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Study of Bianchi type-V Cosmological Models in Self-Creation Theory of Gravitation with Lyra Geometry

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Abstract: In this paper we consider the Bianchi type-V cosmological model in Lyra geometry and self-creation theory of gravitation. We have considered Barotropic perfect fluid. Some physical and geometrical properties are also discussed.

1.1 Introduction

In general theory of relativity, Einstein described gravitation in terms of geometry and it motivated him to geometrize the other physical fields also. Weyl [41] made one of the best attempts in this direction. He introduced a generalization of Riemannian geometry in an attempt to unify gravitation and electromagnetism. Weyl's theory was not taken seriously because it was based on the non- integrability of length transfer. Later Lyra [14] suggested a modification of Riemannian geometry which has a close resemblance to Weyl's geometry, the connection is metric preserving as in Riemannian geometry and length transfer are integrable. Lyra introduced a gauge function which removed the non-integrability condition of the length of a vector under parallel transport. Thus Riemannian geometry was modified by Lyra and was given new name Lyra's geometry.

Soleng [38] has investigated that constant gauge function α in Lyra's geometry either included a creation field or is equal to Hoyle [9] creation field cosmology or contains a special vacuum field which with the gauge vector term, may be considered as a cosmological term. Beesham [5] investigated FRW cosmological models in Lyra's manifold with time dependent displacement vector field, the model so obtained, solve the singularity, entropy and horizon problems which exists in the standard models of cosmology based on Riemannian geometry. Singh and Singh[33, 34] have investigated Bianchi type I, III and Kantowski-Sachs cosmological models with time dependent displacement field and have made a comparative study of R-W models with constant deceleration parameter in Lyra's geometry. Singh and Singh [35, 36] have also investigated Bianchi type I, V and VI_0 cosmological models in Lyra geometry. Bhowmik and Rajput [6] have investigated anisotropic Bianchi type I cosmological models based on Lyra geometry considering deceleration parameter constant and time dependent. Pradhan et al. [19-22] and Rahaman et al. [24 -26] in a series of papers have investigated cosmological models based on Lyra geometry with constant and time dependent displacement field in different contexts. Mohanty et al. [16] have investigated a five dimensional model within the framework of Lyra geometry. Recently Bali and Chandnani [1] have investigated Bianchi type I cosmological models with time dependent gauge function for perfect fluid distribution within the framework of Lyra geometry. Pradhan et al. studied various aspects of cosmological models in light of Lyra geometry [19-22].

Since Einstein himself pointed out that his general theory of gravitation does not account satisfactorily for the inertial properties of matter. i.e. Mach's principle is not substantiated by general relativity, there has been some interesting attempts to generalize the general theory of gravitation by incorporating Mach's principle and other desired features which were lacking in the original theory and hence Barber [3] modified it by coupling the scalar field with the energy momentum tensor so that the Mach's principle is substantially accommodate by the theory. Barber's theory is a variable G-theory and predicts local effects, which are within the observational limits. In it, the Newtonian gravitational parameter G is not a constant but a function of time parameter t. Also the scalar field $\phi(t)$ does not gravitate directly but divides the matter tensor acting as a reciprocal gravitational constant, which is not the case in Einstein theory of gravitation. This theory is capable of verification or falsification. It can be done by observing the behaviour of both bodies of degenerate matter and photons. Pimental [17] and Soleng [37] have presented the Robertson Walker solutions in self-creation theory of gravitation by using power law relation between the expansion factor of the universe and the scalar field.

Reddy and Venkateswarlu [29] have obtained spatially homogeneous and anisotropic Bianchi Type-VI_0 cosmological models in Barber's self-creation theory of gravitation in both vacuum as well as in presence of perfect fluid with pressure equal to energy density. Venkateswarlu and Reddy [40] have also got spatially homogeneous and anisotropic Bianchi Type-I cosmological macro models when the source of gravitational field is a perfect fluid. Bianchi Type-II and III models in self-creation cosmology have been deduced by Shanti and Rao [32]. Rao and Sanyasiraju [27] have discussed Bianchi Type VIII and IX in zero mass scalar fields and self-creation cosmology. S. Ram and Singh [31] studied Bianchi Type-I cosmological models with variable G and Λ. Pradhan and Vishwakarma [23] studied LRS Bianchi Type-I cosmological models in self-creation theory of gravitation. Bali and Upadhaya [2] have presented Bianchi Type-I string dust magnetized cosmological models. Pradhan and Pandey [18] have studied bulk viscous cosmological models in Barber's second self-creation theory. Reddy et al. [30] discussed some vacuum models in self-creation theory.Venkateshwarlu et al. [39] also discussed string cosmological models in self-creation theory. Katore et al. [13] studied plane symmetric cosmological models with negative constant deceleration parameter in self-creation theory. Jain et al. [10] presented Bianchi type-I cosmological model with a varying Λ term in self-creation theory of gravitation.Rao and Vinutha [28] deduced some results on exact Bianchi type II,VIII and IX string models and plane symmetric string models in self-creation theory. Barber [4] again reviewed his theory of self-creation. Recently Jain and Yadav [11] studied LRS Bianchi type I radiation model with a varying Λ term in self-creation theory. Very recently Borkar and Ashtankar [7] studied Bianchi type-I bulk viscous barotropic cosmological models in self-creation theory. Jaiswal and Tiwari [12] explored some of the features of shear free Bianchi-V string models in self-creation.

1.2 Metric and Field Equations

We consider Bianchi V metric in the form

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 e^{2x} dy^2 + C^2 e^{2x} dz^2 \quad (1.2.1)$$

Where A, B, C are functions of 't' alone.

Energy momentum tensor T_i^j for perfect fluid distribution is given by

$$T_i^j = (\rho + p) v_i v^j + p g_i^j \quad (1.2.2)$$

where $v_i = (0,0,0,-1)$, $v_i v^j = -1$, $\alpha_i = (0,0,0,\beta(t))$; $v_4 = -1$; $v^4 = 1$

p is isotropic pressure, ρ the matter density, v^i is fluid flow vector and β the gauge function. Einstein's field equations in normal gauge for Lyra's manifold in self-creation theory of gravitation are given by

$$R_i^j - \frac{1}{2} R g_i^j + \frac{3}{2} \alpha_i \alpha^j - \frac{3}{4} \alpha_k \alpha^k g_i^j = -\frac{8\pi G}{C^4} \varphi^{-1} T_i^j \quad (1.2.3)$$

and $\phi^k_{;k} = \dfrac{8\pi}{3} \eta T \quad (1.2.4)$

where $\phi_{;k}^{k}$ is invariant D'Alembertian and the contracted tensor T is the trace of energy momentum tensor describing all non-gravitational and non-scalar field matter and density.Here η is coupling constant to be determined from experiments, because of homogeneity condition imposed by the metric the scalar field ϕ will be function of 't' only.

For the above line element (1.2.1), theEinstein's field equations (1.2.3) and (1.2.4) are given by

$$\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{B_4C_4}{BC}-\frac{1}{A^2}+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi} \quad (1.2.5)$$

$$\frac{A_{44}}{A}+\frac{C_{44}}{C}+\frac{A_4C_4}{AC}-\frac{1}{A^2}+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi} \quad (1.2.6)$$

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}-\frac{1}{A^2}+\frac{3}{4}\beta^2=-\frac{8\pi p}{\phi} \quad (1.2.7)$$

$$\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{A_4C_4}{AC}-\frac{3}{A^2}-\frac{3}{4}\beta^2=\frac{8\pi\rho}{\phi} \quad (1.2.8)$$

$$\frac{2A_4}{A}-\frac{B_4}{B}-\frac{C_4}{C}=0 \quad (1.2.9)$$

and $$\phi_{44}+\phi_4\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)=\frac{8\pi}{3}\eta(\rho-3p) \quad (1.2.10)$$

Here we have used units in which G =1and C=1.

The energy conservation equation $T_{i;j}^{j}=0$ leads to

$$\rho_4+(\rho+p)\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)=0 \quad (1.2.11)$$

and the conservation of LHS of equation (1.2.3) leads to

$$\frac{3}{2}\beta\beta_4+\frac{3}{2}\beta^2\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)=0 \quad (1.2.12)$$

1.3 Solution of Field Equations

From equations(1.2.5) and (1.2.6) we have

$$\frac{B_{44}}{B}+\frac{B_4C_4}{BC}=\frac{A_{44}}{A}+\frac{A_4C_4}{AC} \quad (1.3.1)$$

From equations (1.2.6) and (1.2.7) we have

$$\frac{C_{44}}{C}+\frac{A_4C_4}{AC}=\frac{B_{44}}{B}+\frac{A_4B_4}{AB} \quad (1.3.2)$$

Equation (1.2.9) leads to

$$A=l\left(BC\right)^{1/2} \quad (1.3.3)$$

where l is the constant of integration.

Multiplying equation (1.2.8) by γ and adding to equation(1.2.7), we get

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\left(1+\gamma\right)\frac{A_4B_4}{AB}-\left(1+3\gamma\right)\frac{1}{A^2}+\frac{3\left(1-\gamma\right)}{4}\beta^2+\gamma\frac{B_4C_4}{BC}+\gamma\frac{A_4C_4}{AC}=\gamma\rho-p \quad (1.3.4)$$

Applying barotropic condition $p=\gamma\rho$, equation (1.3.4) leads to

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\left(1+\gamma\right)\frac{A_4B_4}{AB}-\left(3\gamma+1\right)\frac{1}{A^2}+\frac{3\left(1-\gamma\right)}{4}\beta^2+\gamma\frac{B_4C_4}{BC}+\gamma\frac{A_4C_4}{AC}=0 \quad (1.3.5)$$

With the help of equations (1.2.12) and (1.3.3) we get

$$\beta=0 \quad (1.3.6)$$

or

$$\beta=\frac{k_1}{\left(BC\right)^{3/2}} \quad (1.3.7)$$

where k_1 is constant of integration.

Using equations (1.3.3)and (1.3.7) in equation(1.3.5) we get

$$\frac{3}{2}\frac{B_{44}}{B}+\frac{1}{2}\frac{C_{44}}{C}+\frac{B_4C_4}{BC}\left[\frac{1}{2}+\frac{\left(1+\gamma\right)}{2}+\frac{3\gamma}{2}\right]+\frac{B_4^2}{B^2}\left[\frac{1}{4}+\frac{\gamma}{2}\right]+ \frac{C_4^2}{C^2}\left[-\frac{1}{4}+\frac{\gamma}{2}\right]-\frac{\left(3\gamma+1\right)}{l^2BC}+\frac{3\left(1-\gamma\right)}{4}\frac{W^2}{\left(BC\right)^3}=0 \quad (1.3.8)$$

Let

$$BC = \mu \ , \frac{B}{C} = \upsilon \quad (1.3.9)$$

Then

$$B^2 = \mu\upsilon \text{ and } C^2 = \frac{\mu}{\upsilon} \qquad (1.3.10)$$

Using equations(1.3.9) and (1.3.10) in (1.3.8) we get

$$\frac{\mu_{44}}{\mu} + \frac{1}{2}\frac{\upsilon_{44}}{\upsilon} + \frac{(3\gamma-1)}{4}\frac{\mu_4^2}{\mu^2} + \frac{3}{4}\frac{\mu_4\upsilon_4}{\mu\upsilon} - \left(\frac{\gamma+1}{4}\right)\frac{\gamma_4^2}{\upsilon^2} - \frac{(3\gamma+1)}{l^2}\frac{1}{\mu} - \frac{3}{4}\frac{(\gamma-1)}{\mu^3}k_1^{\,2} = 0 \quad (1.3.11)$$

Combining equations (1.2.9) and (1.3.2) and using equations (1.3.9) and (1.3.10) we get

$$\frac{\upsilon_4}{\upsilon} = \frac{k_2}{\mu^{3/2}} \quad (1.3.12)$$

where k_2 is constant of integration.

Using equation (1.3.12) in equation (1.3.11)we get

$$2\mu_{44} + \left(\frac{3\gamma-1}{2}\right)\frac{\mu_4^2}{\mu} = \frac{(\gamma-1)}{2}\left(3k_1^{\,2} + k_2^{\,2}\right)\frac{1}{\mu^2} + \frac{2}{l^2}(3\gamma+1) \quad (1.3.13)$$

We put

$$\mu_4 = f(\mu) \quad (1.3.14)$$

Then equation (1.3.13) reduces to

$$\frac{d}{d\mu}\left(f^2\right) + \frac{\alpha}{\mu}f^2 = \frac{b}{\mu^2} + k_3 \quad (1.3.15)$$

where $\alpha = \frac{(3\gamma-1)}{2}, \ b = \frac{\gamma-1}{2}(3k_1^{\,2} + k_2^{\,2}) \ \textit{and} \ \ k_3 = \frac{2}{l^2}(3\gamma+1)$

Which on integration leads to

$$f=\left[\left(\frac{3k_1^2+k_2^2}{3}\right)\frac{1}{\mu}+\frac{4\mu}{l^2}+\frac{S}{\mu^\alpha}\right]^{1/2} \quad (1.3.16)$$

where S is constant of integration.

Integrating equation (1.3.12) we have

$$\ln \upsilon=\int \frac{L}{\mu^{3/2}}\frac{1}{\sqrt{\left(\frac{3k_1^2+k_2^2}{3}\right)\frac{1}{\mu}+\frac{4}{l^2}\mu+\frac{S}{\mu^\alpha}}}d\mu \quad (1.3.17)$$

Hence the metric (1.2.1) reduces to

$$ds^2=-\frac{1}{\left[\left(\frac{3k_1^2+k_2^2}{3}\right)\frac{1}{\mu}+\frac{4}{l^2}\mu+\frac{S}{\mu^\alpha}\right]}d\mu^2+l^2\mu dx^2+\mu\upsilon e^{2x}dy^2+\frac{\mu}{\upsilon}e^{2x}dz^2 \quad (1.3.18)$$

After using suitable transformations the metric (1.3.18) reduces to

$$dS^2=-\frac{1}{\left[\left(\frac{3k_1^2+k_2^2}{3}\right)\frac{1}{T}+\frac{4}{l^2}T+\frac{S}{T^\alpha}\right]}dT^2+TdX^2+T\upsilon e^{2x}dY^2+\frac{T}{\upsilon}e^{2x}dZ^2 \quad (1.3.19)$$

Where $lx=X, y=Y, z=Z, \mu=T, s=S$ and υ is given by equation (1.3.17).

The cosmic time t is given by

$$t=\int\frac{dT}{\sqrt{\left(\frac{3k_1^2+k_2^2}{3}\right)\frac{L}{T}+\frac{4T}{l^2}+\frac{S}{T^\alpha}}} \quad (1.3.20)$$

1.4 Some Physical and Geometrical Features

The gauge function (β) is given as

$$\beta=\frac{k_1}{T^{3/2}} \quad (1.4.1)$$

For the model (1.3.19) expansion (θ) is calculated as

$$\theta = \frac{3}{2T^{1+\frac{\alpha}{2}}}\left[\left(\frac{3k_1^{\,2}+k_2^{\,2}}{3}\right)T^{\alpha-1}+\frac{4}{l^2}T^{\alpha+1}+S\right]^{1/2} \quad (1.4.2)$$

The components of shear tensor $\left(\sigma_i^j\right)$ are given by

$$\sigma_1^1 = \frac{1}{3}\left(\frac{2A_4}{A}-\frac{B_4}{B}-\frac{C_4}{C}\right)=0 \quad ,(1.4.3)$$

$$\sigma_2^2 = \frac{1}{3}\left(\frac{2B_4}{B}-\frac{A_4}{A}-\frac{C_4}{C}\right)=\frac{k_2}{2T^{3/2}} \quad ,(1.4.4)$$

$$\sigma_3^3 = \frac{1}{3}\left(\frac{2C_4}{C}-\frac{A_4}{A}-\frac{B_4}{B}\right)=-\frac{k_2}{2T^{3/2}} \quad ,(1.4.5)$$

$$\sigma_4^4 = 0 \quad ,(1.4.6)$$

Hence the shear (σ) is given by

$$\sigma = \frac{k_2}{2T^{3/2}} \quad .(1.4.7)$$

The energy density (ρ) is calculated as

$$\rho = \frac{k_4}{T^{3/2(1+\gamma)}} \quad (1.4.8)$$

Isotropic pressure (p) is evaluated as

$$p = \frac{\gamma k_4}{T^{3/2(1+\gamma)}} \quad (1.4.9)$$

The energy conditions given by Hawking and Ellis [8]

(i) $\rho + p > 0$ (ii) $\rho + 3p > 0$ (1.4.10)

are satisfied if $k_4 > 0$.

Also $$\frac{\sigma}{\theta} = \frac{k_2 T^{1-\alpha/2}}{6}\left[\left(\frac{3k_1^2+k_2^2}{3}\right)T^{\alpha-1}+\frac{4}{l^2}T^{\alpha+1}+S\right]^{-1/2} \quad (1.4.11)$$

Hubble parameter (H) is calculated as

$$H = \frac{1}{2T^{1+\frac{\alpha}{2}}}\left[\left(\frac{3k_1^{\,2}+k_2^{\,2}}{3}\right)T^{\alpha-1}+\frac{4}{l^2}T^{\alpha+1}+S\right]^{1/2} \qquad (1.4.12)$$

The scale factor (R) is given as

$$R^3 = lT^{3/2}e^{2x} \quad (1.4.13)$$

The deceleration parameter (q) is calculated as

$$q = \frac{\frac{2}{3T}\left(3k_1^{\,2}+k_2^{\,2}\right)+\left(\frac{3\gamma+1}{2}\right)\frac{S}{T^{\alpha}}}{\left(\frac{3k_1^{\,2}+k_2^{\,2}}{3}\right)\frac{1}{T}+\frac{4T}{l^2}+\frac{S}{T^{\alpha}}} \quad (1.4.14)$$

From equation (1.2.10), the scalar field (ϕ) is calculated as

$$\phi = \int \frac{k_3}{\left[\left(1-\frac{3}{2}\gamma\right)\mu^{\frac{1+3\gamma}{2}}+\frac{c}{\mu^{3/2}}\right]\left[\left(\frac{3k_1^{\,2}+k_2^{\,2}}{3}\right)\frac{1}{\mu}+\frac{4\mu}{l^2}+\frac{S}{\mu^{\alpha}}\right]^{1/2}}d\mu \quad (1.4.15)$$

1.5 Conclusion

In this paper we have presented Bianchi type-V barotropic perfect fluid cosmological model in Lyra geometry in the framework of self –creation theory of gravitation. The model (1.3.19) starts with a Big-Bang at T=0 and the expansion in the model decreases as the time increases for $\alpha > -2$. The energy conditions given by Hawking and Ellis [8] $\rho + p > 0$, $\rho + 3p > 0$ are satisfied if $k_4 > 0$. The matter density $\rho \to \infty$ when $T \to 0$ and $\rho \to 0$ when $T \to \infty$ provided $\gamma > -1$. The spatial volume (R^3) increases as time (T) increases. The deceleration parameter $q < 0$ if $T > \left[\frac{-3S(3\gamma+1)}{4(3k_1^2+k_2^2)}\right]^{\frac{1}{\alpha-1}}$, hence the model (1.3.19) is in decelerating phase and $q > 0$

if $T < \left[\frac{-3S(3\gamma+1)}{4(3k_1^2+k_2^2)}\right]^{\frac{1}{\alpha-1}}$ then the model (1.3.19) is in accelerating phase. Since $\frac{\sigma}{\theta} \neq 0$ hence anisotropy is maintained throughout. The model (1.3.19) has POINT TYPE singularity (MacCallum [15]) at $T = 0$. Shear tensor (σ) increases as time (T) decreases and σ decreases as T increases.

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Assessment of Turmeric Growers in Udaipur district of Rajasthan

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ABSTRACT

The concept of statistics and research methods remain an important area of advancing the developing and adoption of improved technologies in agriculture. The statistical analysis gives meaning and meaningless numbers thereby breathing life into lifeless data. Keeping it in views, the present investigation was undertaken in Jhadol panchayat samiti of Udaipur district of Rajasthan. Statistical tools and test like mean score, mean percent score, rank correlation, standard deviation and Z test was applied to assess adoption gap and training needs among turmeric growers. It was reported maximum adoption gap in the use of improved varieties of turmeric crop. Besides, plant protection measures were perceived as an important area of training by both tribal and non- tribal category of respondents on top priority.

KEY WORDS- Mean Score, Mean percent score, Z Test, Rank Correlation, standard deviation, Adoption gap, training need

INTRODUCTION

The concept of statistics and research methods remain an important area of advancing the developing and adoption of improved technologies in agriculture. Statistics is a branch of science that deals with collection, organization, analyzing of data and drawing of inferences from the samples to the whole population. This requires a proper design of study, an appropriate selection of study sample and choice of a suitable statistical test. Statistical methods involved in carrying out a study include planning, designing, collecting data, analyzing, drawing meaningful interpretation and reporting of research findings. The statistical analysis gives meaning and meaningless numbers thereby breathing life into lifeless data. The results and inferences are precise only if proper statistical tests are used.

In the state of Rajasthan Udaipur, Bundi and Bhilwara are major turmeric producing districts. Turmeric crop plays an important role in the economy of tribal and non-tribal farmers in the district. The adoption of turmeric production technology at the field level largely depends upon the training programs organized by the training institution for benefit of client system. There are written evidences that intensive training programs have contributed significantly in the adoption of turmeric production technology. To bridge the adoption gap, knowledge of areas where gap lies and perceived training needs of farmers are of paramount importance.

Thus, with this point in view, the present investigation was undertaken with the following specific objectives:

1. To find out the adoption gap pertaining to various package of practices of turmeric cultivation in the study area by using mean percent score.
2. To assess the training needs of turmeric growers with regard to turmeric cultivation through application of statistical tools like mean score, rank correlation.

METHODOLOGY:

The present investigation was under taken in Jhadol panchayat samiti of Udaipur district of Rajasthan. Selection of panchayat samiti was done considering the maximum production of turmeric among all the panchayat samities of the district. Further, three village panchayats having maximum area under turmeric crop and six villages two each from identified were selected for the study purpose. The criterion in the selection of villages was maximum area covered under turmeric crop in the villages. To select the sample of turmeric growers, 20 respondents i.e. 10 from tribal and 10 from non-tribal community of identified villagers were taken on a random basis. Thus, the total sample consisted of 120 respondents, out of which 60 were tribal and 60 were non tribal. The data were collected through a well-structured interview schedule by applying a personal interview technique. Statistical tests like mean score, mean percent score, rank correlation was applied for analysis of data.

RESULT AND DISCUSSION

The turmeric growers both tribal and no-tribal were grouped into three categories based on mean and standard deviation i.e. low, medium and high.

Table-1 Distribution of respondents on the basis of their extent of adoption of turmeric production technology:

S.No.	Adoption level	NT		T		Total	
		F	%	F	%	F	%
1.	Low (85)	0	0.0	24	40.00	24	20.00
2.	Medium (86-92)	31	51.66	36	60.00	67	55.83
3.	High	29	48.33	0	0.00	29	24.17
		60	100	60	100	120	100

F-Frequency

The data in table-1 shows that 67 (55.83%) of total turmeric production were found to be from medium adoption level group whereas 29 (24.17%) respondents were reported from the group of high adoption level and only 24 (20%) respondents should be placed in low adoption level group i.e. poor adoption. While analyzing the case of tribal respondents regarding their level of adoption an improved turmeric production technology. It was alarming to note that none of respondents was found with high level of adoption. On the other side the frequency of non-tribal respondents with high level of adoption of turmeric production technology was reported to be quite encouraging i.e. 29 (49.33%).

A close observation of the data in the table reveals that majority of respondents i.e. 51.66% percent in case of non-tribal and 60 per cent in case of tribal respondents were found with the medium level of adoption. It is further interesting to record that none of the respondents from non-tribal category of respondents was reported with low level of adoption. To find out the extent of adoption, mean per cent score of each major practice was calculated. Thereafter, the adoption gap was reported under each major practice of turmeric cultivation.

Table-2: Adoption gap with respect to improved turmeric cultivation practice among the turmeric growers.

S.No.	Improved practices	NT MPS	T MPS	Total MPS	Adoption%	'Z' Value
1	Improved varieties	33.33	33.33	33.33	66.67	0

2	Soil & Soil preparation	83.61	78.98	87.29	18.71	6.39**
3	Seed and sowing	72.31	69.72	71.01	28.99	3.45*
4	Use of manures and fertilizers	67.77	55.00	61.38	38.62	9.83*
5	Weed control and mulching	78.14	77.77	77.95	22.05	0.93NS
6	Irrigation scheduling	100	100	100	0.00	0.00
7	Plant protection measures	36.45	33.33	34.89	65.11	6.64*
8	Harvesting &curing	91.96	91.96	91.96	8.04	1.20NS
9	Marketing &storage of the seeds	58.88	52.50	55.69	44.31	7.87*

Overall,

11.576*

- Significant at 1% level
- MPS – Mean Per Cent Score

The data in table 2 indicate that maximum adoption gap was reported in the use of improved varieties (66.67%). This was followed by plant protection measures (65.11%), marketing and storage of seed (44.31%), use of manures and fertilizers (38.68%), seed and sowing (28.99%) and weed control and mulching (92.05%). The data further indicate that minimum adoption gap was found in harvesting and curing (8.04%) and soil and soil and preparation (18.71%). It is interesting to note that there was no adoption gap in case of irrigation scheduling among the respondents of both the categories. A close observation of the data indicates in general the adoption gap was higher among tribal respondents in all the major areas accept irrigation scheduling and harvesting and curing of turmeric. It can be seen from the table that the overall calculated 'z' value was found to be greater than the tabulated value at 1 per cent level of significance for all the major areas of turmeric cultivation practices., Hence, the research hypothesis which stated there is difference in adoption of improved turmeric production technology between tribal and non-tribal respondent accepted.

Findings are in line with the findings of Poonia (1995) who reported a significant variation in the adoption of improved package of practices of ginger crop among the literate and illiterate farmers. Similar results were also obtained by Kothari (1996) who found that the level of adoption of post-harvest technology of maize was below average among the tribal farmers while it was above average among non-tribals. A significant gap was found in the adoption of post -harvest technology (PHT) of maize tribal and non-tribals.

To identify the training needs of farmers growing turmeric crop, a suitable schedule was developed. This schedule contained six major training need areas viz, soil preparation, seed and sowing, use of manures & fertilizers, intercultural operations, plant protection measures and harvesting, storage and curing. The required information was analyzed.

Table-3: Perception of training needs by different category of turmeric cultivators:

S.No.	Main areas	NT		T		Total	
		MS	Rank	MS	Rank	MS	Rank
1	Use of improved varieties and method of sowing	6.83	2	7.91	2	7.37	2
2	Soil preparation	5.63	6	6.63	5.5	6.13	6
3	Application of manures & fertilizers	6.26	3	6.68	4	6.43	3
4	Inter-cultural operations	6.73	5	6.63	5.5	6.18	5
5	Plant protection	7.73	1	8.88	1	8.30	1

	measures						
6	Harvesting, curing and storage	6.21	4	6.84	3	6.53	4

rS= 0.92* T= Tribal NT= Non-Tribal

- = Significant at 1% level of significance

The data in table -1 reveals that plant protection measures was perceived as an important area of training by both non-tribal and tribal respondents with top priority. Likewise, use of improved varieties and method of sowing was also perceived equally important these two categories of respondents. Soil preparation emerged as much an area which was perceived as less important by respondents. It appears from figure quoted in the table 3 that there exists a strong accordance between tribal and non – tribal turmeric growers with respect to perception of training need areas of turmeric cultivation. The rank order correlation value between tribal and non-tribal respondents was found to be 0.92, which is statistically significant at 1 per cent level of significance. Which indicate that tribal and non-tribal respondents have perceived the training need areas with the similar magnitude or their ranking as in accordance.

The findings are strongly in line with findings of Poonia (1995) who found that all the three categories of respondents have given top priority to the plant protection measures as important needs perceived by them in improved ginger cultivation. Findings are also similar with that of Jang Bahadur et.al. (1987) who found in their study that small farmers gave given top priority to the plant protection measures as an important training need area for paddy cultivation.

CONCLUSION:

It could be concluded from the above discussion that turmeric growers of panchayat samiti Jhadol in Udaipur district of Rajasthan has poor adoption of some of the recommended turmeric cultivation practices like use of improved varieties, use of plant protection measures, marketing and storage of the seed and use of manures and fertilizers. Consequently, a very wide adoption gap was found with regard to improved varieties and use of plant protection measures. It was also found that non-tribal and tribal respondents had given slightly different priorities to the major areas of turmeric cultivation in which they need training. Therefore, to minimize the adoption gap, the perceived training need areas be taken care of. It is assumed that an intensive training programs on turmeric cultivation be organized for the farmers and extension personnel of the area.

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Evolution Dynamics of 3-Dimensional Memristive discrete FHN Neuron Model

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Abstract:
Neurons are granular cells and are fundamental units of the nervous systems presence in all living creatures.These are electrically excitable cells which fires electric signals, called action potentials, across a neural network.Models describing neuron evolution in cells are mostly of internally multicomponent structure and so are complex systems. In this article investigations performed on the dynamics of evolution of a 3-dimensional memristive discrete FHN neuron model. Complex pattern of bifurcationsshows,initially, a period doublingfollowed by chaos.Within chaos one observes clearly phenomena of bistability, appearance of periodic windows of period 6, period 8 etc, reimergent of chaos adding characteristic. Regular and chaotic attractors obtained for different parameter spaces. Plots of Lyapunov exponents (LCEs) obtained for various cases of regular and chaotic motions. As this memristive neural system exhibits complexity, numerical simulation extended to calculate topological entropy as a measure of complexity and presented through graphics. Thesegraphics show significant increase and fluctuations of topological entropy within certain parameter range which justifies presence of significant complexity in the system.

Key Words: Chaos, Lyapunov Exponents, Bifurcation, Topological Entropy

1. Introduction:

Numerous nonlinear systems articles appearing in recent revealing the fact that during evolution such systems displayregularity &chaos in different parameter spaces as well as special properties like bi-stability, intermittency, cascading effects, coexistence of multiple attractors etc. These special properties, other than regularity and chaos, are due to their internally multicomponent structure. Such nonlinear systems are termed as complex system, [1-7]. Thus, the property complexity is also common in many nonlinear systems. Chaos is a state of a dynamical system and the systems evolve into chaos from regularity during the processes of changes of parameter spaces. The special properties stated above, emerging to independent motions of internal components, called complexities present in the system.

Chaos appearing in a system measured by Lyapunov exponents (LCEs); if at a state the system displays chaos then LCE > 0 and if it shows regularitythen LCE< 0, [8–13]. As the type of movements of internal components are independent and not follow a common and definite rule, the property *complexity*of the system can only be measuredstatistically by using probabilistic rule.Such complexity measure appears as increment (or fluctuations) of topological entropy: more increase or fluctuations of topological entropy implies the system is more complex [4, 14–24].

Neurons are fundamental units of the nervous system and are composed of granular cells. Neuronsare electrically excitable cell which display electric signals called action potentials across a neural network.Neurons are main components of nervous tissue in all living creatures [25–33]. Most of the models describing neuron evolutions in cells are internally multicomponent and so are complex systems. The model displaying the activity of a recurrent two-neurons, [22, 31, 32], also comes under this category.

The objective of this research is to investigateevolutionary dynamics 3-D Memristive discrete FHN neuron model. Memristive, (an adjective of memristor**),**anon-linear two-terminal electrical component relating electric charge and magnetic flux linkage.A memristor is a resistive device with an inherent memory.For clear description on Memristive system some recent articles, [34, 35], referred. During the processes of study, bifurcation analysis performed varying two parameters, one by one, keeping other parameters fixed. This provides dynamic view of evolution.Then, numerical investigationscarried outand sets of regular and chaotic attractors obtained.Numerical simulation extended to obtain Lyapunov exponents (LCEs)for regular and chaotic attractors and presented through graphics. Numerical calculations furtherextended to calculate topological entropies as measures of complexity and to compute correlation dimensions of some chaotic attractors and represented in tabular form. Concluding remark section summarize the finding of the investigation.

2. Description of Rulkov Neuron Model:

The evolutionary equations of 3-D Memristive discrete FHN neuron model be described by following equations:

$$x_{n+1} = x_n - \frac{{x_n}^3}{3} - y_n + I_{ext} + k_1 x_n z_n \ ,$$

$$y_{n+1} = \gamma y_n + \theta\, x_n + \delta$$

$$z_{n+1} = z_n + sin z_n - k_2 x_n \qquad (2.1)$$

where x_n is the neuron membrane potential, y_n, z_n denote the recovery variable related to the conductivity of the ion channels, and I_{ext} is the external excitation; $k_1$and k_2 are control parameters ana are positive and γ, θ and δ are other parameters of the system, , see ref. [35].

3. **Bifurcation Phenomena:**

Bifurcation diagrams for system (2.1), , along all three coordinate axes, aredrawn with same initial conditions $(x_0, y_0, z_0) = (0.01, 0.02, 4.0)$for three cases, for clear understandingof the evolutionary phenomena. In the first case,bifurcation diagrams, Figure 1(a) and 1(b),obtained for$\gamma = -0.2$, $\delta = 2.0$, $I_{ext} = 0.2$, $k_1 = -0.06$, $k_2 = 0.2$ andvarying parameter θ, respectively, $-0.25 \leq \theta \leq 0.4$&$-0.25 \leq \theta \leq 0.4$.Here, Figure 1(b) displaying magnification part near nearby periodic windows. In the second case, bifurcation diagrams,Figure 2(a) and 2(b) obtained again by varying θ, $-0.25 \leq \theta \leq 0.4$&$-0.25 \leq \theta \leq -0.1$ for $\gamma = -0.2$, $\delta = 2.0$, $I_{ext} = 0.2$, $k_1 = -0.04$, $k_2 = 0.2$ and $-0.25 \leq \theta \leq 0.4$. Here, again,Figure 2(b) represents magnification part nearby periodic windows. Case three of bifurcation diagram, Figure 3, obtained when one of the of control parameters, k_1, varied $-0.25 \leq k_1 \leq 0.1$and other parameters are when $\gamma = -0.2$, $\delta = 0.081$, $I_{ext} = 1.8$, $k_2 = 0.2$, $\theta = 0.108$ and $-0.25 \leq k_1 \leq 0.1$.

Looking carefully these pictures of bifurcations one gets impressions that the system during evolution displays the following characteristics:

(i) Existence of bi-stability and multi-stability criteria

(ii) Period doubling and chaos adding criteria

(iii) Within chaotic region of bifurcation appearance of non-similar strange types of multi-periodic periodic windows. Such periodic windows are of different periods viz. periods 6, 7, 8, 10 etc.

(iv) Piecewise types existing of bifurcations, (Figure 3)

These dissimilar multiperiodic windows producing effects like intermittency, cascading effects, coexistence of multiple attractors etc.As stated, emergence of these properties confirming presence of complexity in the system. As the event movements of internal components are of independent type, facts appearing need to explain statistically by using the *law of probability*.

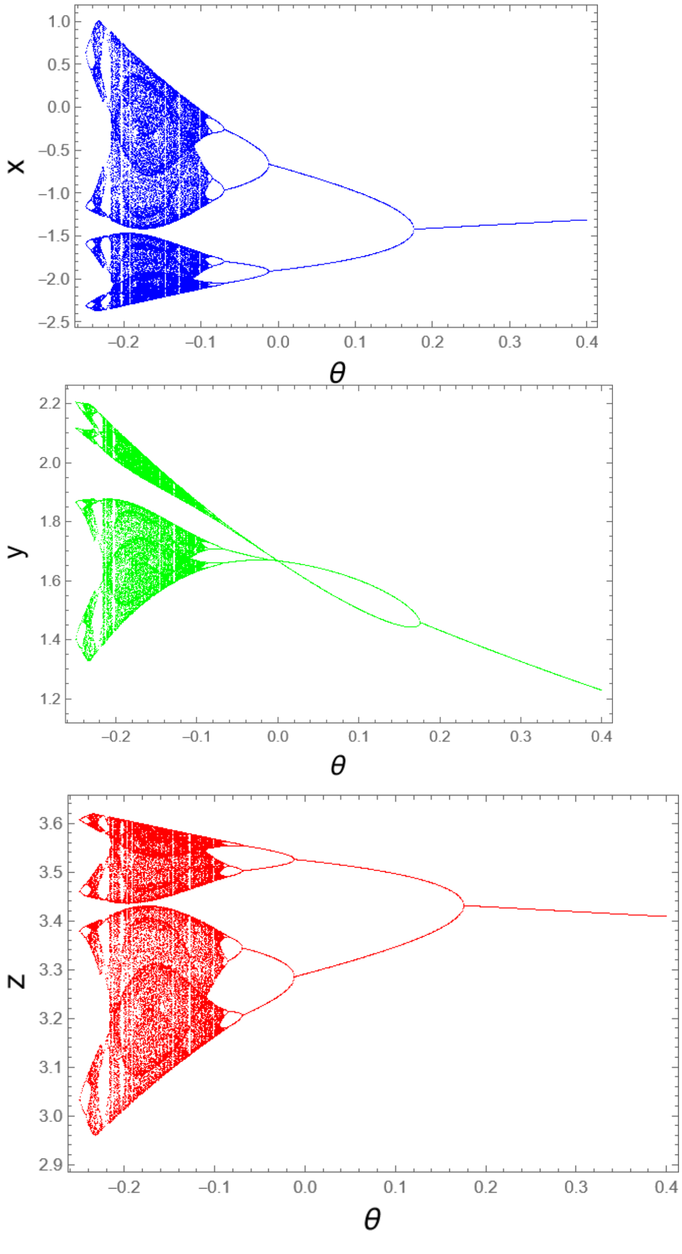


Figure 1(a): Bifurcation of Memristive discrete 3-Dimensional FHN Neuron Model when $\gamma = -0.2$, $\delta = 2.0$, $I_{ext} = 0.2$, $k_1 = -0.06$, $k_2 = 0.2$ and $-0.25 \leq \theta \leq 0.4$. Initial conditions are $(x_0, y_0, z_0) = (0.01, 0.02, 4.0)$.

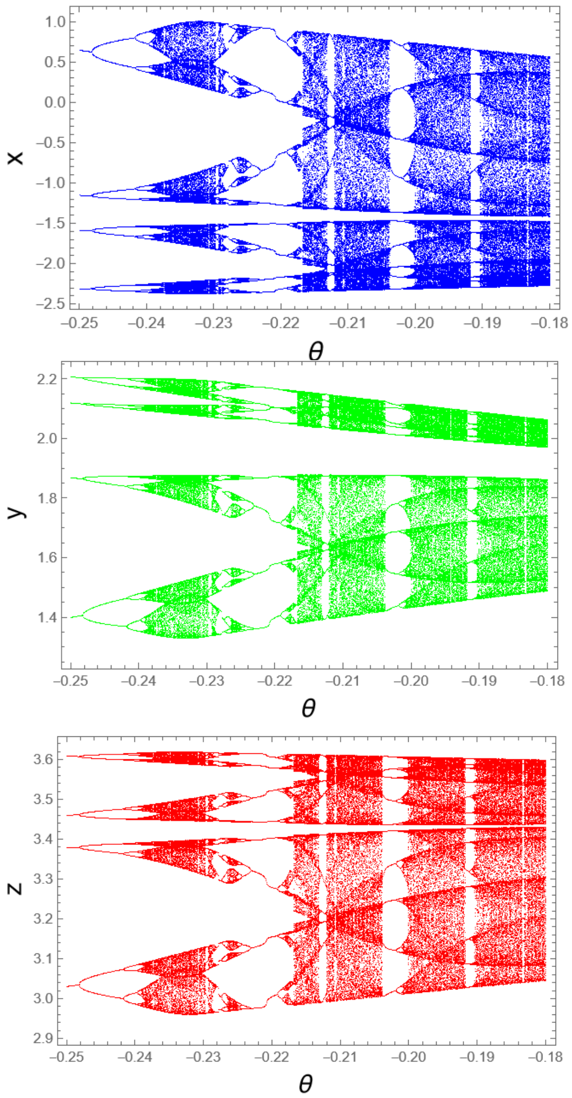


Figure 1(b): Bifurcation of Memristive discrete 3-Dimensional FHN Neuron Model with same parameters and initial conditions as in Figure 1(a) but $-0.25 \leq \theta \leq -0.15$.

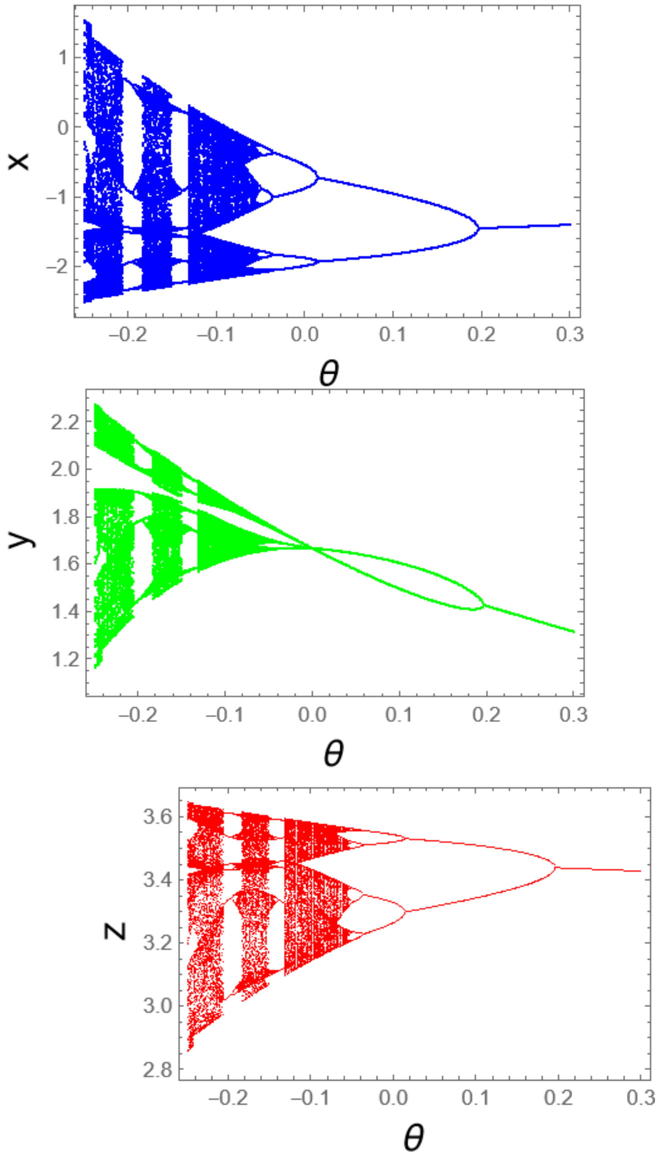


Figure 2(a): Bifurcation of Memristive discrete 3-Dimensional FHN Neuron Model when $\gamma = -0.2$, $\delta = 2.0$, $I_{ext} = 0.2$, $k_1 = -0.04$, $k_2 = 0.2$ and $-0.25 \leq \theta \leq 0.4$. Initial conditions are $(x_0, y_0, z_0) = (0.01, 0.02, 4.0)$.

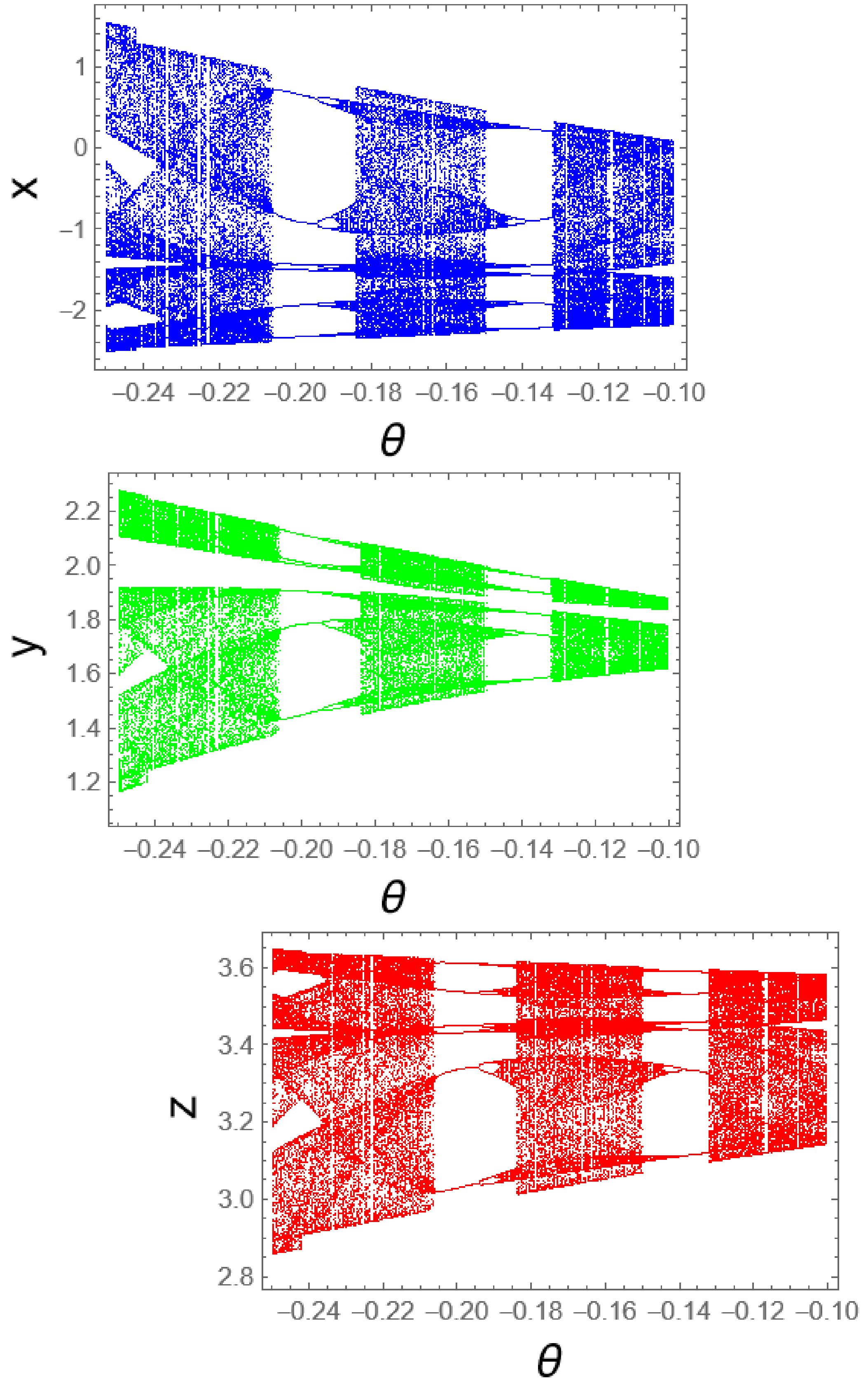


Figure 2(b): Bifurcation of Memristive discrete 3-Dimensional FHN Neuron Model with same parameters and initial conditions as in Figure 1(a) but $-0.25 \leq \theta \leq -0.1$.

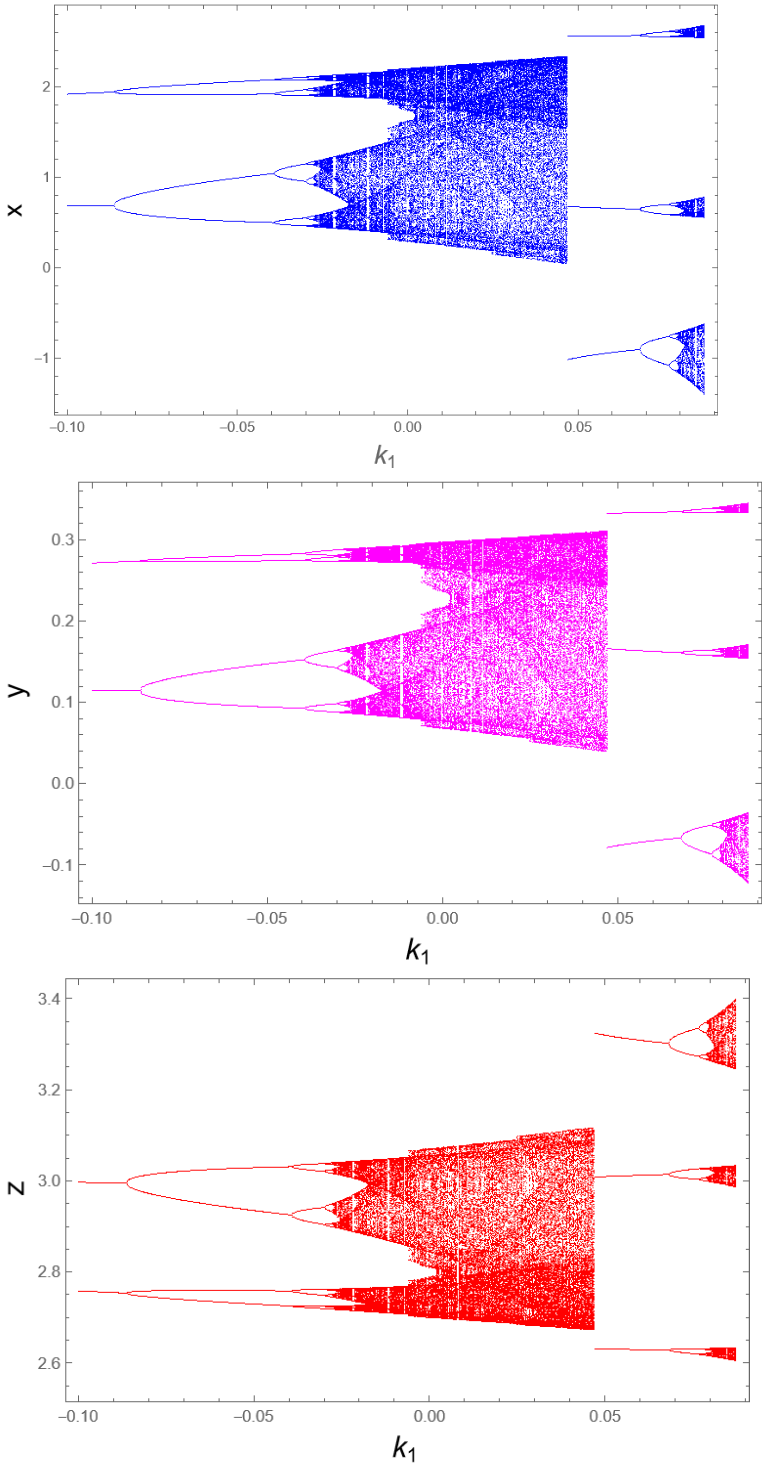
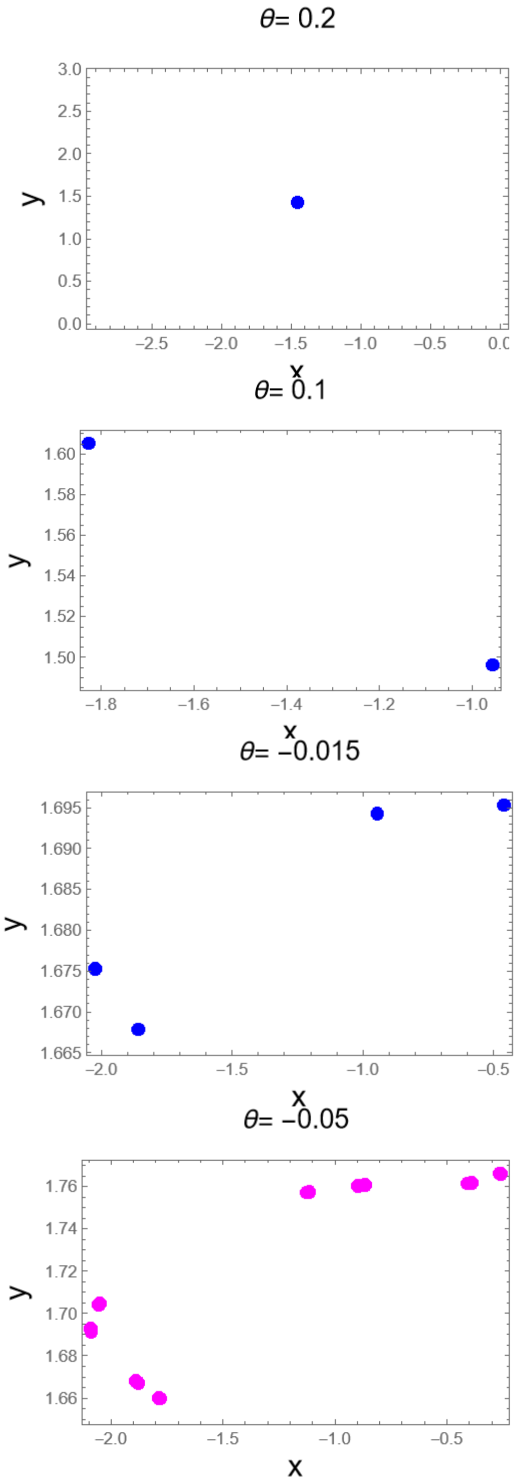


Figure 3: Bifurcation of Memristive discrete 3-D FHN Neuron Model (2.1) when $\gamma = -0.2$, $\delta = 0.081$, $I_{ext} = 1.8$, $k_2 = 0.2$, $\theta = 0.108$ and $-0.25 \leq k_1 \leq 0.1$. Initial conditions are $(x_0, y_0, z_0) = (0.01, 0.02, 4.0)$.

4. Numerical Simulations:

(a):Regular and chaotic attractors: Two sets of planner 2-dimensional regular and chaotic attractors in the $x - y$ plane obtained for system (2.1) corresponding to two cases fordifferent values of $\theta$and of k_1 and presented through Figure 4(a) and Figure

4(b). One also identifies some plots periodic attractors of different periods within the chaotic region where period-doubling ceases to exist.Figure 5, are some 3-dimensionalchaotic attractors for different values of k_1; other parameter values are mentioned there.



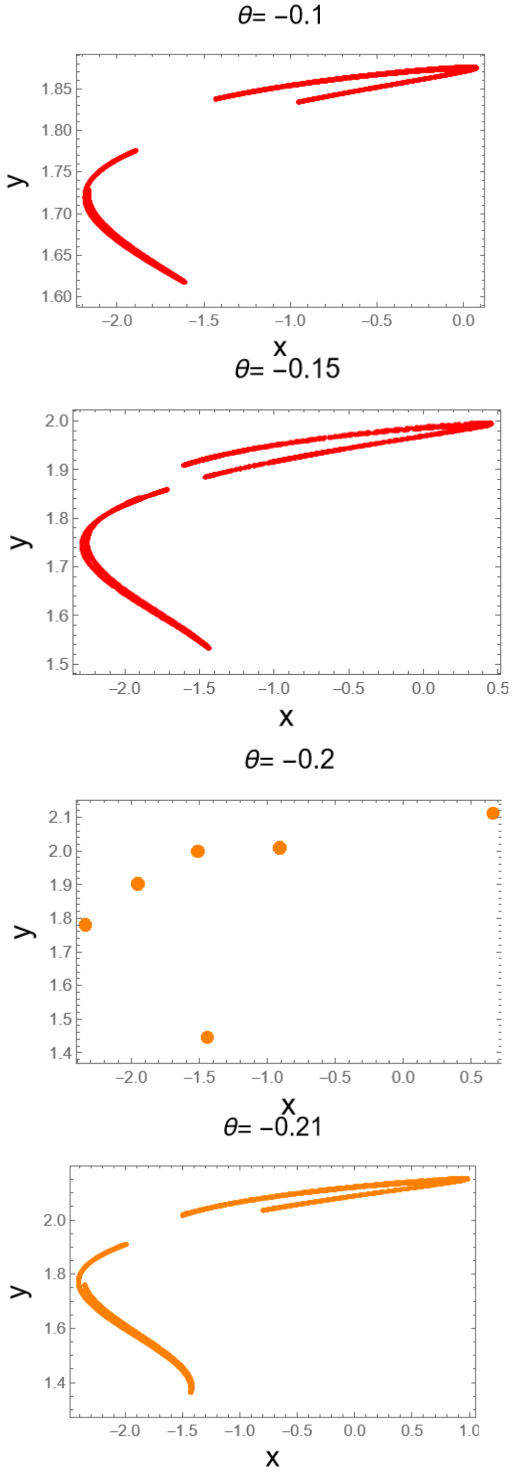


Figure 4(a): Regular and chaotic attractors of 3-D FHN Neuron Model for different values of θ when other parameters are $\gamma = -0.2$, $\delta = 2.0$, $I_{ext} = 0.2$, $k_1 = -0.04$, $k_2 = 0.2$.

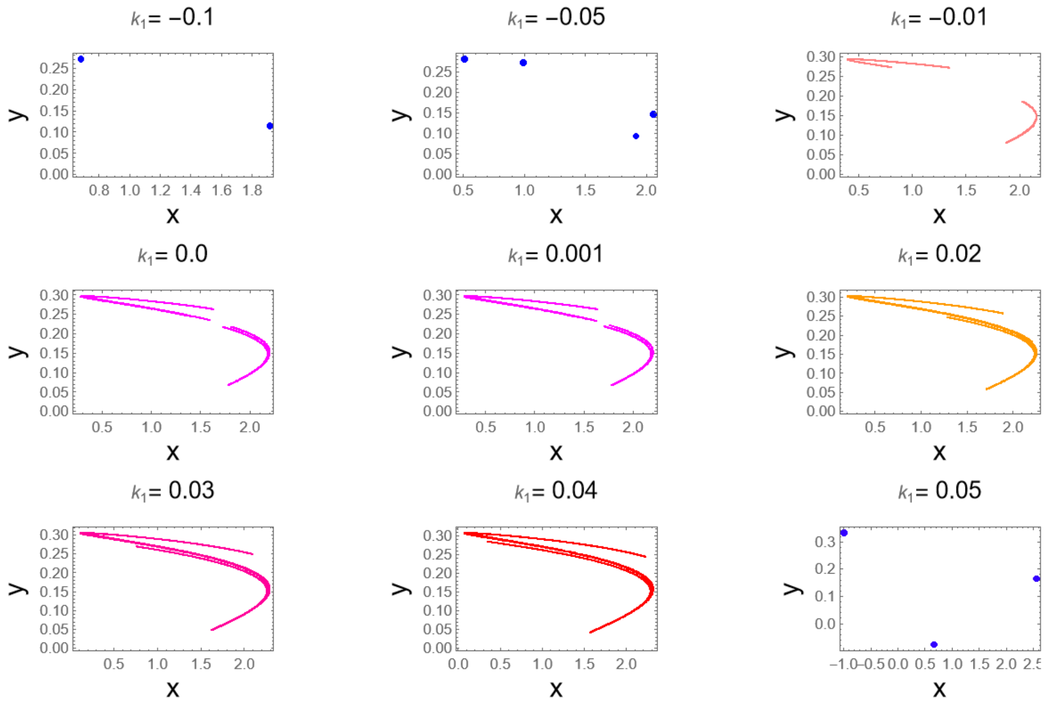


Figure 4(b): Regular and chaotic attractors of 3-D FHN Neuron Model for different values of k_1 when other parameters are $\gamma = -0.15$, $\delta = 0.081$, $I_{ext} = 1.8$, $\theta = 0.108$, $k_2 = 0.2$.

(b): 3-dimensional chaotic attractors:

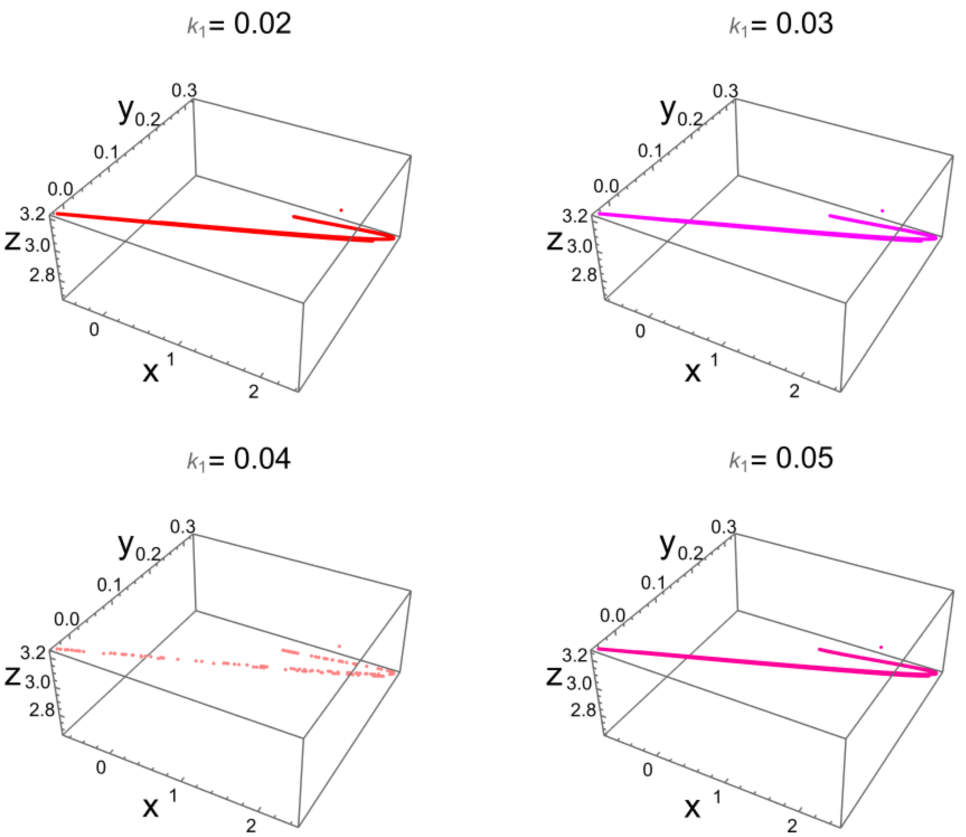
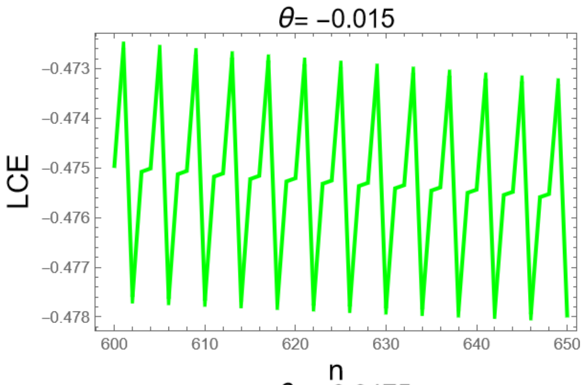
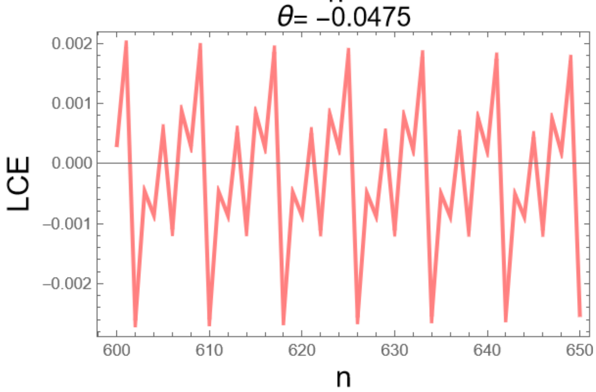


Figure 5: Plots of some 3-dimensional chaotic attractors of FHN Neuron Model (2.1) for different values of k_1 when $\gamma = -0.15$, $\delta = 0.081$, $I_{ext} = 1.8$, $k_2 = 0.2$, $\theta = 0.108$.

(c): Lyapunov Exponents (LCEs):

The universal fact that for chaotic motion, the system must show *sensitivity to initial conditions*,i.e. that two extremely close initiated trajectories display divergence of behaviour during long term evolution. Lyapunov exponents (LCEs) provide a measure of such divergence behaviour and if the system evolves chaotically then LCEs > 0 butif the evolution be regular then LCEs < 0. In Figure 6, plots of Lyapunov exponents shown for cases of regular and chaotic evolution. To calculate Lyapunov exponents and plotting those we have used Mathematica codes of Martelli [36].





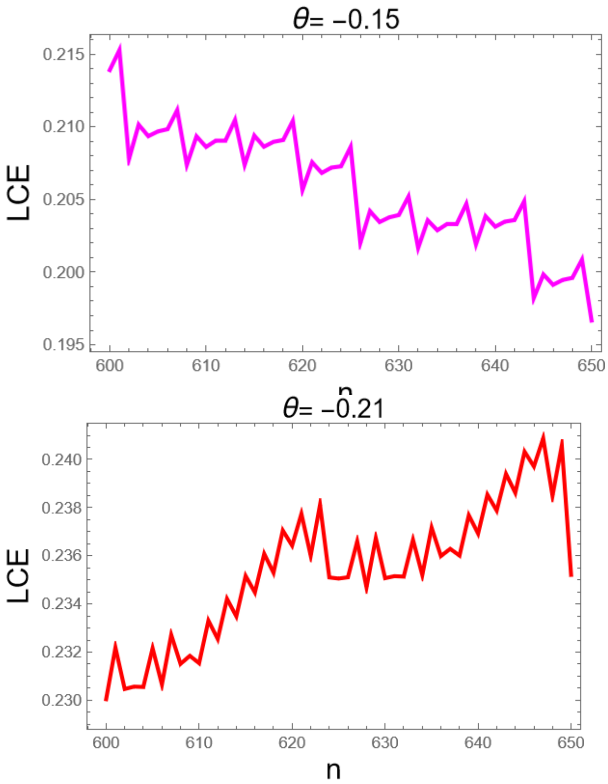


Figure 6: Plots of Lyapunov exponents of FHN Model (2.1) for different values of θ when other parameters are $\gamma = -0.2$, $\delta = 2.0$, $I_{ext} = 0.2$, $k_1 = -0.04$, $k_2 = 0.2$.

(d) Calculation of Topological Entropies:

Measure of complexity be done by applying the law of probability that describes the *rate of mixing of evolutions* due to motions contributed by internal multi-components structure of the complex system. During evolution complex systems display mixed properties of nonlinearity, chaos as well as complexity.Topological Entropy provides the measure of the *rate of mixing* of the evolving nonlinear complex system. Topological entropy is also called Kolmogorov–Sinai entropy [24]. Following is the steps of calculation of topological entropy:

Consider a finite partition of a state space X denoted by

$$P = \{ A_1, A_2, A_3, ... , A_N \}. \tag{4.1}$$

Then a measure μ on X with total measure $\mu(X) = 1$ defines the probability of a given reading as

$$p_i = \mu(A_i), \quad i = 1, 2, . . , N. \tag{4.2}$$

Then the entropy of the partition be given by

$H(P) = -\sum_{i=1}^{N} p_i \; Log \; p_i$

(4.3)

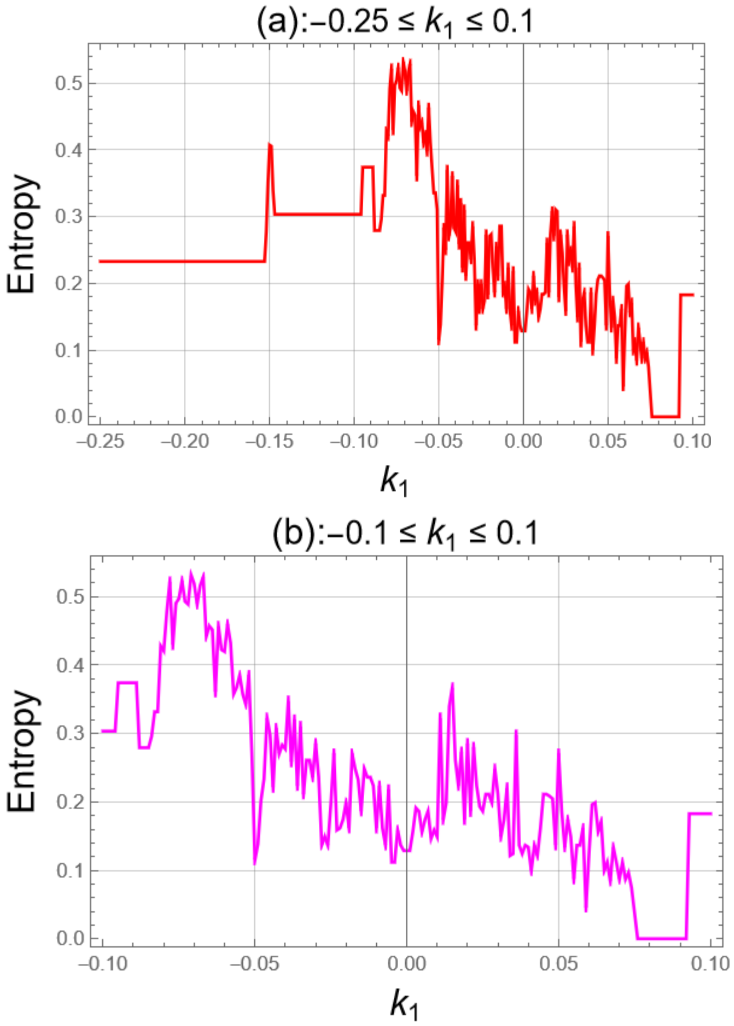


Figure 7: Plots of topological entropies of FHN Model (2.1) for $\gamma = -0.2$, $\delta = 0.081$, $I_{ext} = 1.8$, $k_2 = 0.2$, $\theta = 0.108$ and for k_1 as (i) $-0.25 \leq k_1 \leq 0.1$ and (ii) $-\, 0.1 \leq k_1 \leq 0.1$.

(**e**) **Correlation Dimensions**:

Correlation dimension is a type of fractal dimension and provides the measure of dimensionality of a chaotic attractor.It is calculated statistically with the application of Heaviside function, [36].

In this case, first we have to calculate correlation integral data $C(r)$, for a certain $r \ll 1$. Then, to plot the curve $\frac{log\ C(r)}{log\ r}$ against r. After this, to apply a linear fit criterion to the correlation data and obtained the equation of the straight line fitting the data points. The y-intercept of this line provides the correlation dimension, D_c.

For the chaotic attractor (Strange Attractor) obtained for $\gamma = -0.15$, $\delta = 0.081$, $I_{ext} = 1.8$, $\theta = 0.108$, $k_1 = 0.01$, $k_2 = 0.2$, we have obtained Correlation Dimension approximately as $D_c = 0.87$ follows:

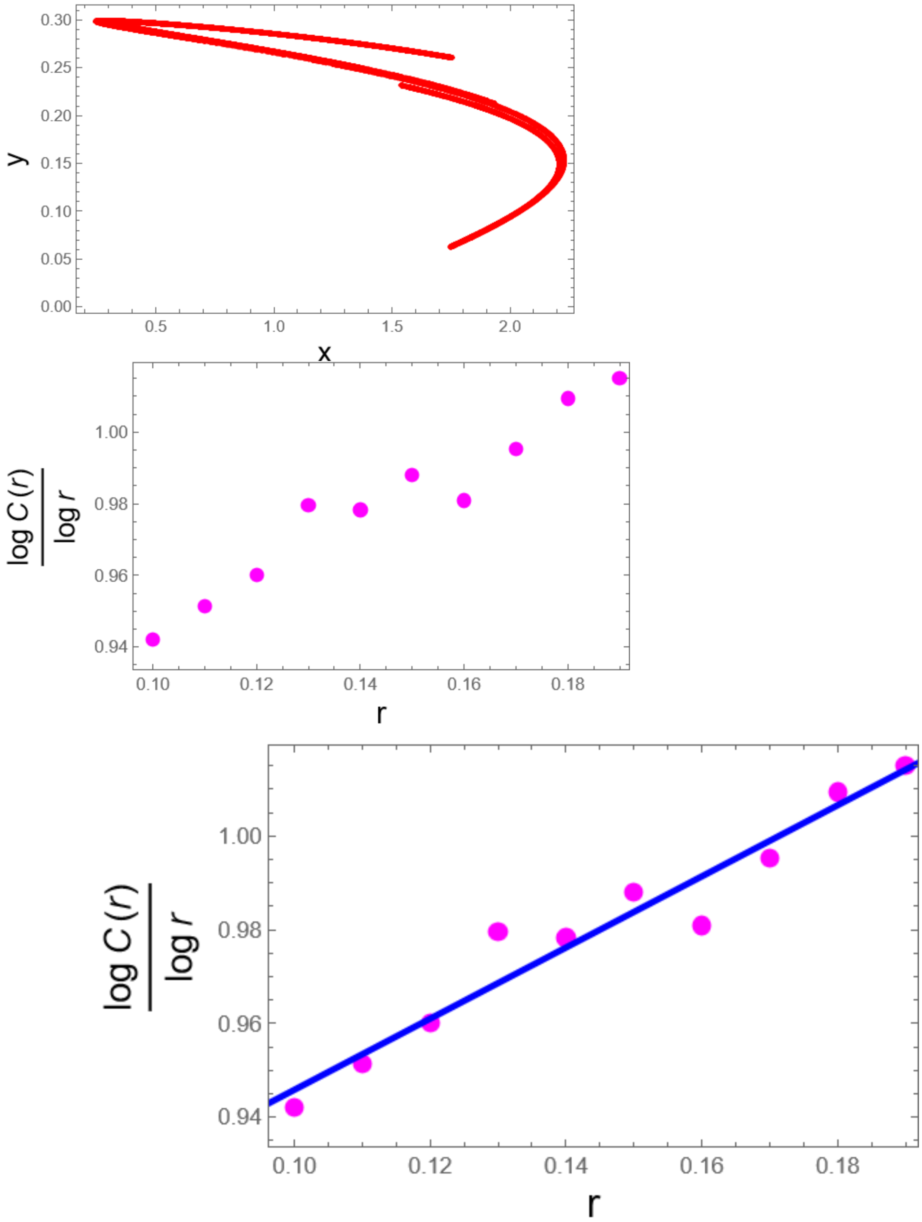


Figure 8: Linear fitting of correlation data of a chaotic attractor.

Straight line fitting data points is

$$y = 0.\ 759002x + 0.869927 \tag{4.4}$$

The y-intercept of this straight line is $0.869927 \cong 0.87$. Hence, Correlation Dimension ofthe chaotic attractor is$D_c = 0.87$. For same set of parameter values and differentk_1,correlation dimensions of chaotic attractors, shown in Figure 4(b) & Figure 5 calculated and presented in Table 1.

Table 1

Parameter k_1	Correlation Dimension D_c
0.01	0.87
0.02	0.77
0.03	0.955
0.04	1.007
0.45	1.2

4. Concluding Remarks:

The evolutionary dynamics of 3-dimensional Memristive discrete FHN neuron system displays very interesting and attractive results.Different sets of bifurcation diagrams, Figure 1-Figure 3, displaying phenomena stated there, confirming the present Memristive discrete neuron system highly complex. This neural system composed internally multicomponent and movements of these components during evolutions are of mixed type since they do not follow any definite rule and show unpredictability. Such mixed manifestation of evolution can only, possibly, be investigated statistically by using the rule of probability. Thus, complexity is measured by increase in topological entropy and described as “more increase and fluctuations in topological entropy implies the system is more complex”.Procedure of calculation of topological entropy discussed in this paper and two lots of topological entropies obtained forour system, Figure 7. These plotsclearly prove complexity in the system.

In addition to the presence of complexity in the system, it shows regular and chaotic behaviour at different parameter spaces shown through the plots of attractors, Figure 2 -Figure 5. Regular and chaotic nature of evolutionof the system can easily be identified by calculating Lyapunov exponents (LCEs) and checking whether LCE < 0 (regular) or LCE > 0 (chaotic). Some cases in this regard shown in Figure 6.

Chaotic attractors are in fact fractals and show proper self-similarity and their correlation dimensions, a type of fractal dimension, are non-integer. Processes of calculation of correlation dimension explained here in detail in section 3(e) and fractal dimensions of few such chaotic attractors given in Table 1.

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Morse potential and $_3F_2$ (1)

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Abstract: We deduce an identity for the hypergeometric function $_3F_2(1)$ which allows show the
equivalence of two formulas for the matrix elements of the Morse potential.

Keywords: Hypergeometric function $_3F_2$, Morse potential, Diatomic molecules, Matrix elements.

1. Introduction

The vibrational Morse interaction for diatomic molecules is given by [1-6]:

$$V(r) = D\,(e^{-2au} - 2\,e^{-au}), \qquad u = r - r_0 \;,(1)$$

where D is the well-depth, r_0 is the equilibrium position, and a is a range parameter, and its matrix elements:

$$< m\left|e^{-\beta au}\right|n > \equiv \int_0^\infty \psi_m^*\, e^{-\beta au}\, \psi_n\, du, \qquad \beta = 0, 1, 2, \ldots (2)$$

are important in quantum mechanics.

Rosen [7] and Vasan- Cross [8] calculated directly the integral (2) to obtain the following hypergeometric expression [9]:

$$< m\left|e^{-\beta au}\right|n > = \frac{(-1)^{n+m}}{k^\beta}\,\frac{\Gamma(\beta+n)\,\Gamma(k-n-1+\beta)}{m!\;\Gamma(k-m)\,\Gamma(\beta)}\sqrt{\frac{b_1 b_2 m!\;\Gamma(k-m)}{n!\;\Gamma(k-n)}}\;\; x$$

(3)

$$x\;\; {}_3F_2(-m, 1-\beta, 1-k+m; 1-n-\beta, 2-k+n-\beta; 1), \qquad n \geq m,$$

where $b_1 = k - 2n - 1,\; b_2 = k - 2m - 1,$ and $k = \frac{2}{a}\sqrt{2D}$.

On the other hand, Berrondo et al [10] used the relationship between (1) and the two-dimensional harmonic oscillator, to deduce the formula [9]:

$$< m\left|e^{-\beta a u}\right|n >= \frac{(-1)^{n+m}}{k^{\beta}} \frac{n!\,\Gamma(\beta+n-m)\,\Gamma(k-n-1+\beta)}{m!\,\Gamma(k-m)\,\Gamma(\beta)(n-m)!} \sqrt{\frac{b_1 b_2 m!\,\Gamma(k-m)}{n!\,\Gamma(k-n)}}\ x$$

(4)

$$x\ {}_3F_2(-m, 1-\beta, 1-\beta+n-m; 1+n-m, 2-k+n-\beta; 1), \quad n \geq m.$$

In Sec. 2 we employ a result of Melvin-Swamy [11] to prove an identity verified by the generalized hypergeometric function ${}_3F_2(1)$, which shows that (3) is completely equivalent to (4).

2. An identity for ${}_3F_2(1)$

We can demonstrate the equivalence of (3) and (4) if we establish the relation:

$${}_3F_2(-m, 1-\beta, 1-k+m; 1-n-\beta, 2-k+n-\beta; 1) = \frac{n!\ \Gamma(n-m+\beta)}{(n-m)!\ \Gamma(n+\beta)}\ x$$

(5)

$$x\ {}_3F_2(-m, 1-\beta, 1-\beta+n-m; 1+n-m, 2-k+n-\beta; 1),$$

for the hypergeometric function ${}_3F_2$ [12].

Now in the interesting expression of Melvin-Swamy [11]:

$${}_3F_2(\alpha_1, \alpha_2, \alpha_3;\ \lambda_1,\ \lambda_2; 1) = \frac{\Gamma(\lambda_1)\,\Gamma(\lambda_1-\alpha_1-\alpha_2)}{\Gamma(\lambda_1-\alpha_1)\,\Gamma(\lambda_1-\alpha_2)}\ {}_3F_2(\alpha_1, \alpha_2,\ \lambda_2-\alpha_3;\ \alpha_1+\alpha_2-\lambda_1+1,\ \lambda_2; 1), \qquad (6)$$

we employ the values:

$$\alpha_1 = -m, \quad \alpha_2 = 1-\beta, \quad \alpha_3 = 1-k+m, \quad \lambda_1 = 1-n-\beta, \quad \lambda_2 = 2-k+n-\beta\ ,(7)$$

then (5) is immediate because the properties of the gamma function [13-15] allow show that:

$$\frac{\Gamma(1-n-\beta)\,\Gamma(-n+m)}{\Gamma(1-n-\beta+m)\,\Gamma(-n)} = \frac{n!\,\Gamma(n-m+\beta)}{(n-m)!\,\Gamma(n+\beta)}\ , \quad n \geq m\ .(8)$$

Hence the formula of Rosen –(Vasan-Cross) [7, 8] is equivalent to the relation of Berrondo et al [10] for the matrix elements of the Morse potential; we consider that our proof is simpler than the one carried out in [9]. The results (3) and (4) allow calculate matrix elements for the Coulomb interaction [16].

Remark 1.-Thomae [17] and Ramanujan [18, 19] discovered a family of “three-term relations” expressing a general ${}_3F_2(1)$ as a linear combination of two other ${}_3F_2(1)$'s with different parameters. In case the first parameter is a negative integer, as we are assuming in (7), then the three-term relation reduces to (6).

Remark 2.- From (3) or (4) is immediate to obtain theexpressions:

$$< m\ |e^{-au}\ \ |n\ > = \frac{(-1)^{m+n}}{k}\left[\frac{(k-2n-1)(k-2m-1)\ n!\ \Gamma(k-n)}{m!\ \Gamma(k-m)}\right]^{1/2},$$

$m \leq n,$ (9)

$$< m\ |e^{-2au}\ \ |n > = \frac{1}{k}[k\ (n-m+1) + m\ (m+1) - n\ (n+1)] < m\ |e^{-au}\ \ |n >,$$

that for the case $m = n$ imply the result [2, 20]:

$$< n\ |e^{-au}\ \ |n > = < n\ |e^{-2au}\ \ |n > = \frac{1}{k}(k-2n-1),\quad(10)$$

which in turn allows us to demonstrate the relationship [21, 22]:

$$< n\ |u\ |n > = \frac{k}{a}[(n-m)(n+m+1-k)]^{-1},\quad m < n.$$
(11)

The book [23] contains an account of how Morse arrived at the potential that bears his name. Finally, we can find absolutely everything we might want to know about the properties of diatomic molecules in the monumental data compilation of Huber-Herzberg [24].

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Numerical Solution of time-fractional Newell-Whitehead-Segel Equation Solved by q – Fractional Homotopy Analysis Transform Method (q- FHATM)

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Abstract:

The objective of the study is to design a numerical scheme of q-fractional homotopy analysis transform method (q-FHATM) to investigate Newell–Whitehead–Segel equations considering fractional order. The solution obtained by this method is in the form of a rapid convergent series where its component can be computed easily. Results are obtained and discussed by taking different fractional values of derivative using figures. These results are also compared with available methods and newly developed methods. Selection of arbitrary appropriate parameters h and n for $n \geq 1$ help to ger more accurate approximations which are found to be identical with the exact solution. The auxiliary parameter h in q-FHATM is found to give a very convenient method to control convergence region of series solution. Also, further test examples are discussed to show the accuracy and competency of proposed numerical scheme. The outcome of this study clearly indicates the attractiveness of numerical scheme, reliable and easy for exploitation and highly effective.

2020 Mathematical Sciences Classification: 26A33, 26B30, 44A40.

Keywords : Non-Linear, Homotopy, Reliable.

1. Introduction

A tremendous variety of application of mathematics can be seen in many natural phenomena. Particularly Physics and engineering applications are found to understand more complex phenomena of real world. To solve completely a nonlinear and extract results from their solution is very difficult task in mathematical analysis. We come across real world models of non linear differential equation to understand many phenomena.

Using fraction order models of differential equations is of a non local property. However, the integer order fractional differential equations are local by nature. It shows that all upcoming state of any real world system is dependent on previous state of the system. Therefore, the realistic models

which can explain the real world phenomenon are based on the fractional order differential equations. To model the scenarios of acoustics, diffusion flux, electromagnet, etc.
[1–7]. In the recent years, several methods have been developed for solving fractional order differential equations containing nonlinear phenomena like application of Adomian decomposition method [8, 9], differential transform method [10], homotopy perturbation method [11], variation iteration method [12, 13], homotopy analysis method [14-16], homotopy analysis transform method [17-20], homotopy analysis Sumudu transform method [21], homotopy perturbation transform method [22], fractional variational iteration method [23-26], q-homotopy analysis transform method [27, 28], fractional iteration method [29, 30] etc.

Further, the Newell–Whitehead–Segel equations have many applications in chemical engineering, ecology, bio-engineering, various disciplines of science engineering [31, 32]. For example, the most important amplitude equations which discover different aspects of stripes patterns like stripes of sea shells and ripples in sand. Also, this has tremendous applications in spatially heterogeneous system. For example, in the Faraday instability, Rayleigh-Benard convection of fluid mixtures, Taylor–Couette flow, and several bio systems.

Further, the equation has an application in spatially extended systems, e.g., depicting the behavior around any bifurcation point in Rayleigh-Benard convection of fluid mixtures. Also, the effect of diffusion term has an overall effect on the system variables in the model.

Newell–Whitehead–Segel equation is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + aU - bU^q \tag{1}$$

where the constants a, b, k ∈ R, with q being a positive integer and k > 0. U(x, t) is a function of time variable t for t ≥ 0 and space variable x, for x ϵ R. Also, U is the non-linear temperature distribution in a thin and infinitely long rod. The derivative term on the left hand side of Eq.(1). The partial derivative $\frac{\partial U}{\partial t}$ denotes the rate of change of U with respect to time variable t. The derivative term on right hand side of Eq. (1) i.e., $\frac{\partial^2 U}{\partial x^2}$ shows the rate of change flux with respect to x at a particular time. So far, Laplace Adomian decomposition method, Differential transform method, Variational iteration method, Adomian decomposition method, Finite difference scheme, Homotopy perturbation method, Iterative method, etc., have been used for the solution of Newell–Whitehead–Segel equation [33-43].

Here, fractional order model of Newell–Whitehead–Segel Eq. (2) is considered in the following form

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + aU - bU^q \qquad 0 < \alpha \le 0 \tag{2}$$

where α is any parameter, which is the order of the derivative of fractional system. The caputo sense of the fractional derivative has been considered. By taking α = 1, the fractional Newell–Whitehead– Segel Eq. (2) becomes classical Newell – Whitehead – Segel Eq. (1).

Kumar and Sharma has published a research paper in which numerical approximation of Newell– Whitehead–Segel equation of fractional order using homotopy analysis is presented [44].

Strongly motivated with the above discussions, we propose here the application of q-(FHATM) to obtain the numerical solution of Newell–Whitehead–Segel equation (2) of fractional order. Also, the obtained results are then compared with the techniques FVIM and HASTM.

2. Preliminaries

Definition 2.1. Consider a real function g(τ), τ > 0. It is said to be in space $C_\beta, \beta \epsilon R$ if ∃ a real no. q (> β), s.t. g (τ) = $\tau^q\, g_1(\tau),\, g_1 \epsilon\, C[0,\infty]$.. It is clear that $C_\beta \subset C_\gamma$ if γ ≤ β.

Definition 2.2. Consider a function g (τ), τ > 0. It is said to be in space $C_\beta^m, m \epsilon N \cup \{0\}$ if $g^{(m)} \epsilon\, C_\beta$.

Definition 2.3. The left sided Caputo fractional derivative of g, $g \epsilon C_{-1}^m, m \in IN \cup \{0\}$,

$$D_t^v g(t) = \{[I^{m-v} g^{(m)}(t)], m-1 < v < m, m \epsilon IN, \frac{d^m}{dt^m} g(t),\ v = m$$

a. $I_t^\beta g(x,t) = \frac{1}{\Gamma\beta}\int_0^t (t-s)^{\beta-1} g(x,s)ds;\ \beta, t > 0,$

b. $D_\tau^v\ U(x,\tau) = I_\tau^{m-v} \frac{\partial^m U(x,\tau)}{\partial t^m}, m-1 < v \le m,$

c. $I_t^\beta D_t^\beta g(t) = g(t) - \sum_0^{m-1} g^k(0+)\frac{t^k}{k!},$

d. $I^v t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(v+\beta+1)} t^{v+\beta}.$

Definition 2.4. The Mittag-Leffler function $E_\beta(z), \beta > 0$ is defined by the following series representation, valid in the entire complex plane:

$$E_\beta(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(1+\beta m)}, \beta > 0, z \in C \tag{3}$$

3. The proposed q-HATM for the time-fractional Newell-Whitehead-Segel equation

Consider a nonlinear fractional non-homogeneous partial differential equation of the form

$$D_x^{\alpha} u(x,y) + Ru(x,y) + Nu(x,y) = g(x,y), \quad n-1 < \alpha \leq n,$$

(4)

where $D_x^{\alpha} u$ is Caputo's fractional derivative; R, N are linear and nonlinear differential operators respectively and $g(x,y)$ is the source term.

Taking transform of Laplace on each side of Eq. (4) & then simplifying, we acquire $L[u(x,y)] - \frac{1}{p^{\alpha}} \sum_{k=0}^{n-1} \quad p^{\alpha-k-1} u^k(0,y) + \frac{1}{p^{\alpha}} [L\{Ru(x,y) + Nu(x,y) - g(x,y)\}] = 0$

(5)

Nonlinear operator is formulated as

$$N[\phi(x,y;q)] = L[\phi(x,y;q)] - \frac{1}{p^{\alpha}} \sum_{k=0}^{n-1} \quad p^{\alpha-k-1} \phi^k(x,y;q)(0^+) + \frac{1}{p^{\alpha}} \{L[R\phi(x,y;q) + N\phi(x,y;q)]\} - \frac{1}{p^{\alpha}} \{L[g(x,y)]\} \tag{6}$$

Here $q \in [0, \frac{1}{n}]$ is embedding parameter and $\phi(x,y;q)$ is a real-valued function of x, y and q.

Now, we build homotopy as:

$$(1-nq)\, L[\phi(x,y;q) - u_0(x,y)] = \hbar q\, H(x,y)\, N[\phi(x,y;q)], \tag{7}$$

Here, L is Laplace transformation operator and n ≥ 1. $H(x,y) \neq 0$ is auxiliary function while $\hbar \neq 0$ is auxiliary parameter. $u_0(x,y)$ is initial approximation. $\phi(x,y;q)$ is an unknown function. For $q = 0$ and $q = \frac{1}{n}$, following result holds:

$$\phi(x,y;0) = u_0(x,y), \phi\left(x,y;\frac{1}{n}\right) = u(x,y) \tag{8}$$

Consequently, as q increases from 0 to $\frac{1}{n}, n \geq 1,$ the solution $\phi(x,y;q)$ swifts from initial approximation $u_0(x,y)$ to the solution $u(x,y)$. Applying Taylor's theorem on $\phi(x,y;q)$ to expand it about q in series form, we get

$$\phi(x,y;q) = u_0(x,y) + \sum_{m=1}^{\infty} \quad u_m(x,y) q^m, \tag{9}$$

where $u_m(x,y) = \frac{1}{m!} \frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}$ (10)

For suitable choice of auxiliary linear operator $u_0(x,y)$, n, h and H (x, y), series (8) converges at $q = \frac{1}{n}$, and we obtain

$$u(x,y) = u_0(x,y) + \sum_{m=1}^{\infty} \quad u_m(x,y) \left(\frac{1}{n}\right)^m, \tag{11}$$

Express vectors in the way

$\vec{u}_m = \{u_0(x,y), u_1(x,y) \ldots\ldots.. u_m(x,y)\}$ (12)

Differentiating zeroth order deformation Eq. (7) m-times w.r.t q then dividing by m! and at last taking q = 0, we obtain the ensuing mth order deformation [40] equation:

$L[u_m(x,y) - k_m u_{m-1}(x,y)] = \hbar H(x,y) æ_m(\vec{u}_{m-1})$

(13)

Taking inverse transform,

$u_m(x,y) = k_m u_{m-1}(x,y) + \hbar L^{-1}[H(x,y) æ_m(\vec{u}_{m-1})]$

(14)

In Eq. (14), we express $æ_m(\vec{u}_{m-1})$ in a new manner as:

$æ_m(\vec{u}_{m-1}) = L\, u_{m-1}(x,y) - \left(1 - \frac{k_m}{n}\right)[\frac{1}{p^\alpha}\sum_{k=0}^{n-1} \quad p^{\alpha-k-1}u^k(0,y) + \frac{1}{p^\alpha}L\{g(x,y)\}] +$

$\frac{1}{p^\alpha}L\{Ru_{m-1}(x,y) + P_{m-1}\}$

(15)

and k_m is presented as $k_m = \{0, m \leq 1, n, m > 1.$

(16)

In Eq. (15), P_m is homotopy polynomial [41] and expressed as

$P_m = \frac{1}{m!}\left[\frac{\partial^m \phi(x,y;q)}{\partial q^m}\right]|_{q=0}$ (17)

And $\phi = \phi_0 + q\,\phi_1 + q^2\,\phi_2 + \ldots$ (18)

Employing results of Eq. (15) in Eq. (14), we get

$u_m(x,y) = (k_m + \hbar)u_{m-1}(x,y) - \hbar\left(1 - \frac{k_m}{n}\right)L^{-1}[\frac{1}{p^\alpha}\sum_{k=0}^{n-1} \quad p^{\alpha-k-1}u^k(0,y) + \frac{1}{p^\alpha}L\{g(x,y)\}] +$

$\hbar\, L^{-1}\left[\frac{1}{p^\alpha}L[Ru_{m-1}(x,y) + P_{m-1}]\right]$

(19)

The advancement in this recommended scheme is that an innovative correction function (19) is established by applying homotopy polynomials. Eventually from Eq. (19), components $u(x,y)$ for $m \geq 1$ can be computed. The q-HATM solution is presented in subsequent form

$$u(x,y) = \sum_{m=1}^{\infty} \quad u_m(x,y)\left(\frac{1}{n}\right)^m$$

(20)

4. Numerical Experiments

In this section, we apply the proposed technique FVIM to some test examples.

Example 1. Consider a linear time–fractional Newell-Whitehead-Segel equation

$$D_t^\alpha U = U_{yy} - 2U, \quad 0 < \alpha \leq 1, \tag{21}$$

with initial condition U(y, 0) = e^y, (22) when $\alpha = 1$, the exact solution of Eqs. (21) - (22) is $U(y,t) = e^{y-t}$.

The initial solution can be taken as $U_0(y, t) = e^y$, then (23)

Using transform of Laplace on each side of Eq. (21) and simplifying, we get

$$L[U] - \frac{y}{p} - \frac{1}{p^\alpha} L\left[U_{yy} - 2U\right] = 0$$

(24)

We state the nonlinear operator as

$$N[\phi(x,y;q)] = L[\phi(x,y;q)] - \left(1 - \frac{k_m}{n}\right)\frac{y}{p} - \frac{1}{p^\alpha} L\left[D_y^2 \phi(x,y;q) - \phi(x,y;q)\right] \quad (25)$$

The m^{th} order deformation equation for H(x, y) = 1 is written as

$$L[u_m(x,y) - k_m u_{m-1}(x,y)] = \hbar æ_m(\vec{u}_{m-1})$$

(26)

where $æ_m(\vec{u}_{m-1}) = L\,[u_{m-1}] - \left(1 - \frac{k_m}{n}\right)\frac{y}{p} - \frac{1}{p^\alpha} L\left[D_y^2 U_{m-1} - 2U_{m-1}\right]$

Taking inverse transform,

$$u_m(x,y) = k_m u_{m-1}(x,y) + \hbar L^{-1} æ_m(\vec{u}_{m-1})$$

(27)

Simplification yields the following approximations of q–FHATM solution:

$U_0 = e^y$,

$U_1 = -\frac{\hbar e^y t^\alpha}{\Gamma(1+\alpha)}$,

$U_2 = \frac{\hbar e^y t^{2\alpha}}{\Gamma(1+2\alpha)}$,

$U_3(x,t) = e^y \hbar \left(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right) + \cdots,$

The solution can be expressed as:

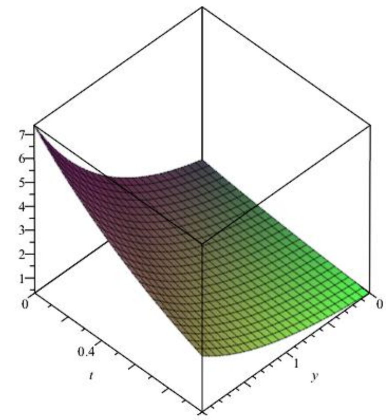
$$U(x,y) = U_0(x,y) + \sum_{m=1}^{\infty} U_m(x,y)\left(\frac{1}{n}\right)^m.$$

(28)

It is observed that if $\alpha = 1, \hbar = -1, n = 1$, the series solution $\sum_{m=0}^{N} u_m(x,y)\left(\frac{1}{n}\right)^m$ converges to exact solution (23) as $N \to \infty$,

$$u(x,y) = e^y \left[1 + \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \cdots\right] = y \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(1+k\alpha)} = y\,E_\alpha(t^\alpha).$$

where $E_\alpha(z)$ is the one parameter Mittag-Leffler function defined in [31].



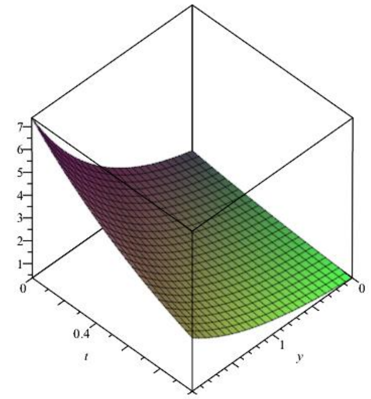


Fig.1

Figure of Exact solution U(x, t) w.r.t x and t, when α =1 for Eqs.(22).

Fig.2

figure of second term approximate solution U (x, t) w.r.t x and t, when α =1 for Eqs. (28).

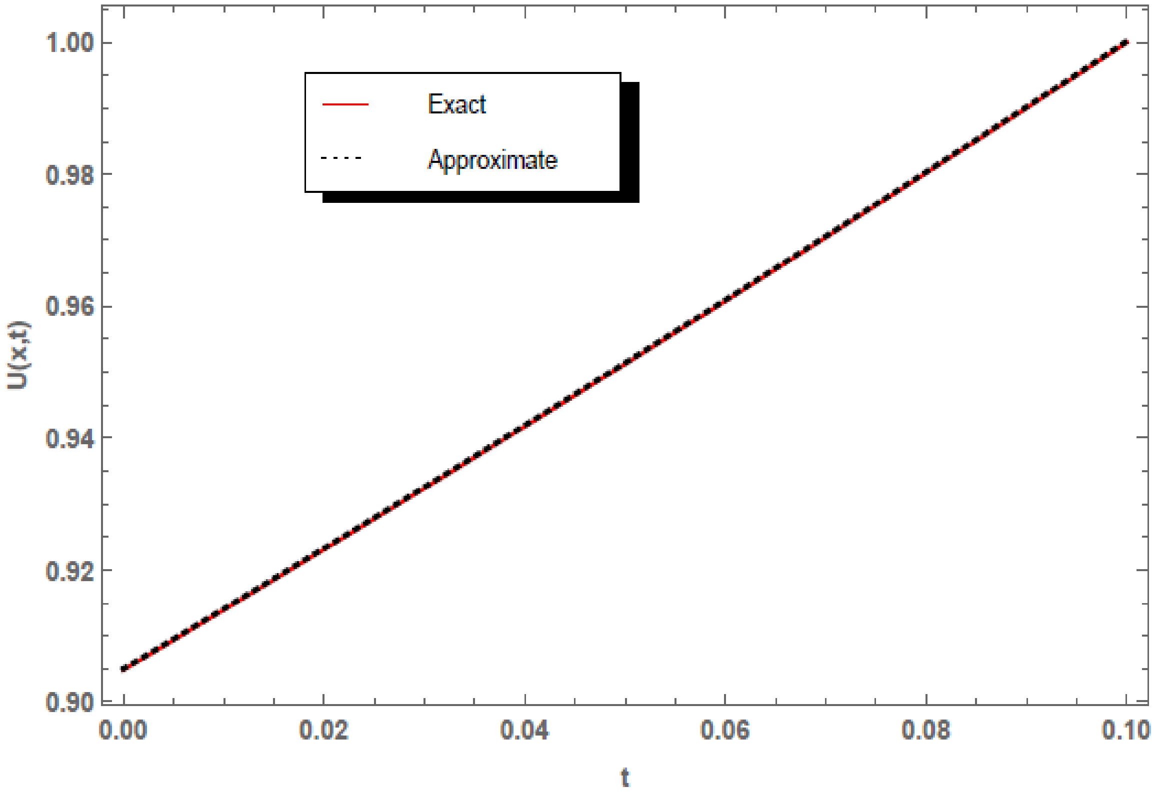


Fig. 3. Comparison between exact and approximate solution when α =1 for Eqs. (21) – (22).

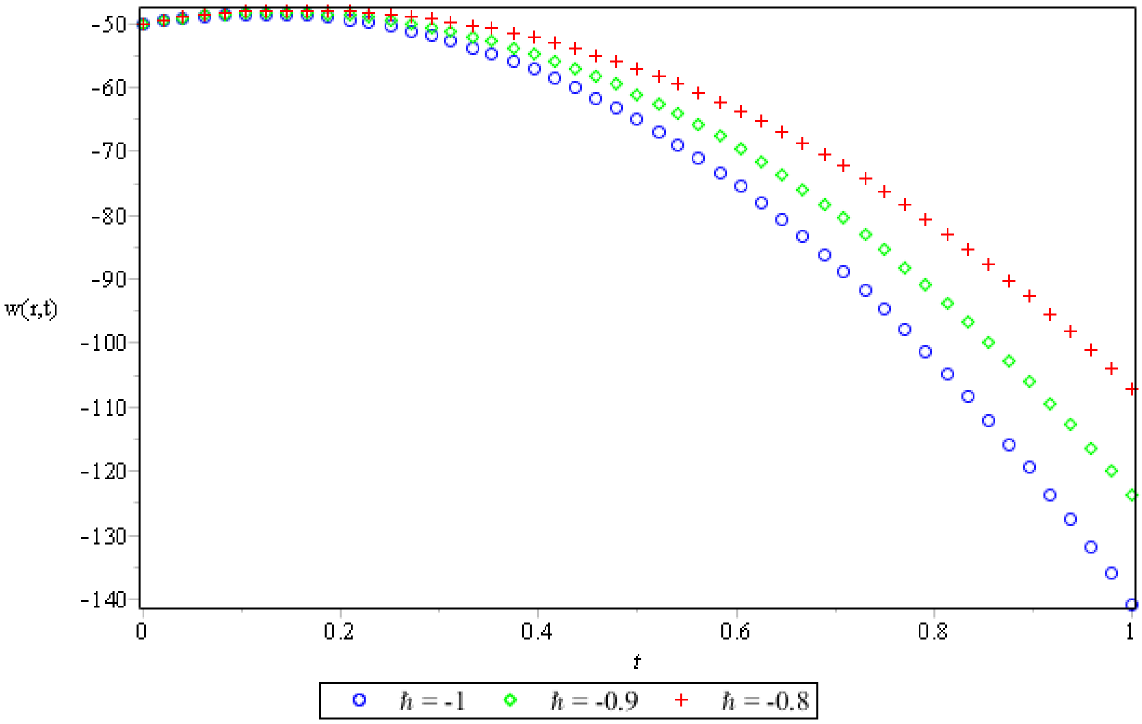


Fig. 4. Comparison between different values for h.

It is observed from here that q- FHATM works efficiently for this problem. However, if we take α = 1, we get the solution of classical Fractional Newell–Whitehead–Segel equation in the form

$$U(x, t) = e^{x}\left[1 - \frac{t}{1!} + \frac{t^2}{2!} + \ldots\ldots\right], \qquad (29)$$

which is very fast to the exact solution U(x, t) = e^{x-t} . (30)

Example 2. Consider the nonlinear time–fractional Newell-Whitehead-Segel equation

$$D_t^{\alpha} U = U_{yy} + 2U - 3U^2, \quad 0 < \alpha \leq 1 \qquad (31)$$

with the initial condition U(x, 0) = λ,
(32)

when $\alpha = 1$, the exact solution of Eqs. (31) - (32) is $(x, t) = \frac{\frac{-2}{3}\lambda e^{2t}}{\frac{-2}{3} + \lambda - \lambda e^{2t}}$.

The initial solution can be taken as

Using transform of Laplace on each side of Eq. (31) and simplifying, we get

$$L[U] - \frac{y}{p} - \frac{1}{p^{\alpha}} L\left[U_{yy} + 2U - 3U^2\right] = 0$$

(33)

We state the nonlinear operator as

$$N[\phi(x, y; q)] = L[\phi(x, y; q)] - \left(1 - \frac{k_m}{n}\right)\frac{y}{p} - \frac{1}{p^{\alpha}} L\left[D_y^2 \phi(x, y; q) - \phi(x, y; q)\right] \qquad (34)$$

The m^th order deformation equation for H(x, y) = 1 is written as

$$L[u_m(x, y) - k_m u_{m-1}(x, y)] = \hbar æ_m(\vec{u}_{m-1})$$

(35)

where $æ_m(\vec{u}_{m-1}) = L\,[u_{m-1}] - \left(1 - \frac{k_m}{n}\right)\frac{y}{p} - \frac{1}{p^{\alpha}} L\left[D_y^2 U_{m-1} + 2U_{m-1} - 3U_{m-1}^2\right]$

Taking inverse transform,

$$u_m(x, y) = k_m u_{m-1}(x, y) + \hbar L^{-1} æ_m(\vec{u}_{m-1})$$

(36)

Simplification yields the following approximations of q–FHATM solution:

$U_0(x, t) = \lambda$, then

$$U_1(x, t) = U_0 - \int_0^t \left[\frac{\partial^{\alpha} U_0}{\partial t^{\alpha}} - \frac{\partial^2 U_0}{\partial x^2} - 2U_0 + 3{U_0}^2\right](d\tau)^{\alpha} = \lambda + (2\lambda - 3\lambda^2)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

$$U_2(x,t) = U_1 - \int_0^t \left[\frac{\partial^\alpha U_1}{\partial t^\alpha} - \frac{\partial^2 U_1}{\partial x^2} - 2U_1 + 3{U_1}^2\right] (d\tau)^\alpha$$

$$= \lambda + (2\lambda - 3\lambda^2)\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{1}{\Gamma(1+\alpha)}\Big[(2\lambda - 3\lambda^2)\frac{t^\alpha}{\Gamma(1+\alpha)} - 2\lambda t^\alpha - 2\frac{(2\lambda-3\lambda^2)}{\Gamma(1+\alpha)}t^{2\alpha} + 3\lambda^2 t^\alpha + \frac{3\lambda^2\,(2-3\lambda)^2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\, t^{3\alpha} + \frac{6\lambda^2(2-3\lambda)}{\Gamma(1+2\alpha)}t^{2\alpha}\Big]$$

Proceeding in this manner, the enduring components can also be obtained using Mathematica.

Hence, we find the solution as $U(x,t) = U_n(x,t)$.

Now, by taking α = 1, we get the solution of classical nonlinear Newell-Whitehead-Segel equation as

$$U(x,t) = \lambda + (2\lambda - 3\lambda^2)\frac{t}{1!} + 2\lambda\,(1-3\lambda)(2-3\lambda)\frac{t^2}{2!} + 6\lambda^2(2-3\lambda)^2\frac{t^3}{3!} + \cdots \ldots..$$
(37)

which converge very fast to the exact solution

$$U(x,t) = \frac{\frac{-2}{3}\lambda e^{2t}}{\frac{-2}{3} + \lambda - \lambda e^{2t}} \tag{38}$$

Which is the same solution as obtained by DTM [35], ADM [36], HPM [39] and HASTM [44].

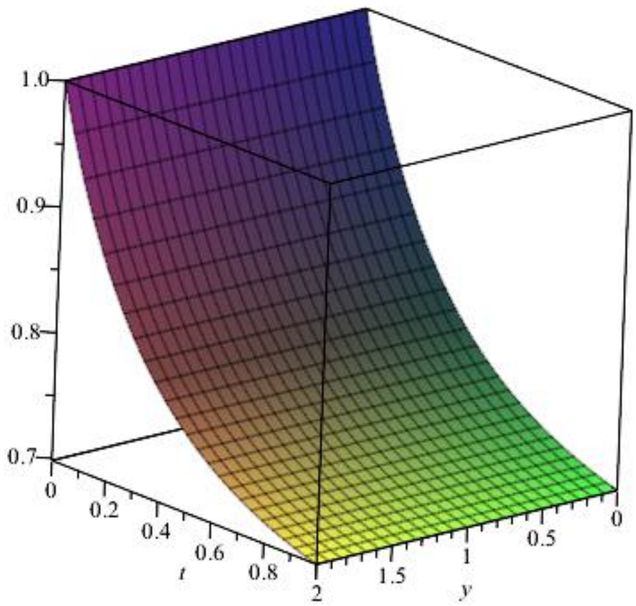


Fig 5. Figure of Exact solution U(x, t)
With approximate solution at alpha = 1.

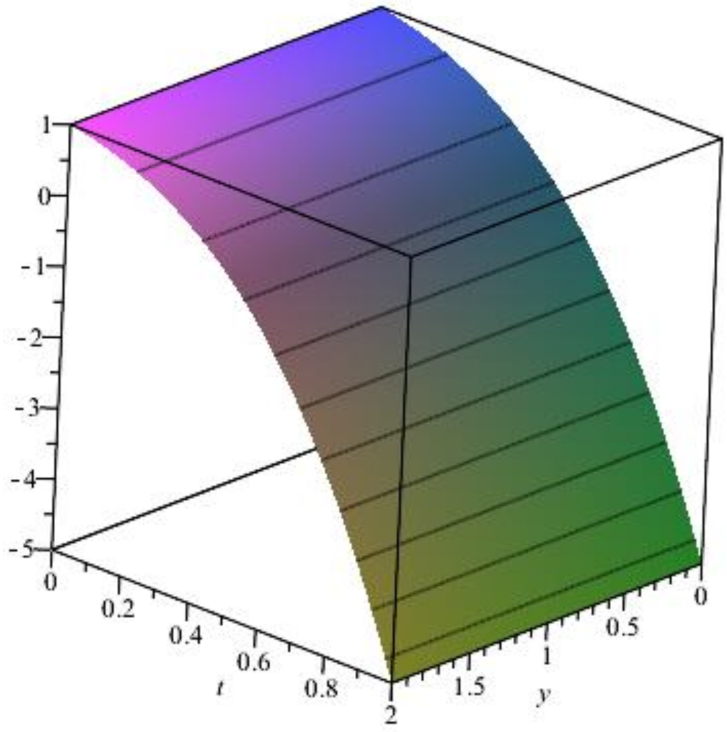


Fig 6. Figure of second term approximate solution U (x, t) w.r.t x and t, when α =1 for Eqs.(22).

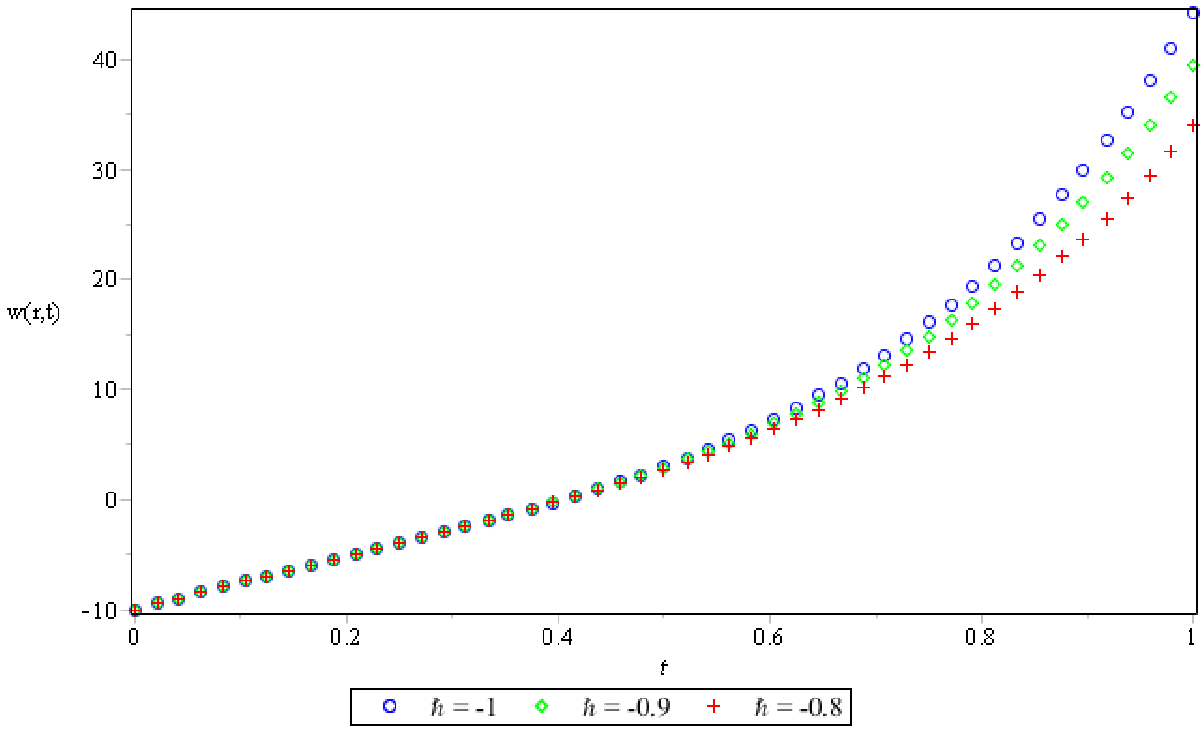


Fig. 7. Comparison between different values for h.

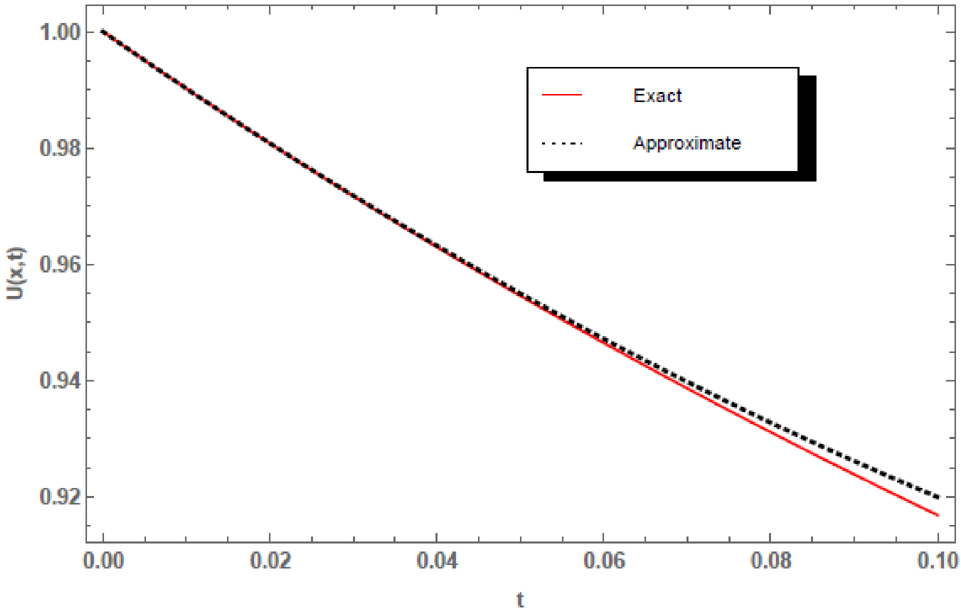


Fig. 8. Plots of U(x, t) vs. t at α =1 and λ =1 for exact solution and approximate solution

6. Conclusion

In this paper, q-FHAPTM is applied to get the solution of fractional model of Newell–Whitehead–Segel equation. The method q-FHAPTM is so successfully applied that it is found to be a very powerful and efficient to obtain approximate solution by taking an illustrative example. It should be also be noted that q-FHAPTM can be used without taking any kind of assumptions.

Hence, q-FHAPTM is proved to be the most convenient method and also easy than any other methods available in the literature.

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QUEUING FOR SUCCESS : A QUICK LOOK AT SERVICE OPTIMIZATION

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ABSTRACT

This comprehensive review explores the application of queuing theory in optimizing service systems across diverse sectors. The studies analyzed delve into the intricacies of minimizing customer wait times and maximizing server utilization. Spanning from Fair Price Shops, banks, and post offices to supermarkets, healthcare centers, and petrol stations, the research employs various queuing models such as M/M/C, GI/M/c, and GI/M/1/N. Methodologies encompass case studies, simulation modeling, and mathematical analysis, providing insights into factors influencing service efficiency and customer satisfaction. Findings underscore the versatility and effectiveness of queuing theory, suggesting avenues for future research, including advanced queuing models, real-time analytics, and the integration of emerging technologies. Practical implementation in real-world service environments remains crucial for continuous improvement.

Keywords: Queuing theory, service systems, queuing models, optimization, customer satisfaction, waiting times, server utilization, simulation modeling, communication networks, healthcare services, banking, customer service, queuing systems, queue management.

INTRODUCTION

Queuing theory, a fundamental paradigm in operations research, plays a crucial role in understanding and enhancing the performance of service systems across various sectors. This article presents a comprehensive review of studies applying queueing theory to optimize service systems, with a focus on minimizing customer wait times and maximizing server utilization. The diverse applications range from Fair Price Shops (FPS) and banks to supermarkets, healthcare centers, and petrol stations. Each study explores specific aspects, such as arrival rates, waiting times, and server configurations, contributing valuable insights to the field of service system management.

OBJECTIVE

To provide a comprehensive overview of the existing literature on queuing theory, focusing on its diverse applications in service systems, and to distill key insights that contribute to optima

RESEARCH METHODOLOGY

The reviewed studies employ diverse research methodologies, including case studies, simulation modeling, mathematical modeling, and data analysis. Researchers utilize queuing models such as M/M/C, GI/M/c, and GI/M/1/N to analyze and optimize service systems. Simulation modeling is prevalent, particularly in studies assessing the optimal number of servers in specific contexts. The research also delves into factors influencing service efficiency, customer satisfaction, and the impact of various strategies on queue management zing service efficiency and customer satisfaction across different sectors

REVIEW OF LITERATURE

In a recent study conducted by Sasi, Subramanian, &Ravichandran (2023), the research explores the application of queueing theory in Fair Price Shops (FPS), government buildings, banks, post offices, and other service sectors. The objective of this study is to achieve an optimal equilibrium between minimising wait times and maximising server utilisation by taking into account several aspects such as server utilisation, arrival rate, and service rate. The primary emphasis of this research is the implementation of the M/M/C queuing model in order to optimise queues at FPS (first-person shooter) venues. Simulation modelling is employed to ascertain the most favourable quantity of servers required to get a desired frame rate (FPS) in the region of Kerala, India. This study investigates the effects of many factors, including arrival rate, waiting time, and server utilisation.

The study conducted by Anand and Arora (2019) aimed to assess the efficacy of customer service and waiting periods in Indian banks using a case study. The objective of their study was to ascertain the variables that influence the efficiency of customer service and to provide an estimation of the duration customers had to wait.

The research conducted in this study was to identify and analyse potential bottlenecks in service delivery inside Indian banks, with the ultimate goal of providing recommendations to improve customer service.

The study conducted by Bhardwaj et al. (2019) examined a voice packetized statistical multiplexing system through the application of a fuzzy queuing model. The research conducted by the authors focused on the optimisation of communication networks, with particular emphasis on cases where voice traffic is predominant.

In their study, Kiataramkul and Neamprem (2019) conducted an investigation of the effectiveness of a queuing model that incorporates many servers. The specific context of their analysis was centred around bank token systems and the impact on client waiting times. The research investigated several configurations of queueing theory in order to compute variables relevant to service.

In their study, Onoja et al. (2018) proposed a mathematical model based on many servers and an exponentially distributed framework. This model aimed to analyse and predict various factors such as client waiting times, service rates, and arrival rates inside a banking environment. The objective of their study was to enhance the allocation of resources and minimise the duration customers spend waiting.

A study was undertaken by Jhala, Bhathawala, and Gujarat (2017) with the objective of examining the utilisation of queueing theory within the context of supermarkets. The objective of their analysis was to maximise the efficiency of queue management for clients waiting outside the establishment. The study conducted a comparison between single queue and multiple queue multiserver systems, revealing the benefits of employing a single queue, multi-server method in terms of minimising client wait times and related expenses.

Thangaraj and Rajendran (2017) conducted a study that investigated a queueing system characterised by batch arrivals and two distinct service patterns. The model takes into account scenarios in which the server would offer bulk service if the queue length surpasses a specific threshold 'a' following a period of inactivity, while otherwise providing single service. The investigation computed the distributions of queue sizes and assessed multiple performance indicators while considering specific situations within the model.

The objective of the study conducted by Agyei, Asare-Darko, and Odilon (2015) was to provide guidance to bank management in optimising the staffing of tellers by identifying the optimal balance between minimising total economic costs (comprising waiting cost and service cost) and reducing customer waiting times. The researchers employed data collected from the Kumasi Main Branch of Ghana Commercial Bank Ltd in order to construct a queuing model with many servers operating in a single line. The findings of the analysis indicate that a five-teller system demonstrated superior performance compared to both four and six-teller systems in terms of average waiting times and overall economic expenses. These results suggest that implementing a five-teller system could be a cost-effective measure to improve customer satisfaction.

In their study, Harley et al. (2014) examined the effects of several customer service components implemented by Nigerian banks on the financial performance of these institutions. A significant relationship was discovered between the mean duration of client waiting in queues and the financial viability of banks in Nigeria, so suggesting that proficient queue management and service provision play a crucial role in determining a bank's level of achievement.

Chandra and Madhu (2013) conducted a study that investigated a Markovian queuing system characterised by the presence of multiple service counters and finite waiting times. In this particular concept, a pair of servers were employed at separate counters in order to deliver a comprehensive level of service to an individual customer. The primary objective of the analysis was to ascertain the distribution of queue sizes in a condition of equilibrium, while also examining the consequences of modifying specific factors. The research emphasised the possibility of achieving more accurate results by integrating state-dependent rates into the multi-counter system model. Moreover, it is underscored that the modelling of queueing systems with blocking is of utmost importance. However, it also highlights the necessity to shift focus towards the factors of bulk arrivals and service.

In a study conducted by Kamau (2012), the focus was on examining the relationship between waiting queue management and customer satisfaction in commercial banks in Kenya. The objective of Kamau's study was to examine the efficacy of Kenyan banks in addressing customer grievances pertaining to extended waiting periods. This study investigated the strategies employed by commercial banks in the management of waiting lines, the obstacles they face in implementing these strategies, and the consequent effects on customer satisfaction. This research enhances the comprehension of queueing management in service systems, particularly in the setting of commercial banks in Kenya.

The study conducted by Ohaneme et al. (2012) focuses on the evaluation of queuing systems at petrol stations. The present study employed petrol stations as a case study to assess the importance of queuing systems in service operations. The researchers noted that petrol stations exhibit a tendency to service consumers in a random manner, resulting in the formation of lengthy queues and prolonged waiting durations. The implementation of the M/M/6 queuing system has been demonstrated to yield substantial improvements in the efficiency of client services when rigorously applied. This study offers valuable insights into the optimisation of queueing systems within service sectors.

In their study titled "Single Working Vacation in GI/M/1/N and GI/M/1/∞," Banik, Gupta, and Pathak (2011) investigate the use of single working vacation policies in the context of GI/M/1/N and GI/M/1/∞ systems. The topic of interest is queueing systems. Banik et al. conducted a study aimed at assessing the effects of a solitary working vacation on queueing systems. The authors employed embedded Markov chain and additional variable techniques to estimate queue length distributions and other essential performance metrics. This study aims to enhance comprehension of the impact of server vacations on the performance of queueing systems.

The study conducted by Rao et al. (2011) focuses on the use of queuing theory in the context of communication networks. The present study has made substantial progress in the utilisation of queuing theory within the realm of communication networks. The research examined the arrival and broadcasting procedures occurring at various network nodes, constructing a comprehensive model for an interdependent communication network. The results of this study have significant significance for the enhancement of network architecture and administration, specifically in relation to optimising data flow inside communication networks.

The field of queueing theory is of paramount importance in comprehending and enhancing the performance of service systems. The research undertaken by Jacob and Szyszkowski (2009) centred on the analysis of call centre data. The researchers noted that the duration of desertion in call centres adheres to a universally applicable and autonomous probability distribution. Based on their empirical findings, the researchers determined that the Poisson distribution exhibited the highest level of suitability as a model for call centres, specifically in relation to service times.

In their study, Adeleke (2009) focused on university health centres and utilised a single-server queuing model to estimate waiting times. The researchers' model, which made the assumption of Poisson arrival with exponential service rates and implemented the First-In-First-Out (FIFO) queue discipline, facilitated the estimation of patient arrivals and waiting durations in emergency departments. These models possess significant value in augmenting the efficiency of healthcare systems.

In their study, Cochran and Roche (2009) conducted an investigation of a range of modelling strategies aimed at mitigating the issues of hospital bed shortages and congestion. The scope of their study included the utilisation of empirical equations, including linear and nonlinear equations, as well as the application of time series forecasting and queuing theory-based models. It is worth noting that models based on queuing theory demonstrated superior performance compared to techniques based on formulas. These queuing theory-based models provided more effective strategies for optimising the distribution of beds and enhancing the quality of healthcare services.

Queueing theory is a fundamental paradigm that provides insights into the dynamics of waiting lines and service systems. The efficiency of queuing in traditional and modern banks in Nigeria was assessed by a comparative analysis undertaken by Kasum et al. in 2006. The collection of primary data was conducted by utilising an inverted-funnel questionnaire that was administered by the bank clients themselves. The results of the study revealed that consumers of contemporary banks encountered notably reduced waiting durations in comparison to customers of conventional banks, thereby emphasising the significance of effective service provision within the banking industry.

In their study, Pei-Chun et al. (2006) utilised queueing theory as a framework to assess the efficacy of different Taiwanese banking institutions, including postal banking services. The research conducted by the authors centred on the examination of several operations of automated teller machines (ATMs), including cash withdrawal, fund transfer, password reset, and balance inquiry. The effectiveness of ATM services was evaluated through the utilisation of a queueing model, which revealed the necessity of augmenting the number of ATMs in certain financial institutions in order to mitigate consumer wait times.

The researchers Green et al. (2006) employed the M/M/s queuing model in their study to investigate the interplay between service delays, patient utilisation, and the optimal number of servers necessary for the functioning of healthcare systems. The aforementioned findings make a valuable contribution to the existing body of knowledge about the optimisation of healthcare services.

The study conducted by Tian and Zhang (2006) examined a queueing system that incorporated several servers and a vacation policy with a (d, N)-threshold. This policy permitted a designated quantity of inactive servers to concurrently engage in vacation periods. The primary aim of their study was to determine the ideal values for the variables d and N. The research conducted by Ke et al. (2009) expanded upon previous studies by investigating the optimal vacation approach (d, c) for an M/M/c/N queue with servers that are prone to failures and require repair processes. This study contributed to the advancement of knowledge in the field of server management inside queueing systems.

Tian and Zhang (2003) conducted a study. Tian and Zhang conducted a study that examined a queueing system of the GI/M/c type, incorporating the concept of vacations, wherein all servers collectively cease operation when the system is devoid of customers. The authors proposed the notion of synchronous vacations, wherein servers resume operation if there are clients in a waiting state. The length of vacations is determined by a random variable that follows a distribution characterised by its phase. The research was centred on the computation of stable probability distributions pertaining to wait durations and queue lengths during arrivals. The work aimed to provide explicit formulas for both measurements.

In their 2002 work, Tian and Zhang examined a GI/Geo/1 queuing model in discrete time, specifically focusing on the presence of server downtime and vacations. The researchers employed a matrix-geometric technique to explicitly compute stationary distributions for both queue length and waiting time.

In their study, Nosek et al. (2001) examined queuing-based methodologies in the field of healthcare administration. The authors placed particular emphasis on the evaluation of hospital practises and the enhancement of pharmaceutical services as means to augment consumer satisfaction.

In the study conducted by Katayama (1995), the primary focus was on a tandem queue system that incorporated cyclic services. The investigation took into account the presence of servers on vacation as well as a whole service load. The objective of the study was to determine the mean durations of stays, accounting for breaks, as well as the mean waiting times, which are relevant for the examination of performance in packet switching systems.

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Table 1: Summary of Reviews

Study	Sector	Queuing Model	Objective	Methodology	Key Findings
Sasi et al. (2023)	Various (FPS, banks, post offices)	M/M/C	Optimal equilibrium between wait times and server utilization	Simulation modeling	FPS optimization using M/M/C model in Kerala, India
Anand and Arora (2019)	Indian banks	Article	Assess customer service and waiting periods	Case study	Identify and analyze bottlenecks for improving customer service
Bhardwaj et al. (2019)	Communication networks	Fuzzy queuing model	Optimize communication networks, focus on voice traffic	Article	Voice packetized statistical multiplexing system
Kiataramkul and Neamprem (2019)	Banks (token systems)	Article	Effectiveness of queuing model with multiple servers	Article	Analyze variables relevant to service in different configurations
Onoja et al. (2018)	Banking	Exponentially distributed model	Analyze waiting times, service rates, and arrival rates	Mathematical model	Enhance resource allocation and minimize customer waiting times
Jhala et al. (2017)	Supermarkets	Single vs. multiple queues	Maximize efficiency of queue management	Article	Single queue, multi-server method minimizes client wait times
Thangaraj and Rajendran (2017)	Queueing system with batch arrivals	Article	Investigate scenarios with bulk service	Article	Assess queue size distributions and performance indicators
Agyei et al. (2015)	Banks	Many servers in a single line	Optimize teller staffing for cost-effectiveness	Queuing model	Five-teller system demonstrated superior performance
Harley et al. (2014)	Nigerian banks	Article	Effects of customer service components on financial performance	Article	Relationship between client waiting durations and financial viability
Chandra and Madhu (2013)	Markovian queuing system	Multiple service counters	Analyze distribution of queue sizes	Article	Emphasize state-dependent rates in multi-counter system models

Kamau (2012)	Commercial banks in Kenya	Article	Relationship between waiting queue management and customer satisfaction	Article	Examine strategies, obstacles, and effects on customer satisfaction
Ohaneme et al. (2012)	Petrol stations	M/M/6 queuing system	Evaluate queuing systems at petrol stations	Article	M/M/6 queuing system improves client services
Banik et al. (2011)	Queuing systems (GI/M/1/N and GI/M/1/∞)	Single working vacation policies	Assess effects of server vacations on queuing systems	Markov chain	Evaluate queue length distributions and performance metrics
Rao et al. (2011)	Communication networks	Article	Queuing theory in communication networks	Article	Construct a comprehensive model for an interdependent communication network
Jacob and Szyszkowski (2009)	Call centers	Poisson distribution	Analysis of call center data	Empirical findings	Poisson distribution suitable model for call centers
Adeleke (2009)	University health centers	Single-server queuing model	Estimate waiting times in emergency departments	Article	Significant value in augmenting efficiency of healthcare systems
Cochran and Roche (2009)	Hospital bed shortages	Queuing theory-based models	Mitigate hospital bed shortages	Article	Queuing theory-based models more effective than formula-based techniques
Kasum et al. (2006)	Banks in Nigeria	Article	Comparative analysis of queuing efficiency	Inverted-funnel questionnaire	Consumers of contemporary banks experience reduced waiting durations
Pei-Chun et al. (2006)	Taiwanese banking institutions	Queuing model	Efficacy of different banking institutions	Article	Augmenting the number of ATMs can mitigate consumer wait times
Green et al. (2006)	Healthcare systems	M/M/s queuing model	Interplay between service delays, patient utilization, and servers	Article	Valuable contribution to optimizing healthcare services

Tian and Zhang (2006)	Queuing system with vacation policy	(d, N)-threshold policy	Determine ideal values for variables d and N	Article	Enhance knowledge in server management in queuing systems
Tian and Zhang (2003)	GI/M/c queuing system	Synchronous vacations	Computation of stable probability distributions	Article	Provide explicit formulas for wait durations and queue lengths
Tian and Zhang (2002)	GI/Geo/1 queuing model	Discrete time	Explicitly compute stationary distributions	Matrix-geometric technique	Examining server downtime and vacations
Nosek et al. (2001)	Healthcare administration	Article	Evaluate hospital practices and enhance pharmaceutical services	Article	Augment consumer satisfaction
Katayama (1995)	Tandem queue system	Cyclic services	Determine mean durations of stays, accounting for breaks	Article	Relevant for examining performance in packet switching systems

CONCLUSION

The findings from these studies underscore the versatility and effectiveness of queueing theory in optimizing service systems. From banking to healthcare and beyond, the application of queuing models provides actionable insights for improving customer satisfaction and operational efficiency. The studies reviewed reveal the importance of tailored queuing strategies, considering factors like server configurations, arrival rates, and waiting times in specific service contexts.

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