conditions surface of sections and Poincaré maps are drawn and shown in Figure 3. One finds, motions started near hyperbolic equilibrium show chaos whereas that started far are regular. Cases (a), (b) are regular and (c),

(d) are chaotic. However, when eccentricity, e, increased to 0.15 motions are entirely chaos, Figure 4.



GANITA SANDESH, Vol. 32 (June & December, 2018)

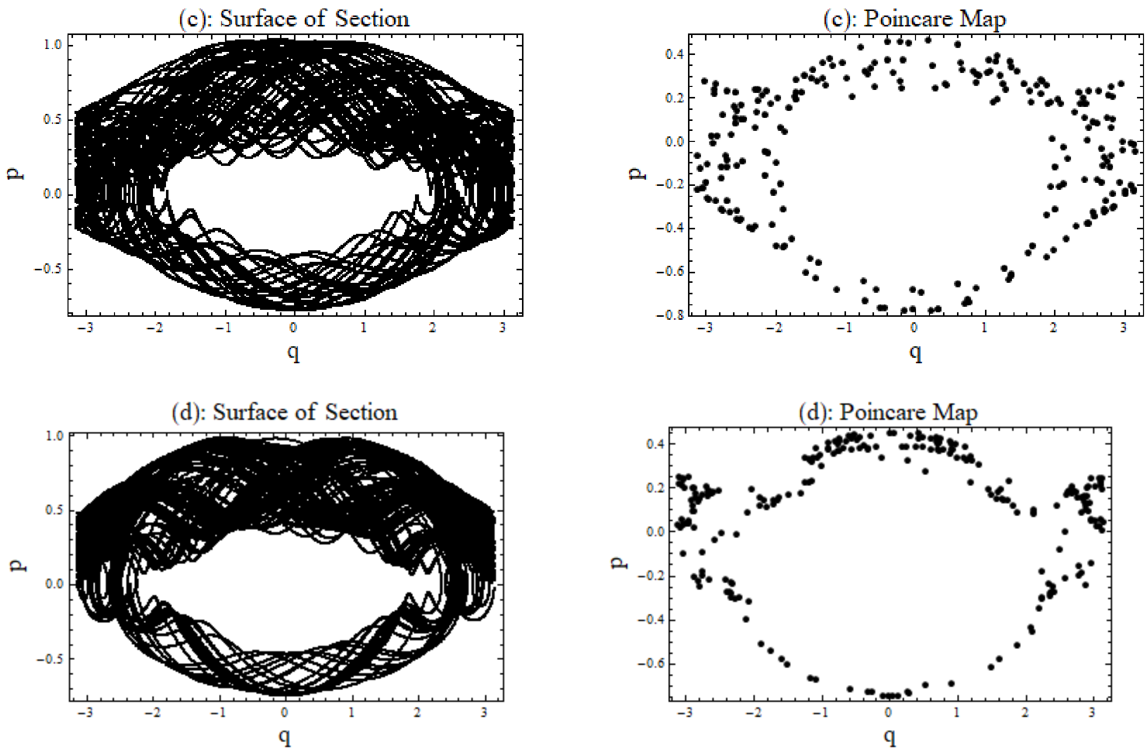


Figure 3: Surfaces of Section and Poincare Maps when $\epsilon = 0$, e = 0.05, $n^2 = 0.1$ and for different sets of initials initial conditions(q_0, p_0)= (0, 0), (0.5, 0.5), (2.8, 0.1) and (3.8, 0.05).

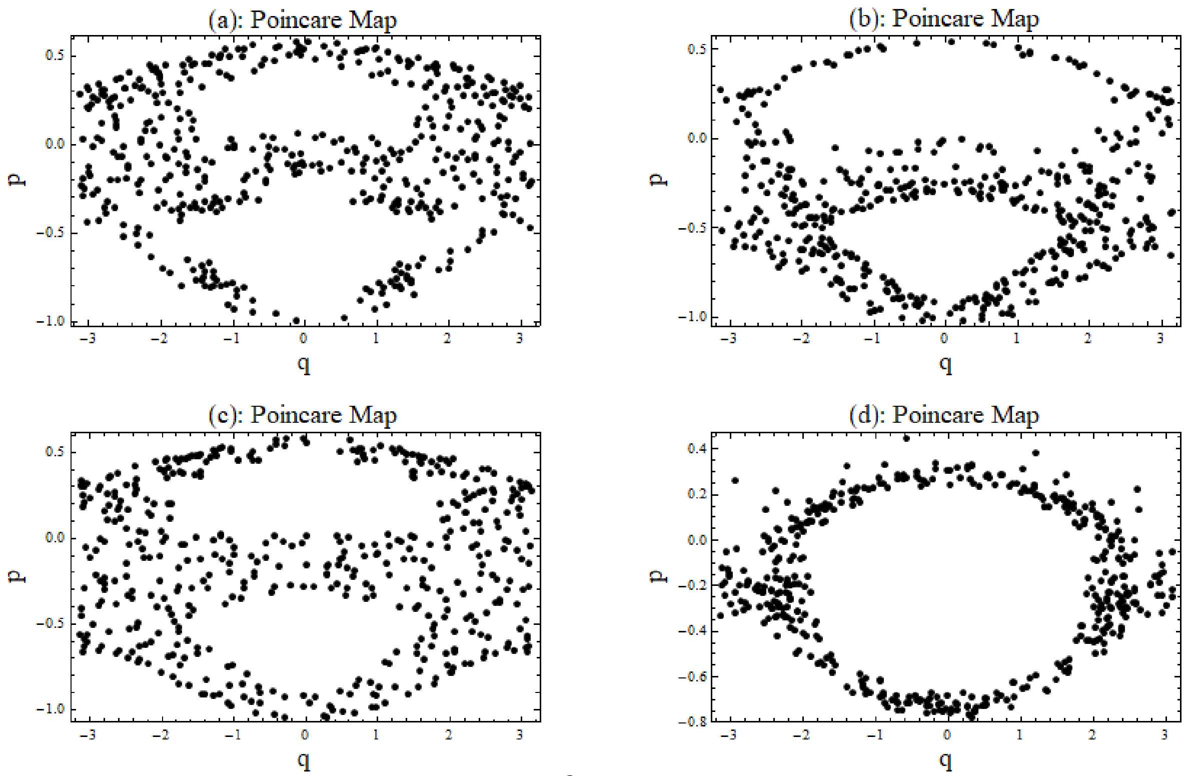


Figure 4: Poincare Maps for ϵ = 0, e = 0.15, n^2 = 0.1 and initial conditions (q_0, p_0) = (0, 0), (0.5, 0.5), (2.8, 0.1) and (3.8, 0.05).

(a) When solar radiation is not taken into account, i. e. when $\epsilon \neq 0$

In this case solar radiation pressure is actively influencing the motion of the satellite. With same initial conditions,(q_0, p_0) = (0, 0), (0.5, 0.5), (2.8, 0.1) and (3.8, 0.05), and forϵ = 0.1, e = 0.05, n^2 = 0.1. Surface of sections and Poincaré maps are drawn as shown in Figure 5. We find in case (a), again, Poincaré map is a dotted closed curve indication quasiperiodic regular motion. However, for other cases Poincaré maps are randomly distributed dots indication chaotic motion. One also finds various islands within the chaotic regime. These may correspond to the periodic windows that appears during bifurcations within chaos.
When the value of the eccentricity increased to 0.15, keeping radiation parameter and that of gravitation same radiation parameter, one finds chaotic oscillations in all cases, Figure 6.

In Figure 7, plots of Lyapunov exponents presented under different initial conditions. For some cases solar radiation pressure is nil and for some cases it is influencing the motion. These are indicated in each of the plots. Value of eccentricity, e, are also cited. The plots below the zero line stand for regularity and those abouve zero line stand for chaos.

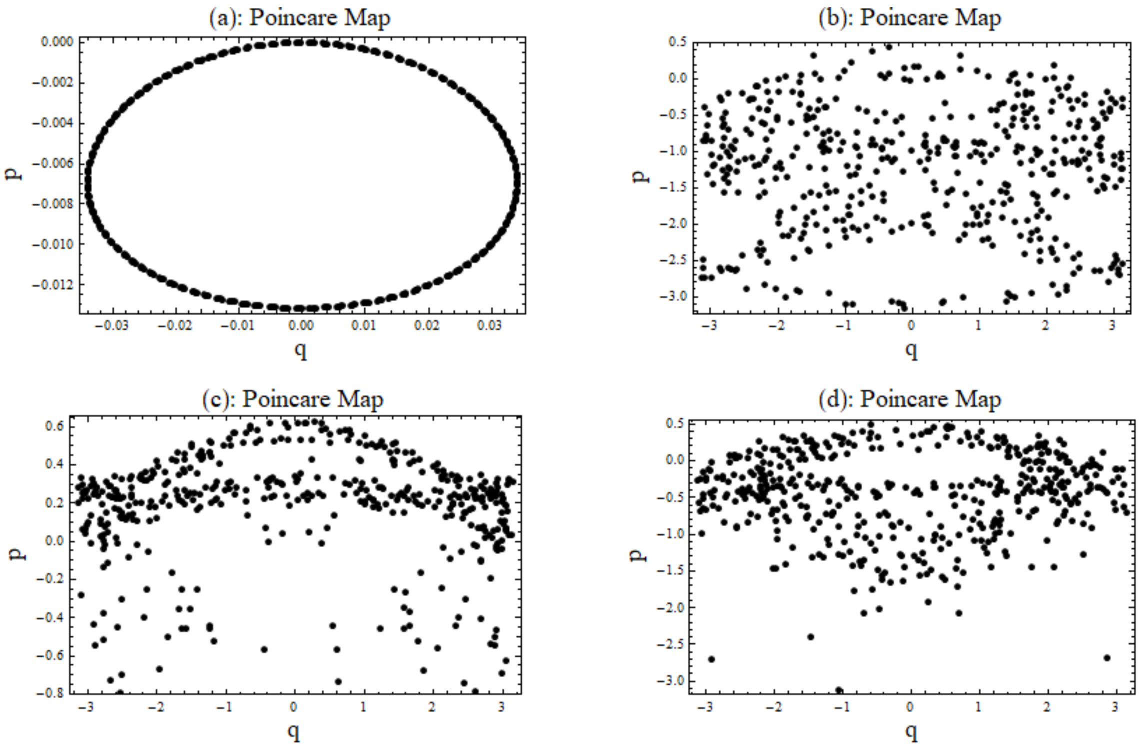


Figure 5: Poincare Maps for $\epsilon = 0.1$, e = 0.05, $n^2 = 0.1$ and initial conditions $(q_0, p_0) =$ (0, 0), (0.5, 0.5), (2.8, 0.1) and (3.8, 0.05).

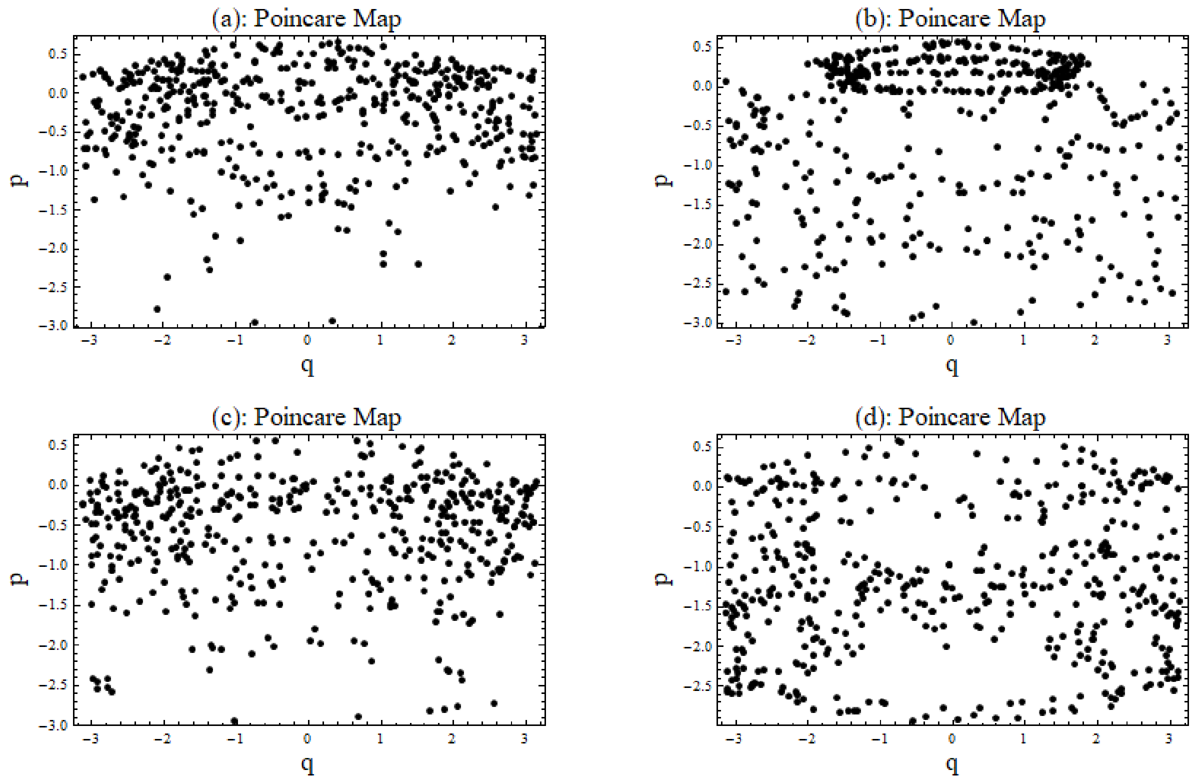


Figure 6: Poincare Maps for $\epsilon = 0.1$, e = 0.15, $n^2 = 0.1$ and initial conditions $(q_0, p_0) =$ (0, 0), (0.5, 0.5), (2.8, 0.1) and (3.8, 0.05).

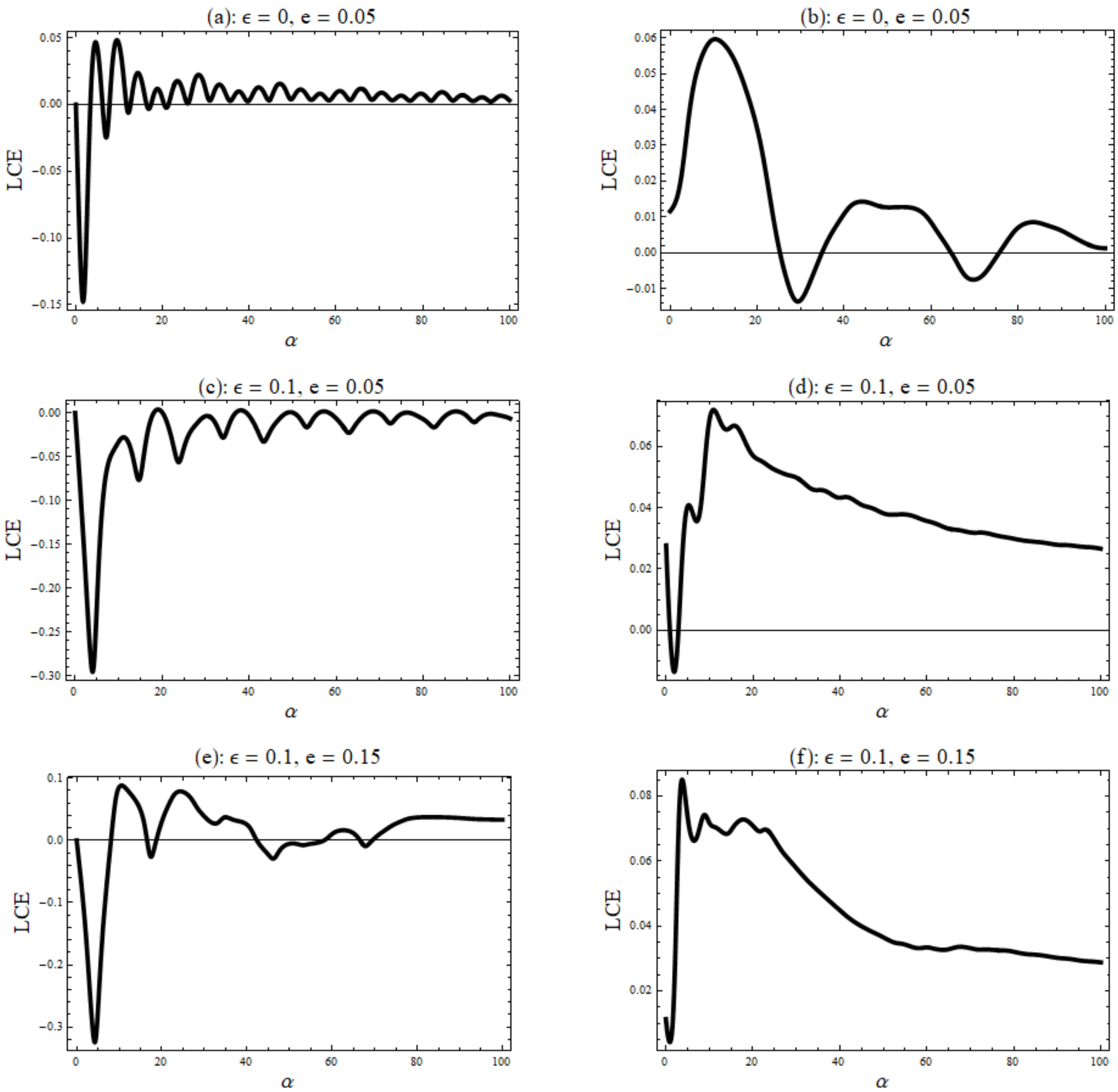


Figure 7: LCE plots when n^2= 0.1 and for (a) ϵ= 0.0, e = 0.05; (b) ϵ= 0.0, e = 0.15

4. **Conclusions**:

Influence of solar radiation and that of eccentricity orbits, both, are influencing the planar motion of the satellite to cause chaotic and regular oscillations of the satellite moving in elliptical orbit around the Earth. These facts explained through various plots. Higher values of the radiation parameterϵ, is more responsible to cause chaos.

Other forces, like tidal torque, atmospheric drag etc. also influence the satellite's motion which are not in consideration in this study. Such effects may be considered in future studies.

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 Ganita Sandesh Vol. 32 (June & December, 2018) pp. 48-55 ISSN : 0970-9169

Some finite Integrals involving the generalized Legendre's associated function, the generalized polynomials and the Aleph function

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ABSTRACT: The aim of this paper is to first evaluate a basic finite integral involving the products of the generalized Legendre's associated function $P_{\gamma}^{\alpha,\beta}$ and the Aleph function. Further we evaluate two more general integrals involving the products of $P_{\gamma}^{\alpha,\beta}$, the generalizedpolynomials $S_{n_1,\ldots,m_s}^{m_1,\ldots,m_s}[x_1\cdots x_s]$ and the Aleph function. All the evaluated integrals are believed to be new and reduce to a large number of simple integrals lying scattered in the literature. We mention here two special cases of the second integral which are also new and of interest by themselves.

Keywords: Generalized Legendre's associated function, Laguerre polynomials, generalized polynomials $S_{n_1,\ldots,m_s}^{m_1,\ldots,m_s}[x_1\cdots x_s]$, I-function,Alephfunction

Mathematics Subject Classification (2010):33C05,33C45,33C47,33C60,33C99

1. Introduction and Preliminaries

The Aleph function introduced by Saxena and Pogány [7, 8] will be define and represented in the following manner:

$$\aleph[z]=\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_j,A_j)]_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_j,B_j)]_{m+1,q_i}\end{matrix}\right.\right]:=\frac{1}{2\pi\omega}\int_L\Omega_{p_i,q_i,\tau_i;r}^{m,n}(\xi)\,z^{-\xi}d\xi\ldots$$

(1.1)

for all $z\neq 0$,where $\omega=\sqrt{-1}$ and

$$\Omega_{p_i,q_i,\tau_i;r}^{m,n}(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j-A_j\xi)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}\xi)}\quad\ldots$$

(1.2)

The integration path $L=L_{\omega\gamma\infty},\gamma\in\Re$ extends from $\gamma-\omega\infty$ to $\gamma+\omega\infty$, and is such that the poles of $\Gamma(1-a_j-A_j\xi),\ j=\overline{1,n}$ (the symbol $\overline{1,n}$ is used for $1,2,\cdots,n$) do not coincide with the poles of $\Gamma(b_j+B_j\xi),\ j=\overline{1,m}$. The parameters p_i, q_i are non-negative integers satisfying the condition $0\le n\le p_i, 1\le m\le q_i$, $\tau_i>0$ for $i=\overline{1,r}$. The parameters $A_j,B_j,A_{ji},B_{ji}>0$ and

$a_j, b_j, a_{ji}, b_{ji} \in C$. An empty product in (1.2) is interpreted as unity. The existence conditions for the defining integral (1.1) are given below

$$\varphi_\ell > 0, \quad |\arg(z)| < \frac{\pi}{2}\varphi_\ell, \quad \ell = \overline{1,r} \qquad \ldots (1.3)$$

$$\varphi_\ell \geq 0, \quad |\arg(z)| < \frac{\pi}{2}\varphi_\ell \text{ and } \Re\{\zeta_\ell\} + 1 < 0, \ldots (1.4)$$

where

$$\varphi_\ell = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_\ell \left(\sum_{j=n+1}^{p_1} A_{j\ell} + \sum_{j=m+1}^{q_1} B_{j\ell} \right), \qquad \ldots (1.5)$$

$$\zeta_\ell = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_\ell \left(\sum_{j=m+1}^{q_\ell} b_{j\ell} - \sum_{j=n+1}^{p_\ell} a_{j\ell} \right) + \frac{1}{2}(p_\ell - q_\ell), \quad \ell = \overline{1,r} \qquad \ldots (1.6)$$

Remark 1.1: For $\tau_i = 1, i = \overline{1,r}$ in (1.1), we get the $I-$function due to Saxena[6], defined in the following manner

$$I^{m,n}_{p_i,q_i;r}[z] = \aleph^{m,n}_{p_i,q_i,1;r}[z] = \aleph^{m,n}_{p_i,q_i,1;r}\left[z \left| \begin{matrix} (a_j, A_j)_{1,n}, \ldots, [1\,(a_j, A_j)]_{n+1,p_i} \\ (b_j, B_j)_{1,m}, \ldots, [1(b_j, B_j)]_{m+1,q_i} \end{matrix} \right. \right]$$

$$:= \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{p_i,q_i,1;r}(\xi) z^{-\xi} d\xi, \qquad \ldots (1.7)$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;r}(\xi)$ is given in (1.2). The existence conditions for the integral in (1.7) are the same as given in (1.3)-(1.6) with $\tau_i = 1, i = \overline{1,r}$.

Remark 1.2: If we put $r = 1$ in (1.7) then it reduces to the familiar H-function [4,12]

$$H^{m,n}_{p,q}[z] = \aleph^{m,n}_{p_i,q_i,1;1}[z] = \aleph^{m,n}_{p_i,q_i,1;1}\left[z \left| \begin{matrix} (a_p, A_p) \\ (b_p, B_p) \end{matrix} \right. \right] := \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{p_i,q_i,1;1}(\xi) z^{-\xi} d\xi, \quad \ldots (1.8)$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;1}(\xi)$ can be obtained from (1.2).

Also, the generalized polynomials $S^{m_1,\ldots,m_s}_{n_1,\ldots,m_s}[x_1 \cdots x_s]$ occurring here in will be defined and represented in the following form which differs slightly from that given by Srivastava [10,p.185,eqn.(7)]

$$S^{m_1,\ldots,m_s}_{n_1,\ldots,m_s}[x_1 \cdots x_s] = \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1}}{k_1!} \cdots \frac{(-n_s)_{m_s k_s}}{k_s!} A[n_1, k_1; \cdots; n_s, k_s] x_1^{k_1}, \ldots, x_s^{k_s}$$

...(1.9)

where $n_i = 0, 1, 2, \ldots;$ $m_i \neq 0$ $[i = 1, \ldots, s]$, m_i is an arbitrary positive integer and the coefficients $A[n_1, k_1; \cdots; n_s, k_s]$ are arbitrary constants ,real or complex.

If we take $s=1$in the equation (1.9) the generalized polynomials $S_{n_1,\ldots,m_s}^{m_1,\ldots,m_s}[x_1\cdots x_s]$ reduce to the well knowngeneral class of polynomials $S_n^m[x]$introduced by Srivastava [11,p.1, eqn (1)].

Finally, the generalized Legendre's associated function $P_\gamma^{\alpha,\beta}(x)$ [5,p.560,eqn.(3); 2,p.81,eqn.(1.1)]occurring in this paper will be define and represented as follows:

$$P_\gamma^{\alpha,\beta}(x)=\frac{(1+x)^{\beta/2}}{(1-x)^{\alpha/2}\ \Gamma(1-\alpha)}\ {}_2F_1\left[\gamma-\frac{\alpha-\beta}{2}+1,-\gamma-\frac{\alpha-\beta}{2};1-\alpha;\frac{1-x}{2}\right]\quad ...(1.10)$$

where β and γ are unrestricted and α is not a positive integer . Further details about this function including its particular cases can be found in the paper of Kuipers et al.[3].

2.Main Integrals

First integral

$$\int_o^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}P_\gamma^{\alpha,\beta}(x)\aleph[z\,x^u(1-x)^v]dx$$

$$=\sum_{t=0}^{\infty}\frac{(\gamma-\frac{\alpha-\beta}{2}+1)_t(-\gamma-\frac{\alpha-\beta}{2})_t}{2^t\,t!\ \Gamma(1-\alpha+t)}$$

$$\aleph_{p_i+2,q_i+1,\tau_i;r}^{m,n+2}\left[z\left|\begin{matrix}(1-\rho,u),(1-\sigma+\frac{\alpha}{2}-t,v),(a_j,A_j)_{1,n},\ldots,[\tau_j(a_j,A_j)]_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_j,B_j)]_{m+1,q_i},(1-\rho-\sigma+\frac{\alpha}{2}-t,u+v)\end{matrix}\right.\right]\quad\ldots(2.1)$$

The integral (2.1) is valid under the following conditions

(i) α is not a positive integer,

$u\geq 0, v\geq 0.$

(ii) $\mathrm{Re}(\rho)+u\max_{1\leq j\leq m}\left[\frac{-\mathrm{Re}(b_j)}{B_j}\right]>0$,

$\mathrm{Re}(\sigma-\frac{\alpha}{2})+v\max_{1\leq j\leq m}\left[\frac{-\mathrm{Re}(b_j)}{B_j}\right]>0$.

Second integral:

$$\int_o^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}P_\gamma^{\alpha,\beta}(x)S_{n_1,\ldots,m_s}^{m_1,\ldots,m_s}\begin{bmatrix} e_1 x^{u_1}(1-x)^{v_1} \\ \vdots \\ e_s x^{u_s}(1-x)^{v_s}\end{bmatrix}\aleph[z x^u(1-x)^v]dx$$

$$=\sum_{t=0}^{\infty}\sum_{k_1=0}^{\left[\frac{n_1}{m_1}\right]}\cdots\sum_{k_s=0}^{\left[\frac{n_s}{m_s}\right]}A[n_1,k_1;\cdots;n_s,k_s]\left(\prod_{j=1}^{s}\frac{(-n_j)_{m_jk_j}e_j^{k_j}}{k_j!}\right)\frac{(\gamma-\frac{\alpha-\beta}{2}+1)_t(-\gamma-\frac{\alpha-\beta}{2})_t}{2^t t!\,\Gamma(1-\alpha+t)}$$

$$\aleph_{p_i+2,q_i+1,\tau_i;r}^{m,n+2}\left[z\left|\begin{matrix}(1-\rho-u_1k_1-\cdots-u_sk_s,u),(1-\sigma+\frac{\alpha}{2}-t-v_1k_1-\cdots-v_sk_s,v),\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_j,B_j)]_{m+1,q_i},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_j,A_j)]_{n+1,p_i}\\(1-\rho-\sigma+\frac{\alpha}{2}-t-(u_1+v_1)k_1-\cdots-(u_s+v_s)k_s,u+v)\end{matrix}\right]$$

…(2.2)

The integral (2.2) is valid under the following conditions

(i) α is not a positive integer,

$u\ge 0, v\ge 0$; $u_j\ge 0, v_j\ge 0$, j=1,…,s.

(ii) $\text{Re}(\rho)+u\max_{1\le j\le m}\left[\frac{-\text{Re}(b_j)}{B_j}\right]>0$,

$\text{Re}(\sigma-\frac{\alpha}{2})+v\max_{1\le j\le m}\left[\frac{-\text{Re}(b_j)}{B_j}\right]>0$.

Third integral:

$$\int_{-1}^{+1}(1-x)^{\rho}(1+x)^{\sigma}P_{\gamma}^{\alpha,\beta}(x)S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\begin{bmatrix} e_1(1-x)^{u_1}(1+x)^{v_1} \\ \vdots \\ e_s(1-x)^{u_s}(1+x)^{v_s}\end{bmatrix}\aleph[z(1-x)^{u}(1+x)^{v}]dx$$

$$=\frac{2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}}{\Gamma(1-\alpha)}\sum_{t=0}^{\infty}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}2^{\sum_{j=1}^{s}(u_j+v_j)k_j}A[n_1,k_1;\cdots;n_s,k_s]$$

$$\left(\prod_{j=1}^{s}\frac{(-n_j)_{m_jk_j}e_j^{k_j}}{k_j!}\right)\frac{(\gamma-\frac{\alpha-\beta}{2}+1)_t(-\gamma-\frac{\alpha-\beta}{2})_t}{(1-\alpha)_t\ t!}$$

$$\aleph_{p_i+2,q_i+1,\tau_i;r}^{m,n+2}\left[z\left|\begin{matrix}(\frac{\alpha}{2}-\rho-t-u_1k_1-\cdots-u_sk_s,u),(-\sigma-\frac{\beta}{2}-v_1k_1-\cdots-v_sk_s,v),\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_j,B_j)]_{m+1,q_i},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_j,A_j)_{1,n},\ldots,[\tau_j(a_j,A_j)]_{n+1,p_i}\\(-\rho-\sigma-\frac{\beta-\alpha}{2}-t-1-(u_1+v_1)k_1-\cdots-(u_s+v_s)k_s,\ u+v)\end{matrix}\right]$$

...(2.3)

The integral (2.3) is valid under the following conditions

(i) α is not a positive integer,

$u \geq 0, v \geq 0$; $u_j \geq 0, v_j \geq 0$, j=1,…,s.

(ii) $\text{Re}(1+\rho-\frac{\alpha}{2})+u\max_{1\leq j\leq m}\left[\frac{-\text{Re}(b_j)}{B_j}\right]>0$,

$\text{Re}(1+\sigma+\frac{\beta}{2})+v\max_{1\leq j\leq m}\left[\frac{-\text{Re}(b_j)}{B_j}\right]>0$.

Proofs:To establish the integral (2.1), we first express the generalized Legendre's associated function occurring in its left hand side in terms of ${}_2F_1$ with the help of (1.10) and the Aleph function in terms of Mellin-Barnes contour integral by (1.1), Now we interchange the order of x and ξ integrals (which is permissible under the conditions stated with (2.1)) in the result thus obtained and get after a little simplification the left hand side of (2.1) (say Δ) as

$$\Delta = \frac{1}{\Gamma(1-\alpha)} \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\, z^{-\xi} \{ \int_o^1 x^{\rho-u\xi-1} (1-x)^{\sigma-\frac{\alpha}{2}-v\xi-1}$$

$${}_2F_1\left[\gamma - \frac{\alpha-\beta}{2}+1, -\gamma-\frac{\alpha-\beta}{2}; 1-\alpha; \frac{1-x}{2}\right] dx\} d\xi \qquad \ldots(2.4)$$

on evaluating the x-integral occurring on the right hand side of (2.4) with the help of a known result [9,p. 60,eqn.(2.16(ii))] and expressing the function ${}_3F_2$ so obtained in terms of series and interchanging the order of summations and integrations (which is permissible under the conditions stated with(2.1)),the equation (2.4)takes the following form after a little simplification

$$\Delta = \sum_{t=0}^{\infty} \frac{(\gamma - \frac{\alpha-\beta}{2}+1)_t (-\gamma-\frac{\alpha-\beta}{2})_t}{2^t\, t!\ \Gamma(1-\alpha+t)} \{ \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\, z^{-\xi} \frac{\Gamma(\rho-u\xi)\,\Gamma(\sigma-\frac{\alpha}{2}+t-v\xi)}{\Gamma(\rho+\sigma-\frac{\alpha}{2}+t-(u+v)\xi)} \} d\xi \ \ldots(2.5)$$

Finally, on reinterpreting the multiple Mellin-Barnes contour integral occurring in the right hand side of (2.5) in terms of the Aleph function, we easily arrive at the desired result (2.1).

To prove (2.2), we first express the generalized polynomials $S^{m_1,\ldots,m_s}_{n_1,\ldots,m_s}[x_1 \cdots x_s]$ occurring in the left hand side of (2.2) in series form with the help of (1.3) and then interchange the order of summations and integration (which is permissible under the conditions stated with (2.2)).Now on evaluating the integral so obtained with the help of the integral (2.1), we easily obtain the desired result (2.2).

To evaluate the integral (2.3), we make use of the following integral

$$\int_{-1}^{+1} (1-x)^{\rho} (1+x)^{\sigma} P^{\alpha,\beta}_{\gamma}(x)\ dx = \frac{2^{\rho+\sigma+\frac{\beta-\alpha}{2}+1}\, \Gamma(1+\rho-\frac{\alpha}{2})\,\Gamma(1+\sigma+\frac{\beta}{2})}{\Gamma(1-\alpha)\,\Gamma(2+\rho+\sigma+\frac{\beta-\alpha}{2})}$$

$${}_3F_2\left[\begin{matrix} \gamma-\frac{\alpha-\beta}{2}+1, -\gamma-\frac{\alpha-\beta}{2}, 1+\rho-\frac{\alpha}{2}; 1 \\ 1-\alpha,\ 2+\rho+\sigma+\frac{\beta-\alpha}{2}; \end{matrix}\right]$$

$\ldots(2.6)$

where α is not a positive integer, $\mathrm{Re}(1+\rho-\frac{\alpha}{2})>0$, $\mathrm{Re}(1+\sigma+\frac{\beta}{2})>0$ and proceed in a manner similar to that given earlier in proofs of (2.1)and (2.2).

3. Special Cases

[i] If we reduce $S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}$ occurring in(2.2) to Laguerre polynomials $L_{n_1}^{\theta}$ [1,p.999,eqn.(8.704); 13,p.159,eqn.(1.8)]. We arrive at the following integral after a little simplification

$$\int_o^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\alpha/2}\, P_\gamma^{\alpha}(x)\, L_{n_1}^{\theta}[e_1 x^{u_1}(1-x)^{v_1}]\,\aleph[z x^u (1-x)^v]dx$$

$$= \sum_{t=0}^{\infty}\sum_{k_1=0}^{[n_1]} \binom{n_1+\theta}{n_1} \frac{e_1^{k_1}(-n_1)_{k_1}(\gamma+1)_t(-\gamma)_t}{(\theta+1)_{k_1}\, k_1!\; 2^t t!\,\Gamma(1-\alpha+t)}$$

$$\aleph_{p_i+2,q_i+1,\tau_i;r}^{m,n+2}\left[z\left|\begin{matrix}(1-\rho-u_1k_1,u),(1-\sigma+\frac{\alpha}{2}-t-v_1k_1,v),(a_j,A_j)_{1,n},\ldots,[\tau_j(a_j,A_j)]_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots,[\tau_j(b_j,B_j)]_{m+1,q_i},(1-\rho-\sigma+\frac{\alpha}{2}-t-(u_1+v_1)k_1,u+v)\end{matrix}\right.\right]$$

…(3.1)

The conditions of existence of (3.1) can be easily obtained with the help of the conditions stated with (2.2).

(ii) If we put $\tau_i = 1, i=\overline{1,r}$ in (2.2), we arrive at the following result in the term of I-function [6]

$$\int_o^1 x^{\rho-1}(1-x)^{\sigma-1}(1+x)^{-\beta/2}\, P_\gamma^{\alpha,\beta}(x)\, S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\begin{bmatrix}e_1 x^{u_1}(1-x)^{v_1}\\ \vdots \\ e_s x^{u_s}(1-x)^{v_s}\end{bmatrix} I_{p_i,q_i;r}^{m,n}[z x^u(1-x)^v]dx$$

$$= \sum_{t=0}^{\infty}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]} A[n_1,k_1;\cdots;n_s,k_s]\left(\prod_{j=1}^{s}\frac{(-n_j)_{m_jk_j}\, e_j^{k_j}}{k_j!}\right)\frac{(\gamma-\frac{\alpha-\beta}{2}+1)_t(-\gamma-\frac{\alpha-\beta}{2})_t}{2^t t!\;\Gamma(1-\alpha+t)}$$

$$I_{p_i+2,q_i+1;r}^{m,n+2}\left[z\left|\begin{matrix}(1-\rho-u_1k_1-\cdots-u_sk_s,u),(1-\sigma+\frac{\alpha}{2}-t-v_1k_1-\cdots-v_sk_s,v),(a_j,A_j)_{1,n},\ldots,[(a_j,A_j)]_{n+1,p_i}\\(b_j,B_j)_{1,m},\ldots,[(b_j,B_j)]_{m+1,q_i},(1-\rho-\sigma+\frac{\alpha}{2}-t-(u_1+v_1)k_1-\cdots-(u_s+v_s)k_s,u+v)\end{matrix}\right.\right]$$

…(3.2)

4. Conclusion

I emphasized on some finite integrals involving Legendre's associated function,the Aleph function and/or the generalized polynomial in terms of Aleph function. Special cases are also discussed in the paper in which the definite integrals reduce to Laguerre polynomials and I-function respectively.

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Vol. 32 (June & December) 2018

RAJASTHAN GANITA PARISHAD

राजस्थान गणित परिषद

The Sequence

M. M. Eltikali, and T. M. Elfrgani	A Note on Weak Bailey Lemma Related to Basic Hypergeometric Series	1-10
J. Bulnes, J. López-Bonilla and S. Vidal-Beltrán	Stirling and Harmonic numbers	11-24
M. A. Pathan Hemant Kumar and Roshni Sharm	On Analytic Continuation of the Appell Type Double Hurwitz – Lerch Zeta Functions	25-30
Bharti Chauhan, Prakriti Rai and Rupakshi Mishra Pandey	Some Properties of Extended τ-Gauss Hypergeometric Function and Its Transformations	31-39
Tilprasad Sarma and L. M. Saha	Regular and Chaotic Motion of a Satellite Influenced by Solar Radiation Pressure	40-47
R P Sharma	Some finite Integrals involving the generalized Legendre's associated function, the generalized polynomials and the Aleph function	48-55