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Mellin – Barnes type of double integrals for the Exton function of two variables

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Abstract. The object of this paper is to derive the Mellin –Barnes type of double integrals for the generalized hypergeomertric function introduced by Exton. The results derived are useful in obtaining the double Mellin transform of the Exton function of two variables. The Mellin-Barnes type integrals have more freedom in parameters and variables in comparison to series definition. We also evaluate double integrals involving the Exton function of two variables, which includes, as special cases , the results for the Kampé de Feriet double hypergeometric function in the modified notation of Srivastava and Panda, additional double hypergeometric function due to Exton and Appell hyprgeometric functions of two variables etc. Finally we obtain the Riemann-Liouville fractional integral and Riemann-Liouville fractional derivative for the Exton function.

The results are of general character and a number of known results follow as special cases of our investigations. The results obtained also include , as special cases, the results given by Reed and in the monograph by Erdélyi et al. The results are useful in the study of statistical distributions .Some special cases of the main results are also discussed.

Keywords and Phrases : Mellin –Barnes type integrals, double Mellin transform, Exton's double hypergeometric function, Kampé de Feriet double hypergeometric function, Appell functions.Humbert's double hypergeometric functions.

2010 AMS Subject Classification : Primary 33C99, Secondary 33C20.

1.Introduction

In an attempt to derive a double integral representation for the well-known Gauss hypergeometric function, Exton [4,p.339(13)] introduced the following double hypergeometric series in two variables as

$$H^{A:B;C:D}_{E:G;M;N}\left[\begin{matrix}(a_A):(b_B);(c_C);(d_D);\\(e_E):(g_G):(m_M);(n_N);\end{matrix}\ x,y\right]$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{[(a_A)]_{2i+j}[(b_B)]_{i+j}[(c_C)]_i[(d_D)]_j}{[(e_E)]_{2i+j}[(g_G)]_{i+j}[(m_M)]_i\,[(n_N)]_j}\frac{x^i y^j}{i!j!}, \quad (0 < |x| < 1), 0 < |y| < 1) \quad (1.1)$$

For convenience in presentation, the symbol (a_A) denotes the array of A parameters given by $a_1, a_2, a_3, \dots, a_A$ in the contracted notation of Slater [10,p.54; 11, p.41]. The symbol $\Delta(N;a)$represents an array of N parameters $\frac{a}{N}, \frac{a+1}{N}, \dots, \frac{a+N-1}{N}$ $[13, p.47, pp.213]$. Some special cases of (1.1) are given below.

By making suitable adjustments in the number of numerator and denominator parameters of (1.1), it gives Kampé de Feriet double hypergeometric function in the modified notation of Srivastava and Panda [14,p.423 (26), see also [15,p.23 (1.2,1.3)] denoted by

$$F^{B;C:D}_{G;M;N}(x,y) = H^{0:B;C:D}_{0:G;M;N}, (x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{[(b_B)]_{i+j}[(c_C)]_i[(d_D)]_j}{[(g_G)]_{i+j}[(m_M)]_i\,[(n_N)]_j}\frac{x^i y^j}{i!j!}, \quad (1.2)$$

where $0 < |x| < 1, 0 < |y| < 1$.

Another special case is additional double hypergeometric function due to Exton [5,p.137 (1.2)] denoted by

$$X^{A;C:D}_{E;M;N} = H^{A:0;C:D}_{E:0;M;N} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{[(a_A)]_{2i+j}[(c_C)]_i[(d_D)]_j}{[(e_E)]_{2i+j}[(m_M)]_i\,[(n_N)]_j}\frac{x^i y^j}{i!j!}, \quad (1.3)$$

where $0 < |x| < 1, 0 < |y| < 1$.

Similarly Appell's four hypergeometric functions of two variables F_1, F_2, F_3 and F_4 [2,p.224 (6,7,8,9)] .Humbert' seven double hypergeometric functions $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \Xi_1, , \Xi_2$ $[2, p.225-226(20-26)$; see also[13,pp.58-59]. Horn's double hypergeometric function $H_3, H_4[2, p.225\ (15,16)]$, and its confluent forms $H_6$and H_7[2,p.226(34.35)] are all special cases of the double hypergeometric function due to Exton defined by (1.1). In what follows

$\Gamma[(a_A)+s+t]$ will stand for the product of A ecxpressions

$\Gamma[(a_1)+s+t]\,\Gamma[(a_2)+s+t] \dots \Gamma[(a_A)+s+t]$. Similar meaning holds for other gamma functions .

In a recent paper, Chaudhry et al [1] obtained certain transformations for the Exton function of two variables defined by (1.1). This has motivated the authors to obtain the double integral representation and other properties for this function.

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Note 1.1. It is interesting to observe that the Horn's function H_2 and H_4 are special cases pf the Exton double hypergeometric function defined by (1.1)as indicated by Exton [4].For certain identities for generalized hypergeometric function, see [12].

2**. Mellin –Barnes type of integral for the function** $\boldsymbol{H^{A:B;C:D}_{E:G;M;N}}$

Theorem 2.1. It will be shown here that

$$H^{A:B;C:D}_{E:G;M;N}\left[\begin{matrix}(a_A):(b_B);(c_C);(d_D);\\(e_E):(g_{G)}:(m_M);(n_N);\end{matrix}\ x,y\right]$$

$$= -\frac{\kappa}{4\pi^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\Omega(s,t)(-x)^s\,(-y)^t\,ds\,dt \qquad (2.1)$$

where

$$\Omega^{\mathrm{A:B;C;D}}_{\mathrm{E:G;M;N}}(s,t)=\frac{\Gamma(-s)\Gamma(-t)\Gamma[(a_A)+2s+t]\Gamma[(b_B)+s+t]\Gamma[(c_C)+s]\Gamma[(d_D)+t]}{\Gamma[(e_E)+2s+t]\Gamma[(g_G)+s+t]\Gamma[\mathrm{m_M}+s]\Gamma[(n_N)+t]}$$

|arg(-x)| < π and |arg(-y)| < π *and* $\kappa=\frac{\Gamma[(e_E)][(g_G)]}{\Gamma[(a_A)][(b_B)]}\frac{\Gamma[(m_M)][(n_N)]}{\Gamma[(c_C)][(d_D)]}$.

Proof. If we evaluate the integral as a sum of the residues by calculus of residues at the simple poles of $\Gamma(-s)$ at the points s=- i ($i\epsilon N_0$) and $\Gamma(-t)$ at the points t = $-j$ ($j\epsilon N_0$), we see that the value of the integral is equal to

$$\kappa\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma[(a_A)+2i+j]\Gamma[(b_B)+i+j]\Gamma[(c_C)+i]\Gamma[(d_D)+j]}{\Gamma[(e_E)+2i+j]\Gamma[(g_G)+i+j]\Gamma[(\Gamma[(m_M)+i])]\Gamma[(n_N)+j]}\frac{x^i}{i!}\frac{y^j}{j!}$$

$$= H^{A:B;C:D}_{E:G;M;N}\left[\begin{matrix}(a_A):(b_B);(c_C);(d_D);\\(e_E):(g_{G)}:(m_M);(n_N);\end{matrix}\ x,y\right]$$

This completes the proof of (2.1).

3. **Special cases of (2.1)**

For The Kampé de Feriet function, in the notation of Srivastava and Panda [14,p.423 (26), see also [15,p.23 (1.2,1.3)], we obtain

Corollary 3.1. The following result holds

$$H^{0:B;C:D}_{0:G;M;N}\left[\begin{matrix}-:(b_B);(c_C);(d_D);\\-:(g_{G)}:(m_M);(n_N);\end{matrix}\ x,y\right]=F^{B;C:D}_{G;M;N}\left[\begin{matrix}-:(b_B);(c_C);(d_D);\\-:(g_{G)}:(m_M);(n_N);\end{matrix}\ x,y\right]$$

$$= -\frac{\kappa_1}{4\pi^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(-s)\Gamma(-t)\Gamma[(b_B)+s+t]\Gamma[(c_C)+s]\Gamma[(d_D)+t](-x)^s\,(-y)^t}{\Gamma[(g_G)+s+t]\Gamma[\mathrm{m_M}+s]\Gamma[(n_N)+t]}\,dsdt\,, \quad (3.1)$$

where |arg(-x)| < π , |arg(-y)| < π *and* $\kappa_1=\frac{[(g_G)]}{\Gamma[(b_B)]}\frac{\Gamma[(m_M)][(n_N)]}{\Gamma[(c_C)][(d_D)]}$. (3.2)

For another additional double hypergeometric function due to Exton [4,p.137 (1.2)] denoted by

$$X^{A;C:D}_{E;M;N} = H^{A:0;C:D}_{E:0;M;N}(x,y)$$,we obtain

Corollary 3.2. The following result holds:

$$H^{0:B;C:D}_{0:G;M;N}\left[\begin{matrix}-:(b_B);(c_C);(d_D); \\ -:(g_G):(m_M);(n_N);\end{matrix}\; x,y\right] = E^{B;C:D}_{G;M;N}\left[\begin{matrix}-:(b_B);(c_C);(d_D); \\ -:(g_G):(m_M);(n_N);\end{matrix}\; x,y\right]$$

$$= -\frac{\kappa_2}{4\pi^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(-s)\Gamma(-t)\Gamma[(b_B)+s+t]\Gamma[(c_C)+s]\Gamma[(d_D)+t](-x)^s\,(-y)^t}{\Gamma[(g_G)+s+t]\Gamma[m_M+s]\Gamma[(n_N)+t]}\,dsdt \quad (3.3)$$

where |arg(-x)| < π and |arg(-y)| < π $and\ \kappa_2 = \frac{\Gamma[(g_G)]}{[(b_B)]}\frac{\Gamma[(m_M)][(n_N)]}{\Gamma[(c_C)][(d_D)]}$.(3.4)

If we set B=C=D=1. G=1, M=N=0, $b_1 = a, c_1 = b,\ g_1 = c, d_1 = b'$ in (2.1),

we obtain [2,page 232, 5.8.3(10)]

Corollary 3.3.

$$F_1(a,b,b';c;x,y) = -\frac{\Gamma(c)}{4\pi^2\Gamma(a)\Gamma(b)\Gamma(b')}$$

$$\times \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(-s)\Gamma(-t)\Gamma[a+s+t]\Gamma[b+s]\Gamma[b'+t](-x)^s\,(-y)^t}{\Gamma[c+s+t]}\,dsdt \quad (3.5)$$

where |arg(-x)| < π and |arg(-y)| < π

If we set B=C=D=1. G=1, M=N=0, $b_1 = a, c_1 = b,\ g_1 = c, d_1 = b'$ in (3.1),

we obtain [2, page 232, 5.8.3(11)]

Corollary 3.4. The following result holds

$$F_2(a,b,b';c,c';x,y) = -\frac{\Gamma(c)\Gamma(c')}{4\pi^2\Gamma(a)\Gamma(b)\Gamma(b')}$$

$$\times \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(-s)\Gamma(-t)\Gamma[a+s+t]\Gamma[b+s]\Gamma[b'+t](-x)^s\,(-y)^t}{\Gamma[c+s]\Gamma[c'+t]}\,dsdt \ , (3.6)\ )$$

where |arg(-x)| < π and |arg(-y)| < π

If we set M=N=B=0,G=1,C=D=2, $b_1 = a, c_1 = b, c_2 = b$, $b_1 = a, d_1 = a', d_2, = b', g_1 = c$,we obtain

[2, page 232, 5.8.3(12)]

Corollary 3.5.

$F_3(\mathrm{a}, a', b, b'; c; x) \quad = - - \frac{\Gamma(c)}{4\pi^2\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}$

$\times \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(-\mathrm{s})\Gamma(-\mathrm{t})\Gamma[\mathrm{a}+s]\Gamma[\mathrm{b}+s]\Gamma[\mathrm{a}'+t]\Gamma[\mathrm{b}'+t](-x)^s\,(-y)^t}{\Gamma[\mathrm{c}+s+t]}\,dsdt$,(3.7)

where $|\arg(-x)| < \pi$ and $|\arg(-y)| < \pi$.

If we set B=2, C=D=G=0, M=N=1, $m_1 = c, n_1 = c'$, $b_1 = a$, $b_{2,} = b$ in (3.1),we obtain [2, page 232, 5.8.3(13)]

Corollary 3.6.

$F_4(\mathrm{a}, a', b, b'; c; x) \quad = -\frac{\Gamma(c)\Gamma[c']}{4\pi^2\Gamma(a)\Gamma(b)}$

$\times \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\frac{\Gamma(-\mathrm{s})\Gamma(-\mathrm{t})}{\Gamma(\mathrm{c+s})}\frac{\Gamma(\mathrm{a+s+t})\Gamma(\mathrm{b+s+t})(-x)^s\,(-y)^t}{\Gamma(\mathrm{c}'+t)}\,dsdt$, (3.8)

where $|\arg(-x)| < \pi$ and $|\arg(-y)| < \pi$.

4. **Double Mellin transform of the Exton function of two variables**

Theorem 4.1.By applying the Reed ‘s Theorem [8, p.565]to (2.1), we immediately obtain the following double Mellin transform of the Exton function of two variables in the form

$$\int_{\mathbf{0}}^{\infty}\int_{\mathbf{0}}^{\infty}\boldsymbol{x^{s-1}}\,\boldsymbol{y^{t-1}}\;H_{E:G;M;N}^{A:B;C:D}\left[\begin{matrix}(a_A):(b_B);(c_C);(d_D);\\(e_E):(g_{G)}:(m_M);(n_N);\end{matrix}\;-x,-y\right]dxdy$$

$$= \kappa\;\Omega_{\mathrm{B:G;M;N}}^{\mathrm{A:B;C;D}}(-s,-t),\ (4.1)$$

where 0<Re(2s+t)<Re((a_A)), $0 < Re(s+t) < Re(b_{\boldsymbol{B}})$, 0<Re($s$) $< Re(c_C), 0 < Re(t) < Re((d_D)$, κ is defined in (2.1),and

$$\Omega_{\mathrm{E:G;M;N}}^{\mathrm{A:B;C;D}}(s,t) = \frac{\Gamma(-\mathrm{s})\Gamma(-\mathrm{t})\Gamma[(a_A)+2s+t]\Gamma[(b_{\boldsymbol{B}})+s+t]\Gamma[(c_C)+s]\Gamma[(d_D)+t]}{\Gamma[(e_E)+2s+t]\Gamma[(g_G)+s+t]\Gamma[\mathrm{m_M}+s]\Gamma[(n_N)+t]}\ (4.2)$$

5. Special cases of (4.1)

Corollary 5.1. For The Kampé de Feriet function, in the notation of Srivastava and Panda [9,p.423 (26), see also [10,p.23 (1.2,1.3)], we obtain

$$\int_{\mathbf{0}}^{\infty}\int_{\mathbf{0}}^{\infty}\boldsymbol{x^{s-1}}\,\boldsymbol{y^{t-1}}\;H_{0:G;M;N}^{0:B;C:D}\left[\begin{matrix}-:(b_B);(c_C);(d_D);\\-:(g_{G)}:(m_M);(n_N);\end{matrix}\;-x,-y\right]dxdy$$

$= \kappa_1 \Omega^{0:B;C;D}_{0:G;M;N}\,(-s,-t)$, (5.1)

where

$\Omega^{0:B;C;D}_{0:G;M;N}(s,t) = \frac{\Gamma(-s)\Gamma(-t)\Gamma[(b_{\boldsymbol{B}})+s+t]\Gamma[(c_C)+s]\Gamma[(d_D)+t]}{\Gamma[(g_G)+s+t]\Gamma[m_M+s]\Gamma[(n_N)+t]}$ (5.2)

$0 < Re(s+t) < Re(b_{\boldsymbol{B}})$, 0<Re$(s) < Re((c_C)$, $0 < Re(t) < Re((d_D)$, κ_1 is defined in (3.2).

Corollary 5.2. If we set B=G=0, we obtain the following result:

$$\int_{\boldsymbol{0}}^{\infty}\int_{\boldsymbol{0}}^{\infty} \boldsymbol{x}^{\boldsymbol{s-1}}\,\boldsymbol{y}^{\boldsymbol{t-1}}\; H^{A:0;C:D}_{E:0;M;N}\left[\begin{matrix}(a_A):-;(c_C);(d_D);\\(e_E):-;(m_M);(n_N);\end{matrix}\; -x,-y\right]dxdy$$

$= \kappa_2\ \Omega^{A:0;C;D}_{B:0;M;N}(-s,-t)$, (5.3)

where

$\Omega^{A:0;C;D}_{B:0;M;N} = \frac{\Gamma(-s)\Gamma(-t)\Gamma[(a_A)+2s+t]\Gamma[(c_C)+s]\Gamma[(d_D)+t]}{\Gamma[(e_E)+2s+t]\Gamma[m_M+s]\Gamma[(n_N)+t]}$, (5.4)

0<Re(2s+t)<Re((a_A)), 0<Re$(s) < Re(c_C)$, $0 < Re(t) < Re((d_D)$, κ_2 is defined in (3.4).

Corollary 5.3. If we set B=C=D=1. G=1, M=N=0, $b_1 = a, c_1 = b, g_1 = c, d_1 = b'$, we obtain the result given by Reed [8]:

$\int_{\boldsymbol{0}}^{\infty}\int_{\boldsymbol{0}}^{\infty} \boldsymbol{x}^{\boldsymbol{s-1}}\,\boldsymbol{y}^{\boldsymbol{t-1}}\,F_1(a,b,b';c;x,y)$dxdy

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(b')}\frac{\Gamma(s)\Gamma(t)\Gamma(a-s-t)\Gamma(b-s)\Gamma(b'-t)}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c-s-t)},\ (5.5)$$

where Re(c)>0, 0<Re(s+t)< Re(a), 0<Re(s)<Re(b), 0<Re(t)<Re(b').

Similarly, if we set B=C=D=1. G=1, M=N=0, $b_1 = a, c_1 = b, g_1 = c, d_1 = b'$ in (3.1)

we obtain the result given by Reed [8]

$$\int_{\boldsymbol{0}}^{\infty}\int_{\boldsymbol{0}}^{\infty} \boldsymbol{x}^{\boldsymbol{s-1}}\,\boldsymbol{y}^{\boldsymbol{t-1}}\,F_2(a,b,b';c,c';x,y)dxdy = \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(b')}\frac{\Gamma(s)\Gamma(t)\Gamma(a-s-t)\Gamma(b-s)\Gamma(b'-t)}{\Gamma(c-s)\Gamma(c'-t)},\ (5.6)$$

where R(c)>0, Re(c$'$) > 0,0<Re(s+t)< Re(a), 0<Re(s)<Re(b), 0<Re(t)<Re(b').

Corollary 5.4. If we set M=N=B=0,G=1,C=D=2,$b_1 = a, c_1 = b, c_2 = b$, $b_1 = a, d_1 = a', d_2, = b', g_1 = c$, we obtain the result given by Reed [8]

$$\int_{\boldsymbol{0}}^{\infty}\int_{\boldsymbol{0}}^{\infty} \boldsymbol{x}^{\boldsymbol{s-1}}\,\boldsymbol{y}^{\boldsymbol{t-1}}\,F_3(a,a',b,b';c;x,y)dxdy = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(a')\Gamma(b')}\frac{\Gamma(s)\Gamma(t)(a-s-t)\Gamma[b-s]\Gamma(b'-t)}{\Gamma(c-s)\Gamma(c'-t)},(5.7)$$

where Re(c)>0, 0<Re(s+t)< Re(a), 0<Re(s)<Re(b), 0<Re(t)<Re(b').

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Corollary 5.5. If we set B=2, C=D=G=0, M=N=1, $m_1 = c, n_1 = c'$, $b_1 = a$, $b_{2,} = b$ in (3.1),we obtainthe following result given by Reed [8]:

$$\int_0^\infty \int_0^\infty \boldsymbol{x^{s-1}}\, \boldsymbol{y^{t-1}}\, F_4(a,b;c,c';x,y)dxdy \;\; = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a-s-t)\Gamma(b-s-t)\Gamma(c')}{\Gamma(c-s)\Gamma(c'-t)}, \quad (5.8)$$

where R(c)>0, Re(c') > 0, 0<Re(s+t)< Re(a), 0<Re(s+t)<Re(b).

6. Fractional integrals and derivatives

The Riemann –Liouville fractional integral of f(t) is defined in [**3,6,7,9,16]**

$$(\mathrm{I}_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \;\; (\alpha \epsilon C. Re(\alpha) > 0, \quad (6.1)$$

The Riemann- Liouville fractional derivative of f (t) is defined as

$$(D_{a+}^{\mu} f)(x) = \begin{cases} \frac{1}{\Gamma(-\mu)} \int_a^x (x-t)^{-\mu-1} f(t)dt \;\; (Re(\mu) < 0) \\ \frac{d^m}{dx^m} D_{a+}^{\alpha-m} f(x) \;\; (m-1 \leq Re(\mu) < m \;(m\epsilon\, N) \end{cases} \quad (6.2)$$

Power function formulas:

If $\alpha, \rho \epsilon C$, $Re(\alpha) > 0$, $Re(\rho) > 0, t > 0$, then [7 , p. 107, eq. (1.122)]

$$(I_{0+}^{\alpha}\, t^{\rho-1})(x) = \frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)} x^{\rho+\alpha-1} \; . \quad (6.3)$$

If $\alpha, \rho \epsilon C$, Re$(\rho) > 0, t > 0$, then [7, p. 107, eq.(1.123)]

$$(D_{0+}^{\alpha} t^{\rho-1})(x) = \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)} x^{\rho-\alpha-1} \; . \quad (6.4)$$

By virtue of the results (6.1) and (6.3), it can be readily seen that

$$\left(I_{0+}^{\alpha}\;\; t^{\rho-1}\, H_{E:G;M;N}^{A:B;C:D} \left[\begin{matrix} (a_A):(b_B);(c_C);(d_D); \\ (e_E):(g_G):(m_M);(n_N); \end{matrix} \;\lambda t,, \mu t \right] \right) (\mathrm{x})$$

$$= \frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)} \;\; x^{\rho+\alpha-1}\, H_{E:G+1;M;N}^{A:B+1;C:D} \left[\begin{matrix} (a_A):(b_B),\rho;(c_C);(d_D); \\ (e_E):(g_G),\rho+\alpha:(m_M);(n_N); \end{matrix} \;\lambda x,, \mu x \right], \quad (6.5)$$

where Re$(\alpha) > 0, Re(\rho) > 0$.

Similarly using (6.4), it is not difficult to establish the formula

$$\left(D_{0+}^{\alpha}\ t^{\rho-1}\ H_{E:G;M;N}^{A:B;C:D}\left[\begin{matrix}(a_A):(b_B);(c_C);(d_D);\\(e_E):(g_{G)}:(m_M);(n_N);\end{matrix}\ \lambda t,,\mu t\right]\right)(x)$$

$$= \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}\ x^{\rho-\alpha-1}\ H_{E:G+1;M;N}^{A:B+1;C:D}\left[\begin{matrix}(a_A):(b_B),\rho;(c_C);(d_D);\\(e_E):(g_{G),\rho-\alpha}:(m_M);(n_N);\end{matrix}\ \lambda x,,\mu x\right], \quad (6.6)$$

where $Re(\rho) > 0$.

7. Special cases of (6.5) and (6.6)

Corollary 7.1. Using (1.2), the following result holds:

$$\left(I_{0+}^{\alpha}\ t^{\rho-1}\ F_{G;M;N}^{B;C:D}\left[\begin{matrix}-:(b_B);(c_C);(d_D);\\-:(g_{G)}:(m_M);(n_N);\end{matrix}\ \lambda t,,\mu t\right]\right)(x)$$

$$= \frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)}\ x^{\rho+\alpha-1}\ F_{G+1;M;N}^{B+1;C:D}\left[\begin{matrix}-:(b_B),\rho;(c_C);(d_D);\\-:(g_{G),\rho+\alpha;}(m_M);(n_N);\end{matrix}\ \lambda x,,\mu x\right], \quad (7.1)$$

where $\mathrm{Re}(\alpha) > 0, Re(\rho) > 0$.

Corollary 7.2. Using (1.3) , the following result holds:

$$\left(I_{0+}^{\alpha}\ t^{\rho-1}\ X_{E;M;N}^{A;C:D}\left[\begin{matrix}(a_A):-;(c_C);(d_D);\\(e_E):-;(m_M);(n_N);\end{matrix}\ \lambda t,,\mu t\right]\right)(x)$$

$$= \frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)}\ x^{\rho+\alpha-1}\ H_{E:1;M;N}^{A:1;C:D}\left[\begin{matrix}(a_A):\rho;(c_C);(d_D);\\(e_E):\rho+\alpha;:(m_M);(n_N);\end{matrix}\ \lambda x,,\mu x\right], \quad (7.2)$$

where $\mathrm{Re}(\alpha) > 0, Re(\rho) > 0$.

Similarly, we can obtain the following special cases of (6.6):

Corollary 7.3. The following result holds:

$$\left(D_{0+}^{\alpha}\ t^{\rho-1}\ F_{G;M;N}^{B;C:D}\left[\begin{matrix}-:(b_B);(c_C);(d_D);\\-:(g_{G)}:(m_M);(n_N);\end{matrix}\ \lambda t,,\mu t\right]\right)(x)$$

$$= \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}\ x^{\rho-\alpha-1}\ F_{G+1;M;N}^{B+1;C:D}\left[\begin{matrix}-:(b_B),\rho;(c_C);(d_D);\\-:(g_{G),\rho-\alpha;}(m_M);(n_N);\end{matrix}\ \lambda x,,\mu x\right] \quad (7.3)$$

where $\mathrm{Re}(\alpha) > 0, Re(\rho) > 0$.

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Corollary 7.4. The following result holds:

$$\left(I_{0+}^{\alpha}\, t^{\rho-1}\ X_{E;M;N}^{A;C:D}\left[\begin{matrix}(a_A):-;(c_C);(d_D);\\(e_E):-;(m_M);(n_N);\end{matrix}\ \lambda t,,\mu t\right]\right)(\mathrm{x})$$

$$= \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}\ x^{\rho-\alpha-1}\, H_{E:1;M;N}^{A:1;C:D}\left[\begin{matrix}(a_A):\rho;(c_C);(d_D);\\(e_E):\rho-\alpha;;(m_M);(n_N);\end{matrix}\ \lambda x,,\mu x\right], (7.4)$$

where $Re(\rho) > 0$.

8. Conclusion. The results obtained in this paper are of general character and results for Mellin –Barnes type integrals and double Mellin transforms of the functions Humbert' seven double hypergeometric functions $\phi_1, \phi_2, \phi_3, \psi_1,\ \psi_2, \Xi_1, , \Xi_2$ $[1, p. 225 - 226(20 - 26)$; see also [8,pp.58-59]. Horn's double hypergeometric function $H_3, H_4[1, p. 225\ (15,16)]$, and its confluent forms $H_6$and H_7[1,p.226(34.35)] are all special cases of the double hypergeometric function due to Exton defined by (1.1) and can be deduced as special cases of findings of the paper. Riemann-Liouville fractional integrals of Humbert' seven double hypergeometric functions etc. can be derived as special cases of (6.5) sand (6.6).

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Complexities in Chaotic and Hyperchaotic Systems

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Abstract

Hyperchaotic systems are those which evolve chaotically for certain set of parameter values and are containing at least two positive Lyapunov exponents. Hyperchaotic systems show complex dynamic behavior. We have studied here complexities in discrete type hyperbolic systems and used proper numerical simulation to obtain Lyapunov exponents, topological and correlation dimension. Positivity of Lyapunov exponents shows chaotic motion. Measure of topological entropy provides information that how complex the system is. Correlation dimension provides the dimensionality of the chaotic attractor. In the processes of investigation, we have drawn the bifurcation diagrams and other interesting graphics to support the evolutionary phenomena.

1. Introduction:

Though there does not exist a unique definition of complexity, however, we can say a system is complex if different parts comprising the system interact with each other in multiple ways. Study of complexity means information that are emerging from a collection of interacting parts. Complexity expresses a condition of numerous elements in a system and numerous forms of relationships among the elements. Usually biological systems are complex and multicomponent. They are spatially structured and their individual elements possess individual properties. Such complexity also effects the system significantly during evolution. In recent years there has been a great emphasis on three concerning phrases: nonlinear dynamics, chaos, and complexity. All natural systems exhibit massive diversity in behavior during evolution. Complex systems are characterized by an internal structure which is built by numerous and varied processes, subsystems and interconnections. Systems featured by complexity display a number of properties such as uncertainty, interactions at different levels, self-organization and nonlinear feedback. Due to its nonlinear structure such systems may display the properties like complexity and chaos. Elaborate descriptions on complexity can be viewed from some well written articles [1 – 6]. Chaos in the system can be better understood by measuring Lyapunov exponents (LCEs), positive LCEs signifies chaos whereas its negative measure means system evolution is regular, [7 – 10].For physical systems, complexity is a measure of the probability of the state vector of the system. Presence of complexity in the system is measured by topological entropy; more topological entropy implies the system is more complex, [11–16].

Hyperchaotic systems are those which contain at least two positive Lyapunov exponentsthroughout a range of parameter space. Hyperchaotic systems are of higher dimensional, (i.e. dimension ≥ 3). Hyperchaotic attractors are importance in various flow pattern, [17–25].The discretized hyperchaotic map proposed by Rossler, [20], was obtained after taking Poincare cross-section of flow of a coupled 4 – D oscillator.

The objective of the present study is based on 3 D folded towel map introduced in [20] which shows hyperchaotic properties. In the processes of investigation, we wish to perform stability analysis of fixed points at a particular parameter space and to do bifurcation analysis on the time evolution of this map. Also, we wish to extend our numerical simulation to obtain Lyapunov exponentsas measure of chaos, topological entropies as a measure of complexities as well as the dimension of the chaotic attractor for certain set of parameter values and initial conditions

2. Mathematical Model, Fixed Points & Bifurcations:

$$\begin{aligned} x_{n+1} &= a\,x_n(1-x_n)-\delta(c+y_n)(1-2z_n) \\ y_{n+1} &= d[(c+y_n)(1+2z_n)-1](1-k\,x_n) \\ z_{n+1} &= r\,z_n(1-z_n)+b\,y_n \end{aligned} \qquad (1)$$

The Jacobian matrix of system (1) be obtained as

$$\mathbf{J}=\begin{pmatrix} \mathbf{a(1-x)-ax} & \mathbf{-\delta(1-2z)} & \mathbf{2\delta(c+y)} \\ \mathbf{-dk[-1+(c+y)(1+2z)]} & \mathbf{d(1-kx)(1+2z)} & \mathbf{2d(1-kx)(c+y)} \\ \mathbf{0} & \mathbf{b} & \mathbf{r(1-z)-rz} \end{pmatrix} \qquad (2)$$

The Jacobian matrix be utilized to determine stability of steady state solutions, i.e., fixed points, and to calculate the Lyapunov exponents for the evolving system.

(a) Fixed Points and Stability:

At parameter values a = 1.9, b = 0.3, c = 0.35, d = 0.1, r = 3.7, δ = 0.05, k = 1.9, one obtains the following five fixed points of system given by:

P_1*(-1.10, 53.14, 1.11), P_2*(0.48, -0.001, 0.73), P_3*(-0.01, -0.02, 0.73),
P_4*(0.45, -0.01, 0.00), &P_5*(0.02, -0.07, 0.001)

Through stability analysis it has been obtained that all above fixed points are unstable. Therefore, orbits originating nearby these fixed points could be unstable orbits also.

(b) Bifurcations:

Bifurcations in the system provide a qualitative change in the behavior of the system during evolution. Bifurcations occur when a particular parameter is allowed to vary within its certain range of values while keeping other parameters constant. We observe different cycles of evolution that leading to chaotic state during the processes of bifurcation. Also, for some cases, phenomena like bistability, periodic windows within chaos etc. may also be observed for some systems. A bifurcation can be taken as a tool to analyze the regular, chaotic as well as complexity within the system. Bifurcation diagrams for system (1) is shown in Fig. 1. We observe here, how the stable solution evolve into chaotic after period doubling phenomena. Within chaos, there are periodic windows, a special characteristics of nonlinear systems. The bifurcations shown in Fig. 1 for parameters b = 0.03, c = 0.35, d = 0.1, r = 3.5, δ = 0.05, k = 1.9 and 2.0 ≤ a ≤ 3.8 in **upper diagrams and** 3.55 ≤ a ≤ 3.65 in lower diagrams.

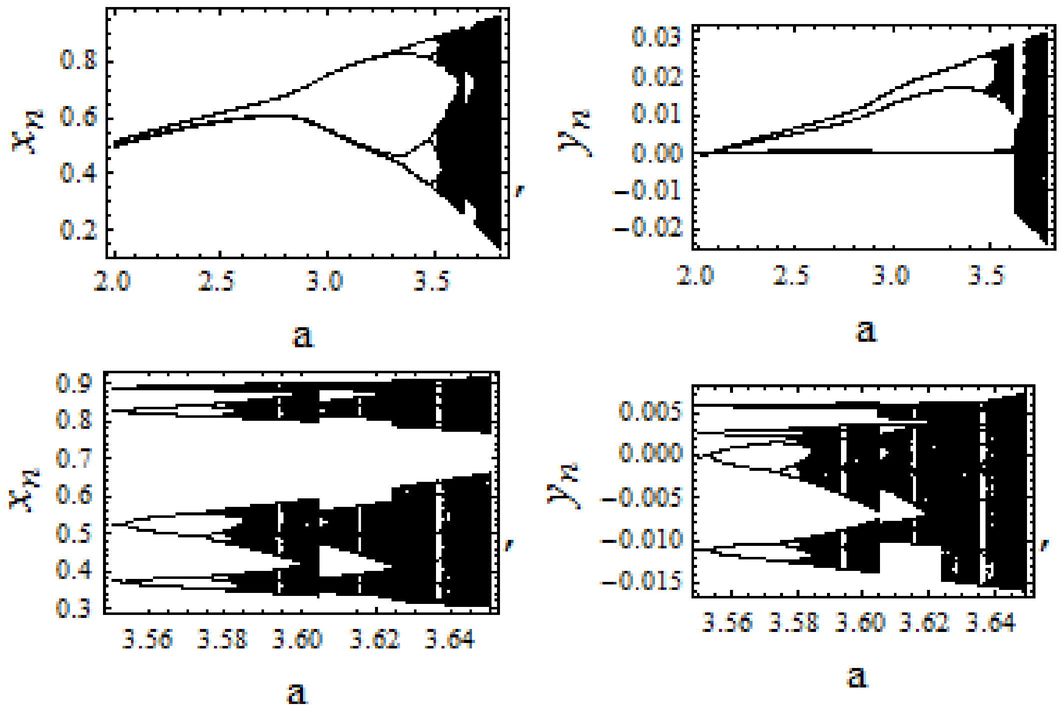

Fig. 1: Bifurcation diagrams for map (1) along x and y directions when b = 0.03, c = 0.35, d = 0.1, r = 3.5, δ = 0.05, k = 1.9 and 2.0 ≤ a ≤ 3.8 in **upper diagrams and** 3.55≤ a ≤ 3.65 in lower diagram.

2. **Numerical Simulations**:

(a) Chaotic Attractor:

The map (1) evolve chaotically of map during evolution. For parameter valuesa = 1.9,b = 0.03, c = 0.35, d = 0.1, r = 3.5, δ = 0.05, k = 1.9. Strange chaotic attractor appear for an orbit with initial conditions (x_0, y_0, z_0) = (0.3, -0.1, 0.1) as shown in Fig. 2. This attractor is of folded type and is a dense chaotic set having fractal structure.

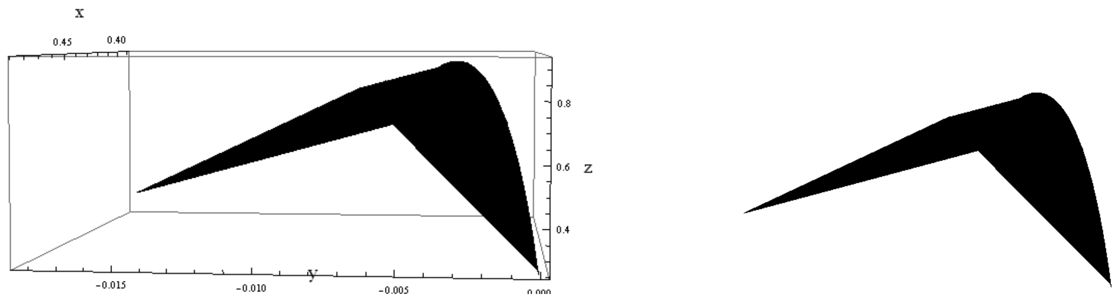

Fig. 2: Sows a folded towel map type chaotic attractor obtained when a = 1.9,b = 0.03, c = 0.35, d = 0.1, r = 3.5, δ = 0.05, k = 1.9 and initial conditions (x_0, y_0, z_0) = (0.3, -0.1, 0.1).

(b) Calculations of Lyapunov Exponents (LCEs):

Lyapunov exponents are perfect indicators of regular and chaotic motion. These are positive for chaotic evolution and negative for regular motion. For map (1), we have calculated LCEs, by using appropriate procedure, [7 – 10, 16, 27, 28]. Plots of LCEs are shown in Fig. 3.

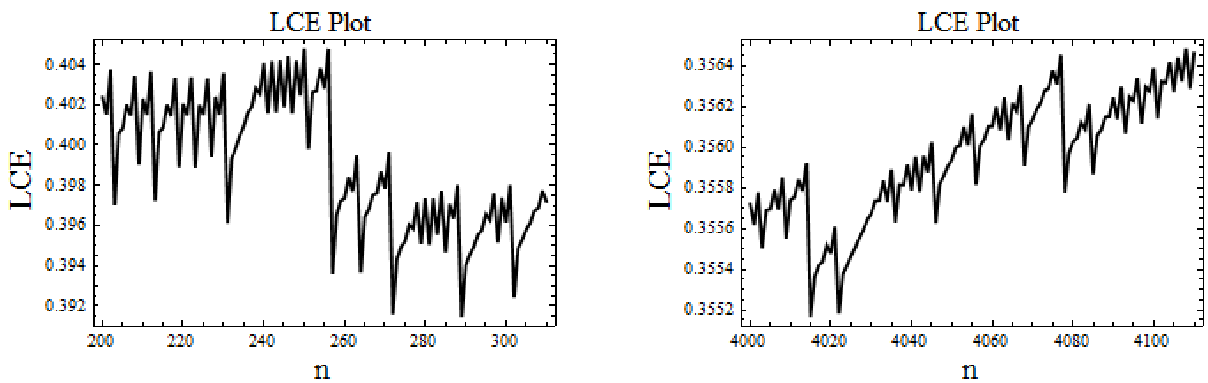

Fig 3: Plots of Lyapunov exponents for a = 1.9,b = 0.03, c = 0.35, d = 0.1, r = 3.5, δ = 0.05, k = 1.9 and initial conditions (x_0, y_0, z_0) = (0.3, -0.1, 0.1).

(c) Topological Entropies:

The system (1) is very complex and can be viewed as that composed on many components which may interact with each other. During evolution, in addition to chaos, it may show some degree of spontaneous order, numerosity and robustness. Complexity in the system is measured by topological entropy which can be calculated statistically in the following way:

Consider a finite partition of a state space X denoted by P = { A_1, A_2, A_3,. . . . , A_N}. Then a measure μ on X with total measure μ(X) = 1 defines the probability of a given reading as

$p_i = \mu(A_i)$, i = 1, 2, . . , N.

Then the entropy of the partition be given by

$$H(p) = -\sum_{i=0}^{N} p_i \log p_i \qquad (3)$$

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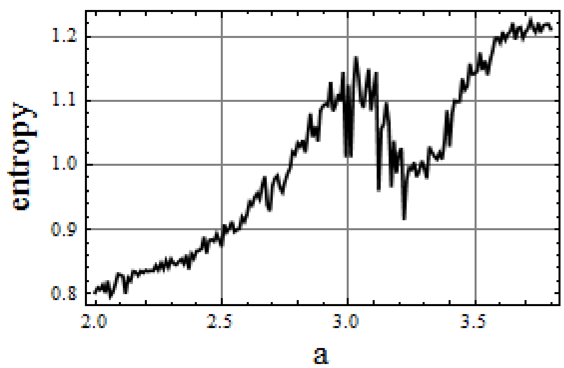
The topological entropy, H (p) is a positive quantity and more topological entropy of a system signifies it is more complex.

Presence of complexity does not mean the system is chaotic and vice versa. In Fig. 4, we have plotted topological entropy for our system for b=0.03, c= 0.35, d= 0.1, r= 3.7, δ=0.05, k = 1.9 and varying parameter a, (2.0≤ a≤ 3.8& 3.0≤ a≤ 3.5). For both the cases, the initial conditions are x = 0.3, y = –0.1 and z = 0.1.



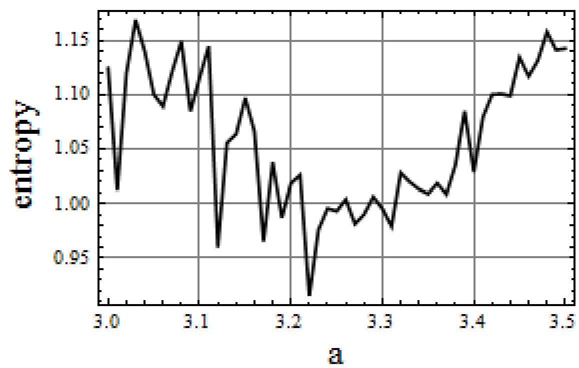


Fig. 4: Plots of topological entropy of the systemfor b=0.03, c= 0.35, d= 0.1, r= 3.7, δ=0.05, k = 1.9 and varying parameter a, (2.0≤ a≤ 3.8& 3.0≤ a≤ 3.5).

(d) Correlation Dimension:

Correlation dimension provides the measure of dimensionality of the chaotic attractor. This is calculated statistically with the application of Heavyside function, [16, 26]. To obtain this, first we have calculated data for correlation integral C(r), for certain r. Then, we have plotted the curve $\frac{\log C(r)}{\log r}$ against r shown in Fig. 5. After this, we have applied a linear fit criterion to the correlation data and obtained the equation of the straight line

$$y = 0.535196 - 0.571383\, x \qquad (4)$$

The y-intercept of this straight line is 0.57066 and so, [16], the correlation dimension of the chaotic attractor Fig. 2(a) is, approximately, given by $\mathbf{D_c \cong 0.535}$.

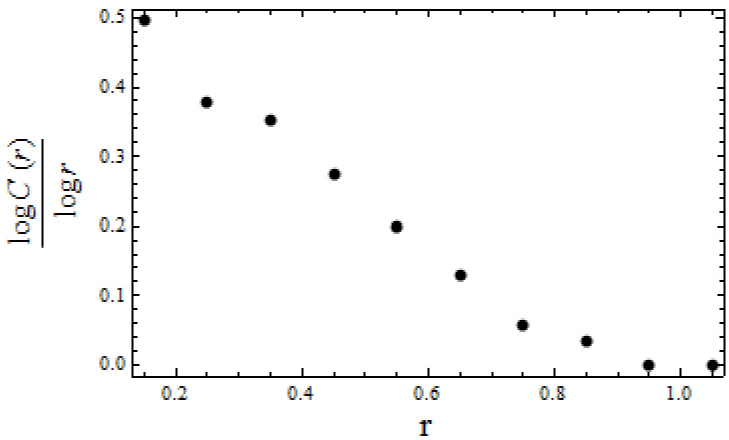


Fig. 5: Plot of correlation integral data

Discussions:

Starting from the bifurcation plots, Fig. 1, and those plots of topological entropy, Fig.4, we can observe the presence of complexity in the system. Also, we find increasing in topological when the system evolution is regular, (e.g. $2.0 \leq a \leq 3.2$). This implies, even if the system is regular, it may exhibit complexity. The correlation dimension of the chaotic attractor is obtained as

$\mathbf{D_c \cong 0.535}$. This shows the fractal property of the chaotic attractor.

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Complexity Investigation In Prey-Predator System with Allee Effect

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Abstract:

A prey-predator system with Allee effect has been studied for chaotic evolution and for complexity behavior. Stability criteria of fixed points have been explained analytically. Bifurcation diagrams have been obtained to analyze the evolutionary phenomena. Regular and chaotic attractors have been obtained. Numerical simulations have been done to obtain plots for Lyapunov exponents (LCEs) which definitely provide regular and chaotic motions. This study further extended to obtain topological entropies and to explain complexities present within the system. Correlation dimension is also obtained for the dimensionality of the chaotic attractor. Interesting graphics are drawn in the processes of study.

1. Introduction:

Most of the biological systems in nature exhibit enormous diversity and structurally multicomponent. The dynamics of such systems are interesting because during evolution, they show chaos and show complexity phenomena. Investigation of dynamic evolution of these system is enables us to find the reasons of extinction of certain species or unbalance in such systems.There has been significant number of articles appeared recently on dynamics of prey-predator and other biological populations under different conditions, [1 – 4].Results emerging from these studies are important and encouraging. Prey-predator problems has been a great emphasis over recent years, [5 – 8], where efforts have been made to study evolution under various conditions of equilibrium of two species.

The Allee effect on prey-predator system is a phenomenon in biology which characterizes certain correlation between population size or density and the mean individual fitness, (often measured as per capita population growth rate), of a population or species, [9 – 12]. In this regard some, discrete models of prey-predator systems with Allee effect, appeared recently are interesting, [13 – 15]. These studies are confined to stability of equilibrium solutions and bifurcation phenomena showng appearing of chaos.

The objective of the present study is to investigate complexity and chaotic evolution in prey-predator system resulting due to Allee effect. Stability of fixed points at certain parameter space are studied in the process of study. Bifurcation diagrams and other numerical simulations are performed for different cases: (i) without Allee effect, (ii) Allee effect on Prey population only, (iii) Allee effect on predator population only and (iv) Allee effect on both, prey and predator, population.

2. **Discrete Prey-Predator Model**:

Most commonly used a prey-predator model can be represented as

$$\begin{aligned} x_{n+1} &= x_n + r x_n (1 - x_n) - a x_n y_n \\ y_{n+1} &= y_n + a y_n (x_n - y_n) \end{aligned} \qquad (1)$$

Where x_n and y_n are, respectively, the densities of prey and predator populations at generation n; parameters r and a are positive constants, a stands for the predation parameter and r is the natural rate of increase in prey population. The terms of model (1) can be interpreted as follows:

- $x_n + r x_n (1 - x_n)$ represents the rate of increase of prey population in absence of predator
- $a x_n y_n$ represents the rate of decrease in prey population due to predation
- $y_n + a y_n (x_n - y_n)$ represents variation of predator density due to prey population

With Allee effect on both, prey and predator population, model (1) can be modified as

$$x_{n+1} = x_n + r x_n (1 - x_n)(1 - e^{-\varepsilon x_n}) - a x_n y_n$$

$$y_{n+1} = y_n + a y_n (x_n - y_n) \left(\frac{y_n}{\mu + y_n} \right)$$

Here,

- $1 - e^{-\varepsilon x_n}$ stands for mate finding Allee effect on prey population, here ε isdefined as the Allee effect constant and the term
- $\dfrac{y_n}{\mu + y_n}$ stand for the Allee effect on predator and here, μ is Allee effect constant. Bigger μ means the stronger the Allee effect on predator population.

Jacobian matrix for system (1) obtained as

$$\mathbf{J_0} = \begin{pmatrix} \mathbf{1 + r - 2rx - ay} & \mathbf{-ax} \\ \mathbf{ay} & \mathbf{1 + ax - 2ay} \end{pmatrix}$$

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The Jacobian matrix for system (2) obtained as

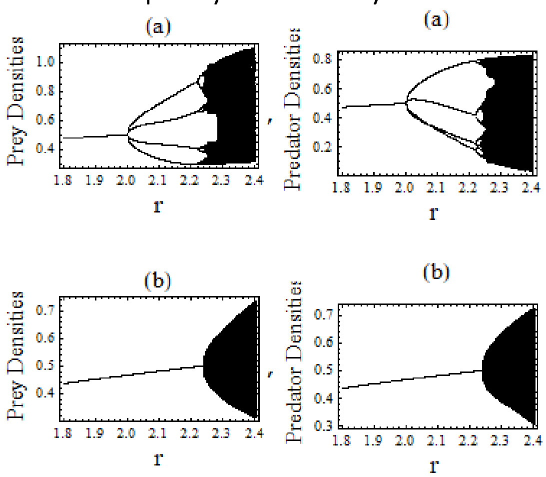
$$J_1 = \begin{pmatrix} e^{-\varepsilon x}[e^{\varepsilon x}(1+r-2rx-ay)+r(x(2+\varepsilon)-1-\varepsilon x^2)] & -ax \\ \dfrac{ay^2}{(\mu+y)} & \dfrac{-2ay^2+y^2[1+a(x-3\mu)]+\mu^2+2\mu y(1+ax)}{(\mu+y)^2} \end{pmatrix}$$

Jacobian matrices J_0 and J_1 play important role during measurement of Lyapunov exponents (LCEs), when the system evolves over time. Here, matrix J_0 has been used when there is no consideration of Allee effect and, similarly, matrix J_1 when Allee effect are applied on both populations. Also, when only prey population is subject to Allee effect or same in case of predator population, one has to re-calculate the Jacobian matrix to obtain LCEs for corresponding cases.

For values of parameters $a=2.0$, $r=2.4$, fixed points of system (1) are obtained, approximately, as $P_1^*(0,0)$, $P_2^*(1,0)$, $P_3^*(0.545455,\ 0.545455)$ and by using stability analysis, we find all are unstable.

Bifurcation Diagrams:

The phenomena of bifurcation provide a qualitative change in the behavior of a system during evolution. Such a change occurs when a particular parameter is allowed to vary while keeping other parameters constant. Bifurcation diagrams show the splitting of a stable solutions within a certain range of values of the parameter. During the processes of bifurcation, one observes different cycles of evolution which leading to the chaotic situation. Phenomena like bistability, periodic windows within chaos etc. may also be observed for some systems. A bifurcation can be taken as a tool to analyze the regular, chaotic as well as complexity within the system.



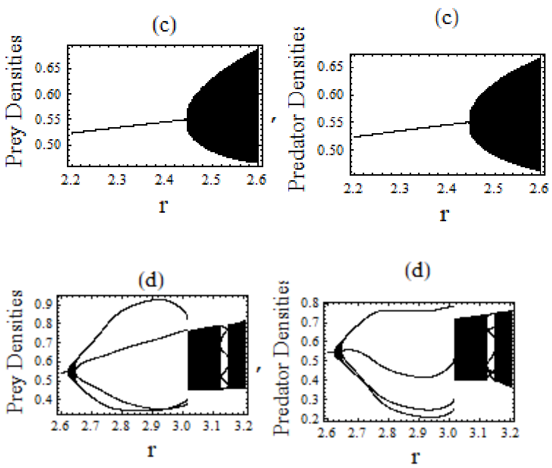


Fig. 1: Bifurcation diagrams for $a = 2.0$ and Fig. 1(a) $: 1.8 \le r \le 2.4$, Fig. 1(b) $: \varepsilon = 4.5.\ \ 1.8 \le r \le 2.4$ Fig. 1(c) $: \mu = 0.1,\ \ 2.2 \le r \le 2.6$, Fig. 1(d) $: \varepsilon = 4.5, \mu = 0.1,\ \ 2.6 \le r \le 3.2$

In Fig. 1, we have presented bifurcation diagrams for the cases: no Allee effect, Allee effect on prey only, Alle effect on predator only and Allee effect on both populations.

3. **Numerical Simulations**:
   (a) **Attractors**:

Keeping parameters a and r fixed, (a = 2.0, r = 2.4), attractors for different cases are obtained through numerical technique, Martelli, [16], and shown in Fig. 2.

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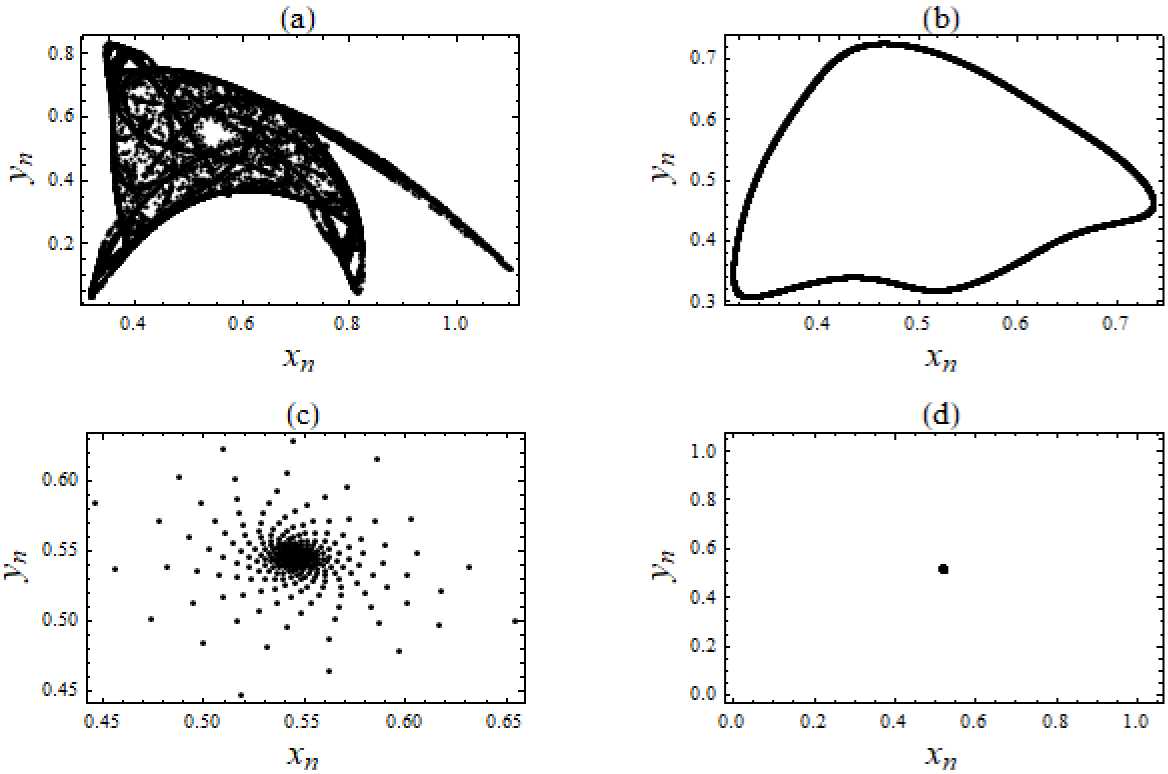

Fig. 2: Plor of regular and chaotic attractors for $a = 2.0, r = 2.4$ and; (i) plot (a) without Allee effect, (ii) plot (b) with Allee effect on prey only, $\varepsilon = 4.5$, (iii) plot (c) Allee effect on predator only $\mu = 0.1$, and (iv) plot (d) Allee effect on prey as well as on predator, $\varepsilon = 4.5$, $\mu = 0.1$.

Looking plots of attractors of Fig. 2, one finds a chaotic attractor, figure (a) when Allee effect is not in consideration, for $a = 2.0, r = 2.4$. But, when Allee effect is applied to either of the population or to both population, system returned to regularity.Attractors shown in figures (b), (c) and (d) are no more chaotic. This also follow from the plots of LCEs given below.

(b) Lyapunov Exponents (LCEs):

To indicate chaotic and regular evolution, an appropriate measure is to find Lyapunov exponents, LCEs, are obtained for different cases by using appropriate procedure, [16 – 21]. Plots of LCEs are shown in Fig. 3.

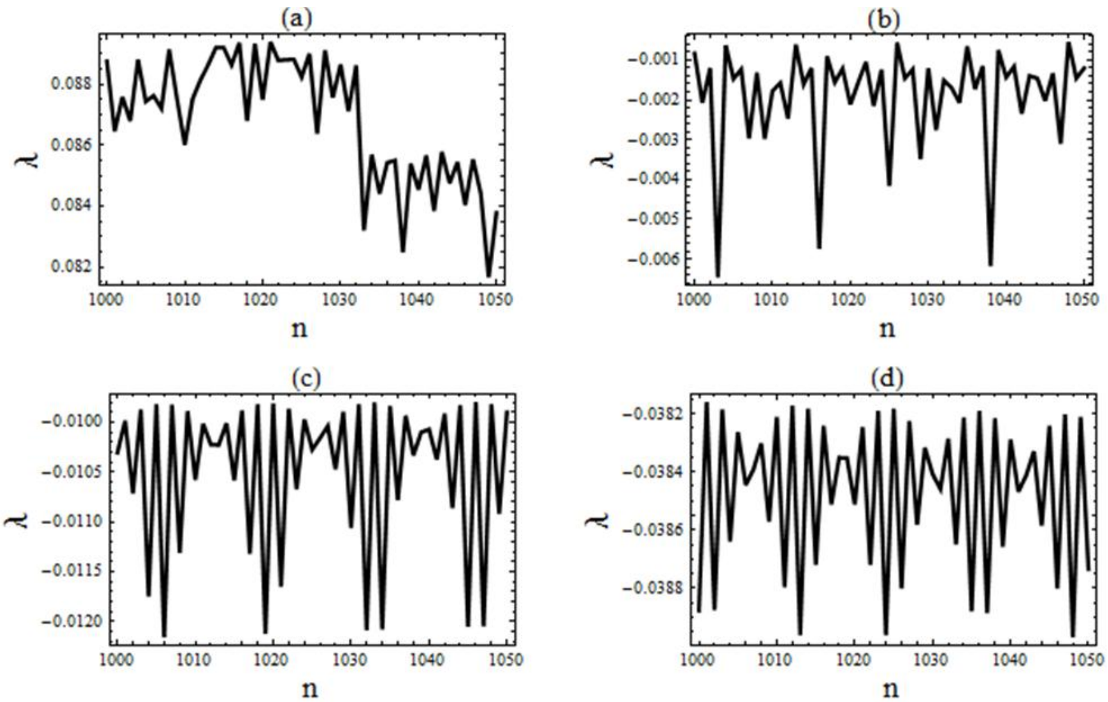

Fig. 3: Plots of Lyapunov exponents for $a = 2.0, r = 2.4$ and (i) figure (a) without Allee effect, (ii) figure (b) with $\varepsilon = 4.5, \mu = 0$, (iii) figure (c) Allee effect on predator only with $\mu = 0.1$, (iv)Allee effect on both population $\varepsilon = 4.5, \mu = 0.1$.

(c) **Correlation Dimension**:

Correlation dimension provides the measure of dimensionality of the chaotic attractor. This is calculated statistically with the application of Heavyside function, [16, 17].To obtain this, using the technique in [16], first we have calculated data for correlation integral C(r), for certain r, (other than the parameter shown in the models (1) & (2)). Then, we have plotted the curve $\frac{\log C(r)}{\log r}$ against r shown in Fig. 4. After this, we apply a linear fit criterion to the correlation

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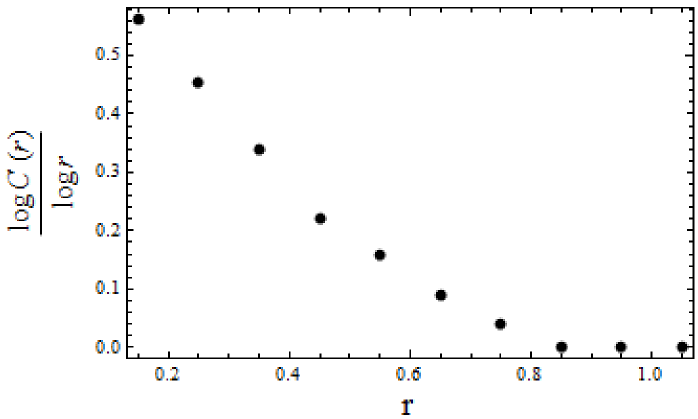


Fig. 4: Plot of correlation integral data.

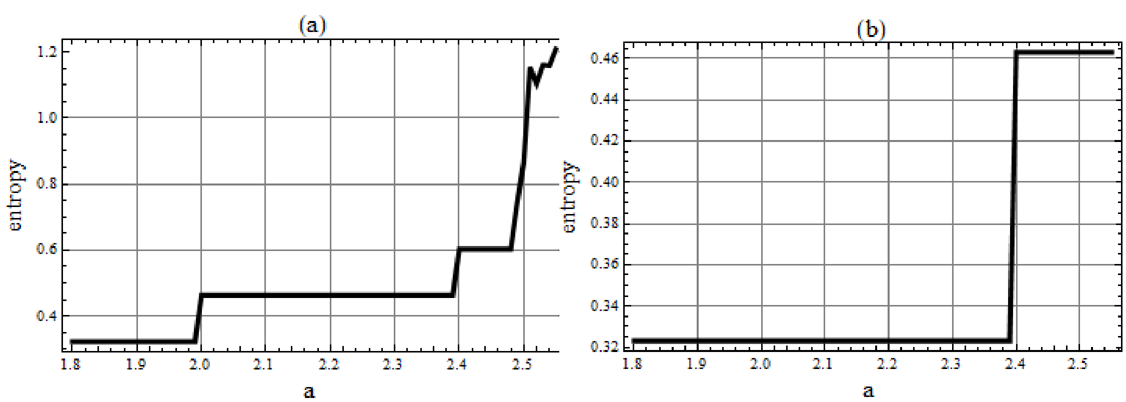
data and obtained the linear fit equation

$$y = 0.57066 - 0.640236\, x$$

The y-intercept of this straight line is 0.57066 and so, [16], the correlation dimension of the chaotic attractor Fig. 2(a) is, approximately, given by $\mathbf{D_c \cong 0.571}$.

(d) **Topological Entropies**:

As explain in the beginning, topological entropy measures the complexity of the system. More topological entropy implies system is more complex. Presence of complexity does not mean the system is chaotic and vice versa. In Fig. 5, we have plots of topological entropy for different cases. In figure (a), topological entropy increases for $r > 2$ but bifurcation diagrams and calculations of LCEs indicate the system is regular within $2.0 \leq r \leq 2.2$. Similar observation can be made looking figures (b) and (c). In figure (d) one finds no fluctuations of topological entropy, it establishes a steady state situation.



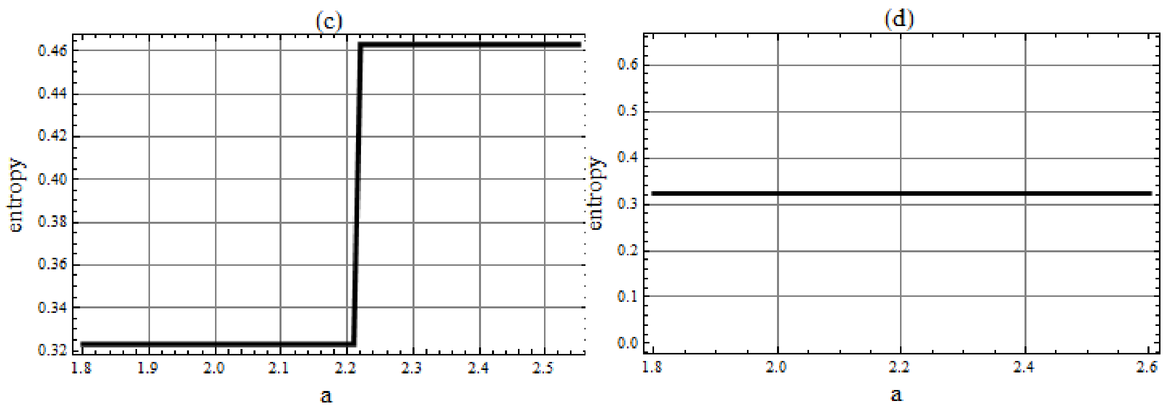


Fig. 5: Plots of topological entropies for $a = 2.0$ and $1.8 \leq r \leq 2.6$: (i) figure (a) with no Allee effect, (ii) figure (b) when $\varepsilon = 4.5$, $\mu = 0$, figure (c) with Allee effect on predator only $\mu = 0.1$, (iv)when $\varepsilon = 4.5, \mu = 0.1$

4. **Discussions**:

The results obtained through bifurcation plots, Fig. 1, and those of LCEs plots, Fig.3, show that the Allee effect stabilize the motion from chaos to regularity. With Allee effect, LCEs are negative in all cases, where these were positive without this effect. The correlation dimension of the chaotic attractor is obtained as $\mathbf{D_c \cong 0.571}$. Also, in this study we find the existence of complexity within the system, Even when system behavior is regular, we find significant amount of increase in topological entropy.This implies the fact that the system may be regular but may exhibit complexity.

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Professional Writing in Mathematics: Removal of Anathemas

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Abstract

During the World Mathematics Year 2000, it was recommended that the publication of research papers and text books in mathematics should be professional. We explain professionalism in the context of writing in science / technology / mathematics. In this paper, we confine to certain anathemas or lapses in mathematics presentation: Megalomania, symbol as the starter of a mathematical sentence, passive voice, obvious claims, prolixes,and cluttering indices in tensors. We give the rectification of each anathema, by citing the professional counterpart. These discussions on professionalwritings will help the researchers to improve the quality of their publications to suit the 21st century demand of professionalism in mathematical writing.

1. Introduction

Evolution of mathematics as a world-wide industry: Mathematics was a *cottage industry* – managed by a few amateurs - before the 20th century. Later on during the years 1901-2000 A.D.researchers have transformed mathematics into a *world-wideindustry*, needing the services of an army of professionals *vide*the presidential address of the World Mathematics Year 2000 by Sir Michael Atiyah in *Mathematics: Frontiers and Prospects,* American Mathematical Society, p. viii, 2000. This implies that *professionalism in reporting of research* is essential in the 21st century due to the exponential proliferation of mathematics: 63 Primary subjects and 7000 Subsidiary subjects (American Subject Classification Scheme 2010). The first book on professional writing in mathematics was published in 2013, by this author, under the title

"Write Mathematics Right:
Principles of Professional Presentation,
Exemplified with Humor and Thrill."

The international edition of the book was published by Alpha Science International Ltd, Oxford, U. K. and the Indian edition is due to Narosa Publishers Delhi. For details see www.alphasci.com and www.narosa.com.

2. Professionalism in Mathematics Writing

In the context of scientific / technical / mathematical writing, we exemplify the word 'professionalism' to connote the three types of pleasantness:

- Etiquettes - pleasantness to the professional colleagues like authors, editors, referees and researchers – accelerate publishability
- Euphony - pleasantness to the ears of the audience - promotes speakability
- Elegance – pleasantness to the eye and the mind of the reader - enhances readability and printability of research.

In the book Write Mathematics Right, 142 explicit principles of professional writing are constructed that help a researcher avoid 142 types of mistakes in his thesis/ publications. On the said book, Cristin Zanella an editor of the American Mathematical Society, U. S. A.,opined:

Important reasoned guidance to the mathematical authors.

Stephen Chang, the Director of Alpha Science International, Oxford, U.K. commented that the book is an *important database*.

3. Anathemas (Casual Writings) & Their Replacement (Professional Writing)

[A] <u>Etiquettes</u>

Etiquettes are rules of formal behaviour, which create pleasantness towards professional colleagues. In fact, they are the international norms for good exposition nurtured in the literature.

(i) **Claim: Megalomania is an anathema in scientific writing:** Something that is vehemently detested in the mathematics literature is referred as

(ii) a mathematical anathema. Accordingly, megalomania - exaggerated importance of one's self- is an anathema and so a distractor of etiquettes. Thus, we have Principle of Professional Writing:*Replace first person singular by first person plural to avoid megalomania.*

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Example of casual writing: I prove Theorem 5 on Page 10.
Example of professional writing: We prove Theorem 5 on Page 10.
Remarks:(1) Here 'we' refers to the writer and the writee (reader).
(2) In reviews of research papers, short notes, in essays - where *personal tastes* matter - we can justify the use of "I". A research paper has to report impersonal reproducible results.

(ii)Claim:The word 'obvious' is a mathematical anathema.

Casual writing: It is obvious that 4 cannot be the 4th term in the sequence <4, 8, 12, …>.
Professional writing: In the sequence

$$<4, 8, 12,\cdots, 4n +(-2) (n- 1) (n – 2) (n – 3) + \cdots>$$

the 4th term is 4.
Verification: Let the general term of the sequence be denoted by

$$t_n = 4n + (-2)(n-1)(n-2)\ (n-3). \quad [1]$$

When we substitute n = 4 in [1], we have

$$t_4 = 4.4 + (- 2). 3. 2.1= 4.$$

Consequently we can present the sequence as

$$<4, 8, 12, 4, \cdots, 4n + (-2)(n-1)(n-2)(n-3), \cdots >. [2]$$

Note: If the general term would have been given ast_n = 4n, then we gett_4 = 16, and the sequence is

$$<4, 8, 12, 16,\cdots, 4n, \cdots> .$$

Remark: (1) Mathematics is an exact science and so needs proof of every claim! For a sequence, the form of the general term isessential.Knowledge of the first three terms is not sufficient, to determine the whole sequence!
(2) In casual writing, 'it is obvious' is an impersonal statement. This is not a responsible statement, as there is no proof.

[B] Euphony

(i) Claim: Symbol as a sentence starter is an anathema

Principle of professional writing: *A mathematical sentence should not start with a symbol, but with the clarification of the symbol.*

(i)Casual writing: F is a very intricate mathematical object and is just the sort of thing that mathematicians delight in.

- Page 4 –

Professional writing: The Cantor set F is a very intricate mathematical object and is just the sort of thing that mathematicians delight in.

Remark: Thereading of casual writing (starting with F) jars the ear right in the beginning, which is not pleasant to the ear. This does not contribute to euphony. Professional writing starts with the explanation of F and thus contributes to pleasantness to hear, that is euphony.

(ii) Use of passive voice is an anathema in writing

Principle of professional writing: For a lively and persuasive style use active voice.

Casual writing:" Every infinite set of real numbers is in one-to-one correspondence with either the set of natural numbers or else the setoff all reals" was asserted by the Continuum Hypothesis.

Professional writing:The Continuum Hypothesis assertsthat 'every infinite set of realnumbers is in one-to-onecorrespondencewith either the set of natural numbers or else the set of all reals'.

Remark: Usage of *neutral subject* like "Continuum Hypothesis" facilitates the avoidance of passive voice.

[C]Elegance

(i) Claim: Clutter of free indices in tensors is an anathema

Principle of Professional Writing: For clarity, in printing of tensors choose *b, d, h, k, t* (letters extending above the line of print) as the superscripts and *g, j, p, q, y* as subscripts (letters extending below the line of print).

Casual writing:We characterize the curvature of space-time by the Riemann-Christoffel tensor field

$$R^{pq}_{hk} \qquad (p, q, h, k = 1,2, 3, 4).$$

Professional Writing:We characterize the curvature of space-time by the Riemann-Christoffel tensor field

$$R^{hk}_{pq} \qquad (p, q, h, k = 1,2, 3, 4).$$

Remarks: By choosing the free indices *p, q, h, k* as inprofessional writing, we improve the appearance and ease of reading the tensor. The clutter of indices - an anathema - in the casual writing is easy to recognise, when hand written!

- Page 5 –

(iii) Claim: Prolixes in Proofsare an anathema.

Introduction:In long derivations, especially in research papers/theses, the non-technical words

'since', 'hence', 'get'

appear and reappear. In order to escape the monotony of repetition, we list below some alternate words – synonyms/elegant variations – to the three words. Not all the synonyms in *thesaurus* – converse of dictionary – are elegant. Long synonyms are called prolixes which are unpopular in mathematical writing.

[1] Synonyms of the adverb 'hence': therefore, consequently, it follows that, so, accordingly, thus.

Prolixes (long synonyms) of 'hence': as a result of, there for, wherefore, whence, thence, in consequence, for this reason, *ergo* (Latin).

[2] Synonyms of the conjunction 'since'; because, as.

Prolixes for 'since': on account of, in view of, in view of the fact that, in consideration of, for the reason that, by virtue of.

[3] Synonyms of the verb 'get': obtain, find, derive, arrive at, reach, gain, achieve.

Prolixes for 'get': beget, proliferate, procure, acquire, deduce, secure, approach, and pop up (colloquialism).

Casual writing: Use prolixes in proofs/ deductions.

Professional writing: Use short synonyms in proofs/ deductions.

Note: For technical words like *set, group, ring, field* thereare no synonyms/prolixes.

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EXTENSIONS AND UNIFICATIONS OF VOIGT FUNCTIONS WITH THEIR REPRESENTATIONS

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Abstract: This paper aims at some representations of generalized Voigt functions due to Klusch [3], M.A. Pathan and M.J. Shahwan [6] and M.A.Pathan [5] and their extensions in terms of series and integrals which are especially useful in situations when the parameters take on particular values. Explicit representations of these functions are given in terms of familiar special functions of one and two variables. The Voigt integrals and series resulting in connections with the Lommel, Struve, Laguerre and parabolic cylinder functions and ultimately the Kampe de Feriet functions will follow as natural consequences for analytical evaluations and uses.

Keywords: Voigt function, Bessel function, parabolic cylinder function, hypergeometric function and Laguerre polynomials.

AMS Subject Classifications:

1. Introduction

The familiar Voigt functions $K(x,y)$ and $L(x,y)$occur in great diversity in astrophysical spectroscopy, neutron physics, plasma physics, physics of stellar atmospheres, probability and statistical communication theory, as well as in some areas of mathematical physics and engineering. Representations (integrals and series) of the Voigt functions have been given by number of workers, for examples, Srivastava and Miller [12], Srivastava, Pathan and Kamarujjama [13], Pathan, Kamarujjama and Alam [7], Klusch [3] and Pathan and Shahwan [6] etcetera.

We begin by recalling here the following representations (see [4],[5] and [8])

$K(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(-yt - \frac{1}{4}t^2\right) \cos(xt)\, dt \; (x \in R; y \in R^+)$ (1.1)

and

$L(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(-yt - \frac{1}{4}t^2\right) \sin(xt)\, dt \; (x \in R; y \in R^+)$(1.2),

so that

$K(x,y) \pm iL(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp[-(y \mp ix)t - \frac{1}{4}t^2]\, dt = \exp[(y \mp ix)^2]\{1 - erf(y \mp ix)\}$(1.3),

where the error function $erf(z)$ is given by (see Srivastava and Kashyap ([14]; p. 17 (71))

$$erf(z) = \frac{2Z}{\sqrt{\pi}}\; {}_1F_1\left[\frac{1}{2}\,;\, \frac{3}{2}\,;\, -z^2\right] = \frac{2Z}{\sqrt{\pi}} \exp(-z^2)\; {}_1F_1\left[1\,;\, \frac{3}{2}\,;\, z^2\right] (|Z| < \infty)(1.4)$$

and ${}_1F_1$ is the familiar confluent hypergeometricfunction Erdelyi et al ([1]; p. 248,Eq.6.1(1)).

An obvious error in the expression $K(x,y) + iL(x,y)$ has been corrected by Srivastava and Miller [12] (see also Srivastava and Chen [10]).The reader's attention is also drawn toward the clearly-stated comments and observations about the inaccurate and incorrect developments surrounding the papers dealing with Voigt functions listed in [10, 13]. These comments and observations were first made by Srivastava and Chen ([10]; p. 69 (footnote)) reiterated in ([13], p. 50 and p. 53].

For the Bessel function J_υ of the first kind (and of order υ), defined by

$J_\upsilon(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (Z/2)^{\upsilon+2m}}{m!\; \Gamma(\upsilon+m+1)} (|Z| < \infty)$(1.5),

it is well known that

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi Z}} \cos(z) \quad \text{and} \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi Z}} \sin(z) (1.6).$$

Motivated by the relationships (1.6), Srivastava and Miller ([12], p.113 (8)) introduced and studied systematically an unification (and generalization) of the Voigt functions $K(x,y)$ and $L(x,y)$ in the form:

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$V_{\mu,v}(x,y) = (\frac{x}{2})^{\frac{1}{2}} \int_0^{\infty} t^{\mu} \exp\left(-yt - \frac{1}{4}t^2\right) J_v(xt)\, dt \ (x, y \in R^+; \mathrm{Re}(\mu + v) > -1)$
(1.7),

so that

$K(x,y) = V_{\frac{1}{2},-\frac{1}{2}}(x,y)$ and $L(x,y) = V_{\frac{1}{2},\ \frac{1}{2}}(x,y)$ (1.8).

Subsequently, following the work of Srivastava and Miller [12] closely, Klusch [3] proposed unification (and generalization) of the Voigt functions $K(x,y)$ and $L(x,y)$ in the form:

$K(x,y,z) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \exp(-yt - z\, t^2) \cos(xt)\, dt \ (x \in R; y, z \in R^+)$(1.9)

and

$L(x,y,z) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \exp(-yt - z\, t^2) \sin(xt)\, dt \ (x \in R; y, z \in R^+)$(1.10),

so that

$$K(x,y,z) \pm iL(x,y,z) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \exp[-(y \mp ix)t - zt^2]\, dt$$

$= \frac{\exp[(y \mp ix)^2]}{2\sqrt{z}} \left\{1 - erf\left(\frac{y \mp ix}{2\sqrt{z}}\right)\right\}$(1.11),

Klusch [3] proposed a unification (and generalization) of the Voigt functions$K(x,y,z)$ and $L(x,y,z)$in the form:

$\Omega_{\mu,v}(x,y,z) = \left(\frac{x}{2}\right)^{\frac{1}{2}} \int_0^{\infty} t^{\mu} \exp(-yt - zt^2) J_v(xt) dt \ (x, y, z \in R^+; Re(\mu + v) > -1)$(1.12),

so that

$$\Omega_{\frac{1}{2},-\frac{1}{2}}(x,y,z) = K(x,y,z) \text{and } \Omega_{\frac{1}{2},\frac{1}{2}}(x,y,z) = L(x,y,z) \text{(1.13).}$$

Relationship (1.12) can indeed be used to obtain many of Klusch's results [3] for $\Omega_{\mu,v}(x,y,z)$ and those given by Srivastava and Miller [12] by taking $Z = \frac{1}{4}$.

Recently, M.A. Pathan and M.J.S. Shahwan [6] defined an extension of $V_{\mu,v}(x,y)$ in the form

$$\Omega_{\mu,\alpha,\beta,v}(x,y) = \sqrt{\frac{x}{2}}\int_0^\infty t^\mu \exp\left(-yt - \frac{1}{4}t^2\right){}_1F_2\left[\alpha\,;\ \beta, 1+v;\ -\frac{x^2\,t^2}{4}\right]\,dt$$

$(\mu, x, y \in R^+;\ \mathrm{Re}(\mu+v) > -1)$(1.14).

Finally, M.A. Pathan [4] has given a generalization of $\Omega_{\mu,v}(x,y,z)$ in the following form:

$$\Omega_{\mu,v}^{(j)}(x,y,z) = \sqrt{\frac{x}{2}}\int_0^\infty t^\mu \exp(-yt - zt^j)J_v(xt)\,dt$$

$(j \in Z^+;\ x, y, Z, \mu \in R^+;\ \mathrm{Re}(\mu+v) > -1)$(1.15).

The purpose of the paper is to provide a natural further step toward the extension of the generalized Voigt functions (1.14), (1.15) in the form $\Omega_{\mu,v}^{(j)}(x,y,z)$. Explicit representations of these functions in integral and series, resulting in connection with Lommel, Struve, Laguerre, parabolic cylinder functions and ultimately ending in the Kampe de Feriet function [11] are presented.

2. An extension of $\Omega_{\mu,v}^{(j)}(x,y,z)$ and $\Omega_{\mu,\alpha,\beta,v}(x,y)$

The function

$$\Omega_{\mu,\alpha,\beta,v}^{(j)}(x,y,z) = \sqrt{\frac{x}{2}}\int_0^\infty t^\mu \exp(-yt - zt^j){}_1F_2\left[\alpha\,;\ \beta, 1+v;\ -\frac{x^2\,t^2}{4}\right]\,dt$$

$(j \in Z^+;\ x, y, Z, \mu \in R^+;\ \mathrm{Re}(\mu+v) > -1)$(2.1),

defines an extension of Eq. (1.15) and exhibits the fact that

$$\Omega_{\mu,\alpha,\alpha,v}^{(j)}(x,y,z) = \Gamma(v+1)\left(\frac{2}{x}\right)^v \Omega_{\mu-v,v}^{(j)}(x,y,z)\ (2.2)$$

or, equivalently that

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$$\Omega^{(j)}_{\mu,\upsilon}(x,y,z) = \frac{\left(\frac{x}{2}\right)^{\upsilon}}{\Gamma(\upsilon+1)}\ \Omega^{(j)}_{\mu+\upsilon,\alpha,\alpha,\upsilon}(x,y,z) \quad (2.3).$$

We recall ([9]; p. 108(1)) that, $J_\upsilon(z) = \frac{\left(\frac{x}{2}\right)^{\upsilon}}{\Gamma(\upsilon+1)}\ {}_0F_1\left(-;\upsilon+1;-\frac{z^2}{4}\right)$,(2.4)

where ${}_0F_1$ is hypergeometric function [9].

In fact, when $\alpha=\beta$, Bessel functions $J_\upsilon(z)$ defined above by (2.4) and ${}_1F_2$ are contained as special cases in the generalized hypergeometric function ([11]; p. 29(4)).

On the other hand, taking $z=\frac{1}{4}$ and $j=2$, Eq. (2.1) reduces to Eq. (1.14).Clearly, we have

$$\Omega_{\mu,\alpha,\beta,\upsilon}(x,y) = \Omega^{(2)}_{\mu,\alpha,\beta,\upsilon}(x,y,\tfrac{1}{4}) \quad (2.5)$$

and

$$V_{\mu,\upsilon}(x,y) = \frac{\left(\frac{x}{2}\right)^{\upsilon}}{\Gamma(\upsilon+1)}\Omega^{(2)}_{\mu+\upsilon,\alpha,\alpha,\upsilon}(x,y,\tfrac{1}{4}) \quad (2.6)$$

3. Explicit representation for $\Omega^{(j)}_{\mu,\alpha,\beta,\upsilon}(x,y,z)$

The well-known Hankel exponential integral

$$\int_0^\infty t^{\mu-1}\exp(-p^2t^2)\,J_\upsilon(xt)dt = \frac{\Gamma\frac{(\upsilon+?)}{2}}{2p^{?}\Gamma(\upsilon+1)})(\frac{x}{2p})^{\upsilon}\ {}_1F_1\left(\frac{\upsilon+?}{2};\upsilon+1;-\frac{x^2}{4p^2}\right)$$

which, upon setting $\upsilon=0$ and $?=2$ reduces to the Weber's first exponential integral

$$\int_0^\infty t\exp(-p^2t^2)\,J_0(xt)dt = \frac{1}{2p^2}\exp(-\frac{x^2}{4p^2})$$

where $\mathrm{Re}(\mu+\upsilon)>0$, $\mathrm{Re}\,p>0$ and $0<x<\infty$.

We can identify the above two integrals and perhaps some more unknown variants of the classical exponential integrals including the following one

$$\int_0^\infty t^{s-1}\exp\left(-at^\lambda - bt^\mu\right)dt = \sum_{r=0}^\infty \frac{\left(-ba^{\frac{-\mu}{\lambda}}\right)^r \Gamma(s+\mu r)}{\lambda a^{\frac{s}{\lambda}} r!}$$

$(\mathrm{Re}(s) > 0,\ \mathrm{Re}\,(a, b, \lambda, \mu > 0\ )$. (3.1)

This result can be easily established, in the usual way, by direct expansion of the exponential functions and term wise integration with the aid of the Mellin transform.

To obtain the various representations for the generalized Voigt function $\Omega^{(j)}_{\mu,\alpha,\beta,v}(x, y, z)$,we first use the series representation of ${}_1F_2$ in (2.1) and integrate the resulting series term by term with the help of the result

$$\int_0^\infty t^\mu \exp\left(-pt - \beta t^\lambda\right)dt = \sum_{r=0}^\infty \frac{(-\beta)^r\,\Gamma(\mu+1+\lambda r)}{r!\; p^{\mu+1+\lambda r}}$$

$$(\mathrm{Re}(\mu+1) > 0, \qquad \mathrm{Re}\,p > 0, \qquad \mathrm{Re}\,\lambda > 0\ )\,,$$

which is a special case of the result (3.1). We thus obtain

$$\Omega^{(j)}_{\mu,\alpha,\beta,v}(x, y, z) = \frac{(x)^{\frac{1}{2}}}{\sqrt{2}(y)^{\mu+1}} \sum_{m,r=0}^\infty \frac{(\alpha)_m\,\Gamma(\mu+jr+2m+1)}{(\beta)_m\,(v+1)_m\;m!\,r!}\left(\frac{-x^2}{4y^2}\right)^m \left(\frac{-z}{y^j}\right)^r \quad (3.2).$$

This concept provides the basis of investigations and further extensions of Voigt and many other integrals by many mathematicians over the years.

4. Special cases

In this section we derive several representations of $\Omega^{(j)}_{\mu,\alpha,\beta,v}(x, y, z)$ in terms of series and integrals which are specially useful in situations when the parameters take on particular values. Since in general, the Voigt function and hypergeometric functions can be decomposed into a sum of two confluent hypergeometric functions or Whittaker functions of conjugate complex arguments, therefore the following special cases are worthy of note.

I. Setting $j = 2$ and then using Legendre's duplication formula ([11]; p.23(26)), Eq (3.2) would reduce to the following explicit expansions of the generalized Voigt function

$$\Omega^{(2)}_{\mu,\alpha,\beta,v}(x, y, z) = \frac{\Gamma(\mu+1)}{(y)^{\mu+1}}\sqrt{\frac{x}{2}}\,F^{2:\;\;1;\;\;0}_{0:\;\;2;\;\;0}\left[\begin{matrix}\frac{\mu+1}{2}, & \frac{\mu+2}{2}: & \alpha;-; \\ & -:\beta, & v+1;-;\end{matrix}\;\frac{-x^2}{y^2},\;\frac{-4z}{y^2}\right] (4.1),$$

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where $F^{p:\ q;\ r}_{l:\ m;\ n}$ is called Kampe de Feriet hypergeometric functions of two variables ([6]; p. 63(16)).

II. Taking $z=\frac{1}{4}$, Eq. (4.1) reduces to ([6]; p.78(3.3)) which is given by M.A.Pathan and M.J.S.
Shahwan.

III. Taking$z=\frac{1}{4}$,replacingμ by $\mu+\upsilon$, letting $\alpha=\beta$, multiplying both sides of Eq.(4.1) by $\frac{\left(\frac{x}{2}\right)^{\upsilon}}{\Gamma(\upsilon+1)}$ and making useof Eq. (2.6), we get the following representation of Voigt function

$$V_{\mu,\upsilon}(x,y)=\frac{\Gamma(\mu+\upsilon+1)}{\Gamma(\upsilon+1)(y)^{\mu+\upsilon+1}}\left(\frac{x}{2}\right)^{\upsilon+\frac{1}{2}}F^{2:\ 0;\ 0}_{0:\ 1;\ 0}\left[\begin{matrix}\frac{\mu+\upsilon+1}{2}, & \frac{\mu+\upsilon+2}{2}: & -; & -; & \frac{-x^2}{y^2}, & \frac{-1}{y^2}\\ & -: & \upsilon+1; & -; & & \end{matrix}\right] (4.2),$$

For $\mu=-\upsilon=\frac{1}{2}$ and $\mu=\upsilon=\frac{1}{2}$, formula (4.2) evidently reduces to representation of Voigt functions
$k(x,y)$ and $L(x,y)$, respectively.

IV. In Eq. (3.2), replacing μ by $\mu+\upsilon$, letting $\alpha=\beta$, multiplying both sides by $\frac{\left(\frac{x}{2}\right)^{\upsilon}}{\Gamma(\upsilon+1)}$ and making use of Eq. (2.3), we get a result of M.A. Pathan ([4]; p. 13(4.2)).

V. Setting $j=2$, taking $\alpha=1,\beta=\frac{3}{2}$, replacing $1+\upsilon$ by β and making use of ([8]; p. 608(13)), then Eq. (2.1)yields to the following result

$$\Omega^{(2)}_{\mu,1,\frac{3}{2},\beta-1}(x,y,z)=\frac{\sqrt{\pi}}{2}\left(\frac{x}{2}\right)^{1-\beta}\Gamma(\beta)\int_0^{\infty}t^{\mu-\beta+\frac{1}{2}}\exp(-yt-z\,t^2)\,H_{\beta-\frac{3}{2}}(xt)dt (4.3),$$

where $H_{\upsilon}(z)=\frac{\left(\frac{z}{2}\right)^{\upsilon+1}}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\upsilon+\frac{3}{2}\right)}\ {}_1F_2\left(\begin{matrix} & 1\,; & \\ \frac{3}{2}, & \upsilon+\frac{3}{2}\,; & \frac{-z^2}{4}\end{matrix}\right)$ is called Struve function ([11]; p.44(16)).

Taking$z=\frac{1}{4}$, Eq. (4.3) reduces to a result of M.A. Pathan and M.J.S.Shahwan([6]; p.78(2.3)).

VI. Setting $j=2$, taking $\beta=\alpha+1$, replacing $1+\upsilon$ by β and making use of ([8]; p.608(13)),

then Eq. (2.1)yields to the following result

$$\Omega^{(2)}_{\mu,\alpha,\alpha+1,\beta-1}(x,y,z) = 2^{\beta}\,\alpha\,\Gamma(\beta)(x)^{1-2\alpha}\int_0^{\infty} t^{\mu-2\alpha+1}\exp(-yt - z\,t^2) \times$$

$$\left[2\alpha J_{\beta-1}(xt)\,S_{2\alpha-\beta-1,\beta-2}(xt) - J_{\beta-2}(xt)\,S_{2\alpha-\beta,\beta-1}(xt)\right] dt \quad (4.4),$$

where $S_{\mu,v}(z) = \frac{(z)^{\mu+1}}{(\mu-v+1)(\mu+v+1)}\ {}_1F_2\left(\begin{matrix} 1; \\ \frac{1}{2}(\mu-v+3), \quad \frac{1}{2}(\mu+v+3); \end{matrix} \frac{-z^2}{4}\right)$ is called Lommel

function ([11]; p. 44(13)). Taking $z = \frac{1}{4}$, Eq. (4.4) reduces to a result of M.A. Pathan and M. J. S.Shahwan

([6]; p.78(2.4)).

VII. Expand ${}_1F_2$ in the integrand of (2.1) after letting $j = 2$, then integrating the resulting (absolutely convergent) series term by term with the use of ([2]; p.416(24)), we get the following result

$$\Omega^{(2)}_{\mu,\alpha,\beta,v}(x,y,z) = \Gamma(\mu+1)(x)^{\frac{1}{2}}(8z)^{\frac{\mu}{2}}\sum_{m=0}^{\infty}\frac{(\mu+1)_m(\alpha)_m\exp(\frac{y^2}{2})}{(\beta)_m(v+1)_m\ m!}\left(\frac{-x^2}{8\,z}\right)^m D_{-\mu-2m-1}\left(\frac{y}{\sqrt{2\,z}}\right)(4.5),$$

where $D_v(z)$ is the parabolic cylinder function ([11]; p. 40(29)). Taking $z = \frac{1}{4}$, Eq. (4.5) reduces to a result of M.A. Pathan and M. J. S.Shahwan ([6]; p.78(3.1)).

VIII. It may be also interesting to observe here that by setting $\alpha = \beta$, letting $y \to 0$ in (4.5) and (2.1)

(after taking $j = 2$) and then equating the equations, we obtain

$$\int_0^{\infty} t^{\mu}\ \exp(-z\,t^2)\,{}_0F_1\left(-; v+1;\ \frac{-x^2t^2}{4}\right) dt = \sqrt{2}\,\Gamma(\mu+1)(8z)^{\frac{\mu}{2}}{}_1F_1\left(\mu+1; v+1; \frac{-x^2}{8z}\right)(4.6),$$

On taking $z = \frac{1}{4}$, Eq. (4.6) reduces to

$$\int_0^{\infty} t^{\mu}\ \exp\left(-\frac{1}{4}t^2\right){}_0F_1\left(-; v+1;\ \frac{-x^2t^2}{4}\right) dt = \sqrt{2}\,\Gamma(\mu+1)(2)^{\frac{\mu}{2}}{}_1F_1\left(\mu+1; v+1; \frac{-x^2}{2}\right)(4.7).$$

5. Further representation of $\Omega_{\mu,v}(x,y,z)$

We derive several further representations of Voigt functions $\Omega_{\mu,v}(x,y,z)$ in terms of series (single and double) of product of Laguerre functions $L_n{}^{(\alpha)}(x)$([11]; p. 41(33)) and parabolic cylinder functions $D_v(x)$ which are essentially useful in situations when the variable values being tacitly excluded in every case. Indeed, if we start from ([8]; p.412(13))

$$(xt)^{\boldsymbol{v}}\exp\left(-\frac{1}{4}t^2\right)\sum_{m=0}^{\infty}\frac{t^{2m}L_m{}^{(v-\alpha)}(x^2)}{\Gamma(v+m+1)2^{2m+v}}\ {}_1F_1\left(a;v+m+1;\frac{t^2}{4}\right)=J_v(xt) \quad (5.1),$$

Expand ${}_1F_1$ in series, multiply both sides by $\sqrt{\frac{x}{2}}t^{\mu}\exp(-yt-zt^2)$, integrate with respect to t over the interval $(0,\infty)$ and then apply the definition of gamma function and making use of Eq. (1.12), we obtain the series representation

$$\Omega_{\mu,v}(x,y,z)$$

$$=\left(\frac{x}{2}\right)^{v+\frac{1}{2}}\exp\left(\frac{y^2}{8z+2}\right)\left(\frac{2}{4z+1}\right)^{\frac{1}{2}\lambda}\sum_{m,n=0}^{\infty}\frac{\Gamma(\lambda+2m+2n)(a)_n\left(\frac{2}{4z+1}\right)^{m+n}}{n!\ 2^{2m+2n}\ \Gamma(v+m+1)(v+m+1)_n}\times$$

$$L_m{}^{(v-a)}(x^2)D_{-\lambda-2m-2n}\left(\frac{y}{\sqrt{\frac{4z+1}{2}}}\right),(\lambda=\mu+v+1) \quad (5.2).$$

For $a=0$ the formula (5.2) reduces to the following result

$$\Omega_{\mu,v}(x,y,z)=\left(\frac{x}{2}\right)^{v+\frac{1}{2}}\exp\left(\frac{y^2}{8z+2}\right)\left(\frac{2}{4z+1}\right)^{\frac{1}{2}\lambda}\sum_{m=0}^{\infty}\frac{\Gamma(\lambda+2m)\left(\frac{2}{4z+1}\right)^{m}}{2^{2m}\ \Gamma(v+m+1)}\times$$

$$L_m{}^{(v)}(x^2)D_{-\lambda-2m}\left(\frac{y}{\sqrt{\frac{4z+1}{2}}}\right),(\lambda=\mu+v+1) \quad (5.3).$$

For $\mu=-v=\frac{1}{2}$ and $\mu=v=\frac{1}{2}$, formulae (5.2) and (5.3) evidently reduce to representations of generalized
Voigt functions $K(x,y,z)$ and $L(x,y,z)$, respectively. Taking $z=\frac{1}{4}$ equations (5.2) and (5.3) reduce to a result
of M.A. Pathan and M. J. S.Shahwan ([6]; p.79(4.2) and (4.3)), respectively.

Again, by appeal to the limiting cases by taking $y\to 0$ and $z=\frac{1}{4}$ in equations (5.2) and (5.3), and adjusting the parameters, we obtain the results of Pathan, Kamarujjama and Alam ([7]; p. 256(3.4)). Furthermore, for $\mu=\frac{1}{2}$, $v=\pm\frac{1}{2}$ and $z=\frac{1}{4}$, the equation (5.2) yields generalizations of the results ([7]; p.256(3.5) to (3.8)).

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C – FIELD COSMOLOGY IN BIANCHI TYPE – III SPACE TIME WITH BULK VISCOSITY AND TIME DEPENDENT COSMOLOGICAL TERM

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ABSTRACT

We have investigated C-field cosmology in Bianchi type-III space time with Bulk viscosity (dust distribution) and time dependent cosmological term. We assume the matter content of the universe is in the form of dust, which leads to p = 0. To get the deterministic model of the universe we assumed that σ (shear) is proportional to θ (expansion) which leads to $B = C^n$, where n is constant and B, C are metric potentials and $\xi\theta$ = constant as considered by Zimdahl. We find that the creation field (C) increases with time which matches with the result of H–N theory. The physical and geometrical aspects of the model are also discussed.

Keywords : Cosmology, C-field, Bianchi - III , Bulk viscosity, Dust, Time dependent Λ

1. INTRODUCTION

Bianchi type cosmological models are important in the sense that these models are homogeneous and anisotropic from which the process of isotropization of the universe is studied through the passage of time. The simplicity of the field equations made Bianchi space-time useful in constructing models of spatially homogeneous and anisotropic cosmologies.

Lorenz [12] has presented tilted electromagnetic Bianchi type III cosmological solution. Tikekar and Patel [19] obtained some exact solutions of massive string cosmology of Bianchi type-III space-time. Bali and Jain [6] have studied Bianchi type-III non-static magnetized cosmological model for perfect fluid distribution in general relativity. Pradhan [15] has presented massive string cosmology in Bianchi type-III space-time with electromagnetic field. Bali and Pradhan [5] have studied Bianchi type-III string cosmological models with time dependent bulk viscosity.

Bali and Dave [9] have investigated Bianchi type-III string cosmological model with bulk viscous fluid in general relativity. Bali and Tinker [4] have studied Bianchi type-III bulk viscous Barotropic fluid cosmological models with variable G and Λ. Singh and Tiwari [18] obtained Bianchi type-III cosmological models with gravitational constant G and the cosmological constant Λ.

In the early universe, all the investigation dealing with physical process use a model of the universe, usually called a big-bang model. The big-bang model based on Einstein field equation successfully explains the three important observation in astronomy : (i) The phenomena of expanding universe, (ii) primordial nucleo-synthesis, (iii) The observed isotropy of the cosmic background radiation.

Thus alternative theories were proposed from time to time. The most well known theory is the 'Steady State Theory' by Bondi and Gold [10]. In this theory the universe does not have any singular beginning nor an end on the cosmic time scale, moreover the statistical properties of the large scale features of the universe do not change. To account for the constancy of the mass density they have envisage a very slow but continuous creation matter in contrast to the one time infinite and explosive creation at t = 0 of the standard model. But it suffers from the serious disqualification that they do not give any physical justification in the form of any dynamical theory for the phenomenon of the continuous creation of matter. Thus the principle of conservation of matter is clearly scarifies in this formulism. To remove this problem Hoyle and Narlikar [11] adopted a field theoretic approach introducing a massless and chargeless scalar field in Einstein Hilbert action to account for creation of matter.

Bali and Tikekar [7] have investigated C-field cosmological model for dust distribution in FRW space-time with variable gravitational constant. Bali and Kumawat [8] have investigated C-field cosmological models for dust and barotropic fluid distribution in non flat FRW space time with variable gravitational constant. Bali and Saraf [3] have investigated Bianchi type I dust field universe with decaying vacuum energy in C-field cosmology. Saraf [17] studied Bianchi type-I cosmological model for dust distribution with variable G and Lambda. Bali and Goyal [1] have investigated inflationary scenario in Bianchi type-V space time with variable bulk viscosity and dark energy in Radiation dominated phase.

Narlikar and Padmanabhan [14] have investigated the solution of Einstein's field equation with admit radiation and negative energy massless scalar C-field as source. Bali and Saraf [2] obtained Bianchi type-III dust filled universe with time dependent Λ in C-field cosmology. Tyagi and Singh [20] studied LRS Bianchi type III barotropic fluid cosmological model in C-field with varying cosmological constant Λ. LRS Bianchi type-V perfect fluid cosmological model in C-field theory with variable Λ have studied by Tyagi and Singh [21]. Parikh and Tyagi [16] have investigated Bianchi type VI_0 cosmological with Barotropic perfect fluid in creation field theory with time dependent Λ. Mehta and Chundawat [13] have studied LRS Bianchi type-II cosmological model with barotropic perfect fluid in C-field theory with time dependent term.

In this Paper, we have investigated C-field cosmology in Bianchi type-III space time with Bulk viscosity (dust distribution) and time dependent cosmological term. We assume the matter content of the universe is in the form of dust, which leads to p = 0. To get the deterministic model of the universe we assumed that σ (shear) is proportional to θ (expansion) which leads to $B = C^n$, where n is constant and B, C are metric potentials and $\xi\theta$ = constant as considered by Zimdahl [22]. We find that the creation field (C) increases with time which matches with the result of H–N theory. The physical and geometrical aspects of the model are also discussed.

2. THE METRIC AND FIELD EQUATIONS

We consider Bianchi type-III metric given in the form

$$ds^2 = dt^2 - A^2 dx^2 - B^2 e^{2x} dy^2 - C^2 dz^2 \quad \text{... (1)}$$

where A, B and C are function of t alone and $\sqrt{-g} = ABCe^x$

Einstein's field equation by introduction of C-field is modified by Hoyle and Narlikar [11] as

$$R_i^j - \frac{1}{2} R g_i^j = -8\pi G \left[\underset{(m)}{T_i^j} + \underset{(c)}{T_i^j} \right] - \Lambda g_i^j \quad \text{... (2)}$$

The energy momentum tensor $\underset{(m)}{T_i^j}$ for Bulk viscous and $\underset{(c)}{T_i^j}$ for creation field are given by

$$\underset{(m)}{T_i^j} = (\bar{p} + \rho) v_i v^j - (\bar{p}) g_i^j \quad \text{... (3)}$$

where $\bar{p} = p - \xi\theta$

$$\underset{(c)}{T_i^j} = -f\left(C_i C^j - \frac{1}{2} g_i^j C^\alpha C_\alpha \right) \quad \text{... (4)}$$

where f > 0 is coupling constant between matter and creation field and $C_i = \frac{dC}{dx^i}$

The co-moving coordinate are chosen such that $v^i = (0, 0, 0, 1)$. The non-vanishing components of energy momentum tensor for matter are given by

$$\underset{(m)}{T_1^1} = \xi\theta = \underset{(m)}{T_2^2} = \underset{(m)}{T_3^3}, \underset{(m)}{T_4^4} = \rho \quad \text{... (5)}$$

The non-vanishing components of energy momentum tensor for creation field are given by

$$\underset{(c)}{T_1^1} = \frac{1}{2} f\dot{C}^2 = T_2^2 = T_3^3,\ \underset{(c)}{T_4^4} = -\frac{1}{2} f\dot{C}^2 \quad \text{... (6)}$$

Hence the Einstein field equation (2) for the metric (1) together with (5) and (6) takes the form

$$\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{B_4C_4}{BC}=8\pi G\left[\xi\theta+\frac{1}{2}f\dot{C}^2\right]+\Lambda \qquad \text{... (7)}$$

$$\frac{A_{44}}{A}+\frac{C_{44}}{C^2}+\frac{A_4C_4}{AC}=8\pi G\left[\xi\theta+\frac{1}{2}f\dot{C}^2\right]+\Lambda \qquad \text{... (8)}$$

$$\frac{A_{44}}{A^2}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}-\frac{1}{A^2}=8\pi G\left[\xi\theta+\frac{1}{2}f\dot{C}^2\right]+\Lambda \qquad \text{... (9)}$$

$$\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{A_4C_4}{AC}-\frac{1}{A^2}=8\pi G\left[\rho-\frac{1}{2}f\dot{C}^2\right]+\Lambda \qquad \text{... (10)}$$

$$\frac{A_4}{A}-\frac{B_4}{B}=0 \qquad \text{... (11)}$$

The suffix 4 by the symbols A, B and C denotes differentiation w.r.t.'t'.

3. SOLUTIONS OF FIELD EQUATIONS

The conservation equation

$$[8\pi G T_i^j+\Lambda g_i^j]_{;j}=0 \qquad \text{... (12)}$$

which leads to

$$\frac{d\dot{C}^2}{dt}+2\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)\dot{C}^2=\frac{2\dot{\rho}}{f}+\frac{2}{f}(\rho-\xi\theta)\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)+\frac{\dot{\Lambda}}{4\pi G f} \qquad \text{... (13)}$$

following Hoyle and Narlikar [11], we have taken p = 0, the source equation of C-field $C_{;i}^{i}=\frac{n}{f}$ leads to C = t for large r

Thus $\dot{C}=1$.

Using $\dot{C}=1$, equations (7), (8), (9) and (10) leads to

$$\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{B_4C_4}{BC}=8\pi G\left[\xi\theta+\frac{1}{2}f\right]+\Lambda \qquad \text{... (14)}$$

$$\frac{A_{44}}{A}+\frac{C_{44}}{C^2}+\frac{A_4C_4}{AC}=8\pi G\left[\xi\theta+\frac{1}{2}f\right]+\Lambda \qquad \text{... (15)}$$

$$\frac{A_{44}}{A^2}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}-\frac{1}{A^2}=8\pi G\left[\xi\theta+\frac{1}{2}f\right]+\Lambda \qquad \text{... (16)}$$

$$\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{A_4C_4}{AC}-\frac{1}{A^2}=8\pi G\left[\rho-\frac{1}{2}f\right]+\Lambda \qquad \text{... (17)}$$

Equation (11) leads to

$$A=\alpha B \qquad \text{... (18)}$$

where α is constant of integration. The condition $\sigma\propto\theta$ leads to

$$B = C^n \qquad \text{... (19)}$$

where n is constant and B, C are metric potentials

Equation (15) and (16) leads to

$$\frac{B_{44}}{B}-\frac{C_{44}}{C^2}+\frac{A_4B_4}{AB}-\frac{A_4C_4}{AC}-\frac{1}{A^2}=0 \qquad \text{... (20)}$$

Equation (11), (18), (19) and (20) leads to

$$2C_{44}+4n\frac{C_4^2}{C}=\frac{2}{\alpha^2(n-1)}C^{1-2n} \qquad \text{... (21)}$$

To get the deterministic value of C, we assume that $C_4 = F(C)$

This leads to $C_{44} = FF'$ where $F' = \frac{dF}{dC}$

Equation (21) leads to

$$\frac{dF^2}{dC} + \frac{4n}{C}F^2 = \frac{2}{\alpha^2(n-1)}C^{1-2n} \qquad \text{... (22)}$$

Equation (22) leads to

$$F^2 = \frac{1}{\alpha^2(n^2-1)}C^{2-2n} = \left(\frac{dC}{dt}\right)^2 \qquad \text{... (23)}$$

The constant of integration has been taken zero for simplicity.

Equation (23) leads to

$$C^n = \frac{n}{\alpha\sqrt{n^2-1}}t + nl$$

$$C^n = (at + b) \qquad \text{... (24)}$$

where $a = \frac{n}{\alpha\sqrt{n^2-1}}$, $b = n\ell$ and n is positive integer.

Equation (19) and (24) leads to

$$B = (at + b) \qquad \text{... (25)}$$

Equation (24) leads to

$$C = (at + b)^{1/n} \qquad \text{... (26)}$$

from equation (18) and (25), we have

$$A = \alpha(at + b) \qquad \text{... (27)}$$

Equation (14) leads to

$$\Lambda = \frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} - 8\pi G\xi\theta - 4\pi Gf \qquad \text{... (28)}$$

Using equation (25) and (26) in (28), we have

$$\Lambda = \frac{1}{n^2} \cdot \frac{a^2}{(at+b)^2} - L - K \qquad \text{... (29)}$$

where $8\pi G\xi\theta = L$ and $4\pi Gf = K$

Equation (17), (25), (26), (27) and (29) leads to

$$8\pi G\rho = \frac{2a^2}{n(at+b)^2} + 2K \qquad \text{... (30)}$$

Thus the metric (1), after using (25), (26) and (27) takes the form

$$ds^2 = dt^2 - (at+b)^2[\alpha^2 dx^2 - e^{2x} dy^2] - (at+b)^{2/n} dz^2 \qquad \text{... (31)}$$

Using equation (25), (26), (27), (29) and (30) in equation (13) leads to

$$\frac{d}{dt}\dot{C}^2 + \left[2\left(2+\frac{1}{n}\right)\frac{a}{at+b}\right]\dot{C}^2 = 2\left(2+\frac{1}{n}\right)\left(\frac{a}{at+b}\right)\left(1-\frac{\xi\theta}{f}\right)$$

which leads to

$$\frac{d\dot{C}^2}{dt} + \left[2\left(2+\frac{1}{n}\right)\frac{a}{at+b}\right]\dot{C}^2 = 2\left(2+\frac{1}{n}\right)\frac{a}{at+b}\beta \qquad \text{... (32)}$$

from equation (32), we have

$$\dot{C}^2 = \beta \qquad \text{where } \beta = \left(1 - \frac{\xi\theta}{f}\right)$$

which leads to

$$C=\sqrt{\beta}t \qquad \text{... (33)}$$

Thus creation field increases with time which matches with the result obtained by Hoyle and Narlikar [11].

4. PHYSICAL AND GEOMETRICAL FEATURES

The homogeneous mass density (ρ), the cosmological constant term, the creation field (C), spatial volume (R^3), the deceleration parameter (q), shear tensor (σ) and expansion (θ) of the model (31) are given by

$$8\pi G\rho=\frac{2a^2}{n(at+b)^2}+2K \qquad \text{... (34)}$$

$$\Lambda=\frac{1}{n^2}\cdot\frac{a^2}{(at+b)^2}-L-K \qquad \text{... (35)}$$

$$C=\sqrt{\beta}t \qquad \text{... (36)}$$

$$R^3=\alpha(at+b)^{(2+1/n)} \qquad \text{... (37)}$$

$$q=(-)\frac{\ddot{R}/R}{\dot{R}^2/R^2}=\frac{(n-1)}{(2n+1)} \qquad \text{... (38)}$$

$$\theta=\left(2+\frac{1}{n}\right)\frac{a}{(at+b)} \qquad \text{... (39)}$$

$$\sigma^2=\frac{a^2}{(at+b)^2}\left[\frac{1}{3n^2}-\frac{2}{3n}+\frac{1}{3}\right] \qquad \text{... (40)}$$

$$\frac{\sigma}{\theta}=\text{constant} \qquad \text{... (41)}$$

5. CONCLUSION

The scale factor R increase with time, since deceleration parameter q > 0 for n > 1, hence the model (31) represent decelerating universe. The density decrease as time increase. The model (31) passes through a singular state at t = $-\frac{b}{a}$, this is explained as creation exists all the time, so there is a big crunch between $-\frac{b}{a}$ to ∞ and creation is going on front t = $-\frac{b}{a}$ to ∞. During this period the model exist. Since $t \to \infty, \frac{\sigma}{\theta}$ is constant, therefore model (31) does not approaches to isotropy in late time. Since $\theta \neq \infty$ at t = 0, hence model (31) is free from initial singularity.

The coordinate distance γ_H to the horizon is the maximum distance a null ray could have travelled at time t starting from infinite past i.e.

$$\gamma_H = \int_{-\infty}^{t} \frac{dt}{R^3(t)}$$

We could extent the proper time t to in the past because of non-singular nature of space time, thus

$$\gamma_H = \int_{0}^{t} \frac{dt}{R^3(t)} = \int_{0}^{t} \frac{dt}{\alpha(at+b)^{(2+1/n)}}$$

The integral of diverge at lower limit shows that the model is free from event horizon.

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BIANCHI TYPE-VIII STRING COSMOLOGICAL MODELS FOR BAROTROPIC FLUID DISTRIBUTION WITH DARK ENERGY

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ABSTRACT

We have investigated Bianchi type-VIII string cosmological model for barotropic fluid distribution with dark energy Λ .To get the deterministic model of the universes, we suppose expansion θ is proportional to the shear σ and the dark energy (Λ) is assumed to be proportional to V^{-3} where V is scale factor. The physical and geometrical aspects of the model are also discussed.

Keywords: Bianchi VIII, string, barotropic fluid, dark energy, perfect fluid.

1. INTRODUCTION

It is still anintractable problem before us to know the exact physical situation at very early stages of the genesis of our universe. The string theory is a utile concept before the creation of the particle in the universe. The string are the important topological stable defects due to the phase transition that occurs as the temperature lower down below some critical temperature at the very early stages of the universe.

It is believed that cosmic strings give rise to density perturbation which leads to the formation of galaxies (Zel'dovich [25]). The general relativistic treatment of strings was obtained by Letelier [9,10] and Stachel [21]. Bali et al. [1, 2, 3, 5] have obtained Bianchi type IX, type V & type I string cosmological models in general relativity. Exact solutions of string cosmology for Bianchi type II,VI_0, VIII and IX space-times have been obtained by Krori et al.[8]. Tyagi et al. [23] have studied Inhomogeneous Bianchi type-VI_0 string dust cosmological model of perfect fluid distribution in general relativity.Singh [20] also studied string cosmology with electromagnetic fields in Bianchi type-II, VIII & IX space times. Tyagi et al. [24] have studied Bianchi Type-IX String Cosmological Models for Perfect fluid Distribution in General Relativity.

Fluids have played important role in the entire history of the Universe. Their components are relatively simple and behave as perfect fluids, at least at the background level. A perfect fluid is an inviscid fluid with no heat conduction. It is analogous to an ideal gas in standard thermodynamics. The matter distribution is satisfactorily described by perfect fluids due to the large scale distribution of galaxies in our universe. However, a realistic treatment of the problem requires the consideration of material distribution other than the perfect fluid.

The homogeneous and anisotropic Bianchi models play remarkable role in the present day universe. The advantages of the anisotropic models are that they have a significant role in the description of the evolution of the early phase of the universe and they help in finding more general cosmological models than the isotropic FRW models. Reddy et al. [16] studied Bianchi type II, VIII and IX models in scale covariant theory of gravitation. Chhajed et al. [6] have studied Bianchi type VIII cosmological model with Quadratic equation of state.Also Rao and Sanyasi Raju[15] and Sanyasi and Raju [19] have studied Bianchi type VIII & IX models in Zero mass scalar fields and in Self – Creation theory of Cosmology. Bali and Swati [4] have investigated Inflationary scenario in Bianchi type VIIIspace-time for a massless scalar field with flat potential. Rao et al. [12-14] have studied Bianchi type-II,VIII & IX string cosmological models , perfect fluid cosmological models in Saez Ballester scalar tensor theory of gravitation and string cosmological models in general relativity as well as self-creation theory of gravitation respectively.

Motivated by the above discussion, in this chapter, we have investigated Bianchi type-VIII stringcosmological model with barotropic fluid distribution and dark energy Λ. The present study deals with Bianchi VIII string cosmological models for perfect fluid distribution. We consider two cases (i)$\boldsymbol{\rho + \lambda = 0}$(Reddy string) (ii)$\boldsymbol{\rho - \lambda = 0}$ (Nambu string). To get the deterministic model of the universe, we assume that the shear (σ) is proportional to expansion (θ) as considered by Thorne [22] and Collins et al. [7]**.** The physical and geometrical properties of the models are discussed.

2. METRIC AND FIELD EQUATIONS

The line –element for Bianchi type-VIII space time is considered as

$$ds^2 = -dt^2 + R^2(t)[d\theta^2 + cosh^2\theta d\emptyset^2] + S^2(t)[d\Psi + sinh\theta d\emptyset]^2 \quad (1)$$

in which *R (t), S (t)* are cosmic scale functions.

The energy momentum tensor T_i^j in the presence of perfect fluid is defined by

$$T_i^j = (\rho + p)v_i v^j + p g_i^j - \lambda x_i x^j \quad (2)$$

Where ρ is proper energy density, p is pressure and λ is string tension density. Also x^i, the unit space like vector specifying the direction of strings and v^i, the unit time like vector specifying the following conditions

$v_i v^i = -1 = -x_i x^i$ and $v^i x_i = 0$

The co-moving coordinate system is chosen as

$v^i = (0,0,0,1)$ and $x^i = \left(\frac{1}{S}, 0,0,0\right)$

The Einstein's field equation in the geometrized unit $(c = 8\pi G = 1)$ is given by

$$R_i^j - \frac{1}{2} R g_i^j + \Lambda g_i^j = -T_i^j \quad (3)$$

Where R_i^j is Ricci tensor, $R = g^{ij} R_{ij}$ is Ricci scalar.

The Einstein's field equations (3) for metric (1) leadto:

$$\frac{2R_{44}}{R} + \frac{{R_4}^2}{R^2} - \frac{1}{R^2} - \frac{3S^2}{4R^4} + \Lambda = -(p - \lambda) \quad (4)$$

$$\frac{R_4 S_4}{RS} + \frac{S^2}{4R^4} + \frac{S_{44}}{S} + \frac{R_{44}}{R} + \Lambda = -(p) \quad (5)$$

$$\frac{{R_4}^2}{R^2} + \frac{2R_4 S_4}{RS} - \frac{1}{R^2} - \frac{S^2}{4R^4} + \Lambda = \rho \quad (6)$$

The scalar expansion θ and shear σ are given by

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$\theta = 3\mathrm{H}$ (7)

$\sigma^2 = \frac{1}{2}\left[\sum_{i=1}^{3} H_i^2 - \frac{1}{3}\theta^2\right]$ (8)

3. SOLUTION OF FIELD EQUATIONS

The field equations (4) to (6) represent a system of three independent equations in six unknowns *R, S,* λ*,* *p,* ρ and Λ.

In order to overcome the undeterminacy of six unknowns involved in three independent field equations, we consider the following two cases (i) $\rho + \lambda = 0$, i.e. the sum of the rest energy density and tension density for a cloud of strings vanishes (Reddy [17, 18] and Mohanty[11]) (ii)(Nambu) strings given by $\rho - \lambda = 0$.

Two additional constraints related to these parameters are required to obtain the explicit solution of the system.

Cosmic shear σ represents an effect of distortion of the image of distant galaxies due to deflection of light by matter, as predicted by general relativity.

Metric expansion θ is a key feature of Big-Bang cosmology and is modeled mathematically with the Friedmann- Lemaitre- Robertson-Walker (FLRW) metric. The metric expansion of space is the averaged increase of metric (i.e.) measured distance between distant objects in the universe with time.

We assume that shear σ is proportional to expansion θ

Thus we have

$S = R^n$ (9)

and $\wedge$ is proportional to

$\wedge = \frac{\alpha}{SR^2}$(10)

Where $\alpha$is a constant of proportionality.

We assume the above conditions under two cases

(i) **Case I: $\rho + \lambda = 0$**

From (4) and (6)

$$\frac{2R_{44}}{R} + \frac{2R_4^2}{R^2} - \frac{2}{R^2} - \frac{S^2}{R^4} + \frac{2R_4S_4}{RS} + 2\wedge = -p \quad (11)$$

Using (5), we get

$$\frac{R_{44}}{R} + \frac{2R_4^2}{R^2} - \frac{2}{R^2} - \frac{5}{4R^{4-2n}} + \frac{R_4S_4}{RS} - \frac{S_{44}}{S} + \wedge = 0 \quad (12)$$

Using condition (9) and (10), we get

$$\frac{(1-n)R_{44}}{R} + \frac{(2+2n-n^2)R_4^2}{R^2} = \frac{2}{R^2} + \frac{5R^{2n-4}}{4} - \frac{\alpha}{R^{n+2}} \quad (13)$$

$$R_{44} + \frac{(n^2-2n-2)R_4^2}{(n-1)R} = \frac{2}{(1-n)R} + \frac{5R^{2n-3}}{4(1-n)} - \frac{\alpha}{(1-n)R^{n+1}} \quad (14)$$

Now let us consider R_4 = *f (R)* and R_{44}= *ff'* in equation (14) we get

$$ff' + \frac{(n^2-2n-2)f^2}{(n-1)R} = \frac{2}{(1-n)R} + \frac{5R^{2n-3}}{4(1-n)} - \frac{\alpha}{(1-n)R^{n+1}} \quad (15)$$

Equation (15) can be written in the form

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$$\frac{df^2}{dR}+\frac{2(n^2-2n-2)f^2}{(n-1)R}=\frac{4}{(1-n)R}+\frac{5R^{2n-3}}{2(1-n)}-\frac{2\alpha}{(1-n)R^{n+1}} \quad (16).$$

After integration, Eq. (16) leads to

$$f^2=\frac{M}{R^{\frac{2(n^2-2n-2)}{(n-1)}}}-\frac{2}{(n^2-2n-2)}-\frac{5R^{\frac{(2n^2-4n+2)}{(n-1)}}}{2(4n^2-8n-2)}+\frac{2\alpha}{(n^2-3n-4)R^n} \quad (17)$$

where M is the integrating constant.

from equation (17), we have

$$\int \frac{dR}{\sqrt{\frac{M}{R^{\frac{2(n^2-2n-2)}{(n-1)}}}-\frac{2}{(n^2-2n-2)}-\frac{5R^{\frac{(2n^2-4n+2)}{(n-1)}}}{2(4n^2-8n-2)}+\frac{2\alpha}{(n^2-3n-4)R^n}}}=\int dt+M'=t+M' \quad (18)$$

Where M' is the integrating constant. Value of R can be obtained from equation (18). Hence by appropriate transformation of co-ordinates, the metric (1) leads to the form

$$ds^2=-\frac{dT^2}{\left[\frac{M}{T^{\frac{2(n^2-2n-2)}{(n-1)}}}-\frac{2}{(n^2-2n-2)}-\frac{5T^{\frac{(2n^2-4n+2)}{(n-1)}}}{2(4n^2-8n-2)}+\frac{2\alpha}{(n^2-3n-4)T^n}\right]}+T^2[d\theta^2+cosh^2\theta d\emptyset^2]\;+T^{2n}[d\Psi+ sinh\theta d\emptyset]^2 \quad (19)$$

(ii) Case II: $\boldsymbol{\rho-\lambda=0}$

From (4) and (6)

$$\frac{2R_4S_4}{RS}+\frac{S^2}{2R^4}-\frac{2R_{44}}{R}=p \quad (20)$$

Using (5), we get

$$\frac{2nR_4^2}{R^2}+\frac{3}{4R^{4-2n}}-\frac{R_{44}}{R}+\frac{R_4S_4}{RS}+\frac{S_{44}}{S}+\frac{\alpha}{R^{n+2}}=0 \quad (21)$$

Using condition (9) and (10), we get

$$\frac{(n-1)R_{44}}{R}+\frac{(2n+n^2)R_4^2}{R^2}=-\frac{3R^{2n-4}}{4}-\frac{\alpha}{R^{n+2}} \quad (22)$$

$$R_{44}+\frac{(n^2+2n)R_4^2}{(n-1)R}=-\frac{3R^{2n-3}}{4(n-1)}-\frac{\alpha}{(n-1)R^{n+1}} \quad (23)$$

Now let us consider $R_4 = f(R)$ and $R_{44} = ff'$ in equation (23) we get

$$ff'+\frac{(n^2+2n)f^2}{(n-1)R}=-\frac{3R^{2n-3}}{4(n-1)}-\frac{\alpha}{(n-1)R^{n+1}} \quad (24)$$

Equation (24) can be written in the form

$$\frac{df^2}{dR}+\frac{2(n^2+2n)f^2}{(n-1)R}=-\frac{3R^{2n-3}}{2(n-1)}-\frac{2\alpha}{(n-1)R^{n+1}} \quad (25)$$

After integration, Eq. (25) leads to

$$f^2=\frac{N}{R^{\frac{2(n^2+2n)}{(n-1)}}}-\frac{2\alpha}{(n^2+5n)R^n}-\frac{3R^{\frac{(2n^2-4n+2)}{(n-1)}}}{2(4n^2+2)} \quad (26)$$

Where *N* is the integrating constant.

From equation (26), we have

$$\int \frac{dR}{\sqrt{\frac{N}{R^{\frac{2(n^2+2n)}{(n-1)}}} - \frac{2\alpha}{(n^2+5n)R^n} - \frac{3R^{\frac{(2n^2-4n+2)}{(n-1)}}}{2(4n^2+2)}}} = \int dt + N' = t + N' \quad (27)$$

Where N' is the integrating constant. Value of R can be obtained from equation (27). Hence by appropriate transformation of co-ordinates, the metric (1) leads to the form

$$ds^2 = -\frac{dT^2}{\left[\frac{N}{T^{\frac{2(n^2+2n)}{(n-1)}}} - \frac{2\alpha}{(n^2+5n)T^n} - \frac{3T^{\frac{(2n^2-4n+2)}{(n-1)}}}{2(4n^2+2)}\right]} + T^2[d\theta^2 + cosh^2\theta d\emptyset^2] \ + T^{2n}[d\Psi + sinh\theta d\emptyset]^2 \quad (28)$$

4. **PHYSICAL AND GEOMETRICAL CHARACTERISTICS**

For the model (19), energy density (ρ), pressure (p), expansion (θ), shear tensor (σ), string tension density (λ) are given by

$$\rho = \frac{M(2n+1)}{T^{\frac{(2n^2-2n-6)}{(n-1)}}} + \frac{\alpha(n+2)(n-1)}{(n-4)(n+1)T^{n+2}} - \frac{1}{4T^{4-2n}} - \frac{n(n+2)}{(n^2-2n-2)T^2} - \frac{5(2n+1)T^{\frac{(2n^2-6n+4)}{(n-1)}}}{4(2n^2-4n-1)} \quad (29)$$

$$\lambda = \frac{1}{4T^{4-2n}} - \frac{M(2n+1)}{T^{\frac{(2n^2-2n-6)}{(n-1)}}} - \frac{\alpha(n+2)(n-1)}{(n-4)(n+1)T^{n+2}} + \frac{n(n+2)}{(n^2-2n-2)T^2} + \frac{5(2n+1)T^{\frac{(2n^2-6n+4)}{(n-1)}}}{4(2n^2-4n-1)} \quad (30)$$

The directional Hubble parameters H_x, H_y, H_z are given by

$$H_x = H_y = \frac{\dot{R}}{R} \quad (31)$$

$H_z = \frac{\dot{S}}{S} = \frac{n\dot{R}}{R}$(32)

The mean Hubble parameter (H) is given by

$H = \frac{1}{3}\left(H_x + H_y + H_z\right) = \frac{(n+2)\dot{R}}{3R}$(33)

$$\theta = (n+2)\left[\frac{M}{T^{\frac{2(n^2-n-3)}{(n-1)}}} - \frac{2}{(n^2-2n-2)T^2} - \frac{5}{2(4n^2-8n-2)T^{4-2n}} + \frac{2\alpha}{(n^2-3n-4)T^{n+2}}\right]^{\frac{1}{2}} \qquad (34)$$

$$\sigma^2 = \frac{(n-1)^2}{3}\left[\frac{M}{T^{\frac{2(n^2-n-3)}{(n-1)}}} - \frac{2}{(n^2-2n-2)T^2} - \frac{5}{2(4n^2-8n-2)T^{4-2n}} + \frac{2\alpha}{(n^2-3n-4)T^{n+2}}\right] (35)$$

Isotropic pressure p is determined if fluid is known to obey an equation of state of the form

$p = \gamma\rho$, $where\ \ 0 \le \gamma \le 1$(36)

Equation (36) lead to

$$p = \gamma\left(\frac{M(2n+1)}{T^{\frac{(2n^2-2n-6)}{(n-1)}}} + \frac{\alpha(n+2)(n-1)}{(n-4)(n+1)T^{n+2}} - \frac{1}{4T^{4-2n}} - \frac{n(n+2)}{(n^2-2n-2)T^2} - \frac{5(2n+1)T^{\frac{(2n^2-6n+4)}{(n-1)}}}{4(2n^2-4n-1)}\right) (37)$$

For the model (28), energy density (ρ), pressure (p), expansion (θ), shear tensor (σ), string tension density (λ) are given by

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$$\rho = \lambda = \frac{N(2n+1)}{T^{\frac{(2n^2+6n-2)}{(n-1)}}} - \frac{\alpha(n^2+n-2)}{(n^2+5n)T^{n+2}} - \frac{(n+2)(n+1)}{2(2n^2+1)T^{4-2n}} - \frac{1}{T^2} \quad (38)$$

$$p = \gamma\left(\frac{N(2n+1)}{T^{\frac{(2n^2+6n-2)}{(n-1)}}} - \frac{\alpha(n^2+n-2)}{(n^2+5n)T^{n+2}} - \frac{(n+2)(n+1)}{2(2n^2+1)T^{4-2n}} - \frac{1}{T^2}\right) \quad (39)$$

$$\theta = (n+2)\left[\frac{N}{T^{\frac{(2n^2+6n-2)}{(n-1)}}} - \frac{3}{2(4n^2+2)T^{4-2n}} - \frac{2\alpha}{(n^2+5n)T^{n+2}}\right]^{\frac{1}{2}} \quad (40)$$

$$\sigma^2 = \frac{(n-1)^2}{3}\left[\frac{N}{T^{\frac{(2n^2+6n-2)}{(n-1)}}} - \frac{3}{2(4n^2+2)T^{4-2n}} - \frac{2\alpha}{(n^2+5n)T^{n+2}}\right] \quad (41)$$

$$\Lambda = \frac{\alpha}{T^{n+2}} \quad (42)$$

5. CONCLUSION

The models (19) and (28) start expanding with big-bang at *T=0.*The expansion θ decreases as time increases for $-2 < n < 2$. We also observe that it approaches to zero as $T \to \infty$ and stops when $n = -2$. Since $\to \infty$,$\frac{\sigma}{\theta} \neq 0$, therefore the model does not approach isotropy for large value of T, however the model is isotropize for $n = 1$.

The energy density(ρ), string tension density (λ) and pressure(p) for both models are found to be a decreasing function of time T for $-2 < n < 2$ and approaches to 0 as$T \to \infty$.

By equation (42) , we observe that the cosmological term Λ for the model is also decreasing function of time T for $n > -2$ and approaches to zero at late time , which in agreement with present astronomical observations.

Hence, in general, the present model represents expanding, shearing and non-rotating, anisotropic universe.

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Generalized gamma-type function involving $\omega -$ confluent hypergeometric function and associated probability distributions

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Abstract: In view of their importance and usefulness in diffraction theory and probability distributions, serval generalizations of Gamma-type functions were investigated extensively in recent years. The main object of this paper is to present some results on a new generalized gamma- type function

$$S\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \\ u, v, \omega\end{matrix} p, k\right) := v^{-\lambda} \int_0^\infty t^{u-1} \emptyset^{\omega}(\alpha, \beta; -pt) \quad {}_3R_2\left(\lambda, a, b; c, d; k; -\frac{t}{v}\right) dt$$

where $\phi^{\omega}(\alpha, \beta; z)$ is $\omega -$ confluent hypergeometric function and ${}_3R_2^k$ is a new generalized hypergeometric function which has been defined and studied by Saxena, Ram, Naresh [1] in the following form

$${}_3R_2(\lambda, a, b; c, d; k; z) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \sum_{r=0}^{\infty} \frac{(\lambda)_r \Gamma(a+kr)\Gamma(b+kr)}{\Gamma(c+kr)\Gamma(d+kr)} \frac{z^r}{r!}$$

A probability density function associated with the generalized gamma-type function investigated in the paper, together with serval other related in the theory of probability and statistics are also considered. The results investigated earlier by AL-Musallam and Kalla [2], Kobayashi[3,4], Saxena, Ram, Naresh and Kalla [5] etc follow, as special cases.

Introduction and preliminaries

Recently Saxena, RamNaresh [1]introduce the following generalized hypergeometric function in mathematical analysis:

$${}_3R_2(\lambda, a, b; c, d; k; z) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \sum_{r=0}^{\infty} \frac{(\lambda)_r \Gamma(a+kr)\Gamma(b+kr)}{\Gamma(c+kr)\Gamma(d+kr)} \frac{z^r}{r!} \tag{1}$$

where $|z| < 1, k > 0$,

This function can also be represented in the integral form as

$$ {}_3R_2(\lambda,a,b;c,d;k;z)=\frac{\Gamma(d)}{\Gamma(a)\Gamma(d-a)}.\int_0^1 t^{a-1}(1-t)^{d-a-1}\ {}_2R_1\left(\lambda,b;c;k;zt^k\right)dt $$

$$ =\frac{\Gamma(d)}{\mathrm{k}\,\Gamma(a)\Gamma(d-a)}.\int_0^1 t^{\frac{a}{k}-1}\left(1-t^{\frac{1}{k}}\right)^{d-a-1}\ {}_2R_1(\lambda,b;c;k;zt)dt \quad (2) $$

where $Re(d)>Re(a)>0, Re(c)>Re(b)>0, k>0, |z|<1.$

Certain analytic properties of (1), such as integral representations, recurrence relations etc.as considered in [1].

In the present paper the authors introduce a further generalization of the gamma-type function associated with $\omega-$ confluent hypergeometric function $\phi^{\omega}(\alpha,\beta;z)$ in the form

$$ S\begin{pmatrix}\lambda,a,b;c,d;\alpha,\beta & p,k\\ u,v,\omega & \end{pmatrix}:=v^{-\lambda}\int_0^{\infty} t^{u-1}\emptyset^{\omega}(\alpha,\beta;-pt)\quad {}_3R_2\left(\lambda,a,b;c,d;k;-\frac{t}{v}\right)dt \quad (3) $$

$Re(u)>0, Re(p)>0, k>0, |argv|<\pi$, the $\omega-$ confluent hypergeometric function $\phi^{\omega}(\alpha,\beta;z)$

Is defined as

$$ \phi^{\omega}(\alpha,\beta;z)=\frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+\omega k)}{\Gamma(\beta+\omega k)}\frac{z^k}{k!} \quad (4) $$

$$ |z|<\infty, \omega>0, (\beta+\omega k)\neq 0,1,2,\ldots $$

The Mellin –Barnes integral representation of ${}_3R_2^k(z)$ can be expressed in the form

$$ {}_3R_2^k(z)=\frac{1}{2\pi\omega}\frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(\lambda)}\int_{-\omega\,\infty}^{\omega\,\infty}\frac{\Gamma(-s)\Gamma(\lambda+s)\Gamma(a+ks)\Gamma(b+ks)}{\Gamma(c+ks)\Gamma(d+ks)}(-z)^s ds \quad (5) $$

Where $\omega=\sqrt{-1}$, $|\arg(-z)|<\pi, k>0$ and the poles of the gamma functions appearing in the integrand of equation (5) are assumed to be simple. Equation (5)can be established by the application of calculus of residues ,if we calculate the sum of residues at the the poles of $\Gamma(-s)$at the points $s=\nu(\nu=0,1,2\ldots)$.

If, however, we calculate the sum of the residues at the poles of the gamma functions $\Gamma(\lambda+s), \Gamma(a+ks)\ and\ \Gamma(b+ks)$ at the points $s=-\lambda-\nu; s=-\frac{a}{k}-\nu\ and\ s=-\frac{b}{k}-\nu\ (\nu=0,1,2\ldots)$ repectively, in the integrand of equation (5),we find that

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$${}_3R_2^k(z) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(\lambda)}\cdot\left[(-z)^{\lambda}\sum_{v=0}^{\infty}\frac{\Gamma(\lambda+v)\Gamma(a-k\lambda-kv)\Gamma(b-k\lambda-kv)}{\Gamma(c-k\lambda-kv)\Gamma(d-k\lambda-kv)}\left(\frac{1}{z}\right)^{v} + (-z)^{-\frac{a}{k}}\sum_{v=0}^{\infty}\frac{\Gamma\left(\frac{a}{k}+v\right)\Gamma\left(\lambda-v-\frac{a}{k}\right)\Gamma(b-a-kv)}{\Gamma(c-a-kv)\Gamma(d-a-kv)}\left(\frac{1}{z}\right)^{v} + (-z)^{-\frac{b}{k}}\sum_{v=0}^{\infty}\frac{\Gamma\left(\frac{b}{k}+v\right)\Gamma\left(\lambda-v-\frac{b}{k}\right)\Gamma(a-b-kv)}{\Gamma(c-b-kv)\Gamma(d-b-kv)}\left(\frac{1}{z}\right)^{v}\right] \quad (6)$$

Where $|arg(-z)| < \pi, |z| > 1.$

Equation (6) gives the analytic continuation formula for the function ${}_3R_2^k(z)$.

Generalized gamma-type function

Let us assume that $\Omega_v = \{ v \in C; |arg(v)| < \pi \}$ and $\Omega_u = \{ u \in C;\ Re(u) > 0\}$.

DEFINITION Let $\lambda, a, b, c, d, \alpha, \beta, p \in C$ with $c, d \neq 0, -1, -2, \ldots; k > 0, \omega > 0$ and $Re(p) > 0$,we define

$$S\left(\begin{matrix}\lambda, a, b; c, d; \alpha, \beta; \\ u, v, \omega\end{matrix}\ p, k\right) := v^{-\lambda}\int_0^{\infty} t^{u-1}\emptyset^{\omega}(\alpha, \beta; -pt)\ \ {}_3R_2\left(\lambda, a, b; c, d; k; -\frac{t}{v}\right)dt \quad (7)$$

where $u \in \Omega_u, v \in \Omega_v$ and ${}_3R_2^k\left(\lambda, a, b; c, d; k; -\frac{t}{v}\right)$ is the generalized hypergeometric function.

For $\omega = 1$, the ω- confluent hypergeometric function reduces to confluent hypergeometric function[6]

$${}_1F_1(\alpha; \beta; z) = \sum_{k=0}^{\infty}\frac{(\alpha)_k z^k}{(\beta)_k k!} \quad (8)$$

where $|z| < \infty, \alpha, \beta > 0, \beta \neq 0,1,2, \ldots$

For $\omega = 1$,b=d ,(7) reduces to the following result given earlier by Saxena [5]

$$S\left(\begin{matrix}\lambda, a, b; c, b; \alpha, \beta; \\ u, v\end{matrix}\ p, k\right) := v^{-\lambda}\int_0^{\infty} t^{u-1}\phi(\alpha, \beta; -pt)\ \ {}_2R_1\left(\lambda, a; c; k; -\frac{t}{v}\right)dt \quad (9)$$

where $Re(u) > 0, Re(p) > 0, k > 0, |argv| < \pi.$

For $\alpha = \beta, \omega = 1$ the ω- confluent hypergeometric function reduces to an exponential function e^{-pt} and consequently for $\alpha = \beta, \omega = 1, b = d$,(7) reduces to the following result given by Virchenko et. al.[7].

$$S\left(\begin{matrix}\lambda, a, b; c, b; \alpha, \alpha; \\ u, v\end{matrix}\ p, k\right) = \Gamma\left(\begin{matrix}\lambda, a; c; \\ u, v\end{matrix}\ p, k\right) = v^{-\lambda}\int_0^{\infty} t^{u-1}e^{-pt}\ \ {}_2R_1\left(\lambda, a; c; k; -\frac{t}{v}\right)dt \quad (10)$$

where $Re(u) > 0, Re(p) > 0, and\ |argv| < \pi.$

If we set $\alpha = \beta, \omega = 1 and\ k = 1$, equation (7) reduces to the generalized gamma type function involving Clausenian hypergeometric series introduced and studied by Saxena and Kalla [8] in the form

$S\left(\begin{matrix}\lambda,a,b;c,d \\ u,v\end{matrix}\ p\right)=v^{-\lambda}\int_0^{\infty}t^{u-1}e^{-pt}\quad {}_3F_2\left(\lambda,a,b;c,d;-\frac{t}{v}\right)dt$(11)

where $Re(u)>0, Re(p)>0, and\ |argv|<\pi$.

For $\alpha=\beta,\omega=1,b=d\ and\ k=1$, equation (7) yields the following generalized gamma function studied by Al-Mussallam and Kalla[9,10].

$D\left(\begin{matrix}\lambda,a;c \\ u,v\end{matrix}\ p\right)=v^{-\lambda}\int_0^{\infty}t^{u-1}e^{-pt}\quad {}_2F_1\left(\lambda,a;c;-\frac{t}{v}\right)dt$ (12)

where $Re(u)>0, Re(p)>0, and\ |argv|<\pi$.

For a=c, b=d, $\alpha=\beta,\omega=1 and\ k=1$, equation (7) reduce to the following generalized gamma function studied by Kobayashi [3,4].

$\Gamma_m(u,v)=\int_0^{\infty}\frac{t^{u-1}e^{-t}}{(t+v)^m}dt$ (13)

where $Re(u)>0, |argv|<\pi\ and\ m\in N_0$.

Theorem1. S is analytic in the domain $\Omega_u\times\ \Omega_v$.

The proof is similar to the corresponding theorem for the generalized gamma function given by Saxena and Kalla [8,pp.191-192], if we employ the asymptotic estimate[2].

$${}_3R_2(\lambda,a,b;c,d;k;z)=A_1z^{-\lambda}+A_2z^{-\frac{a}{k}}+A_3z^{-\frac{b}{k}}+O\left(z^{-\lambda-1}\right)+O\left(z^{-\frac{a}{k}-1}\right)+O\left(z^{-\frac{b}{k}-1}\right)\qquad(14)$$

which holds for large z,$|arg(-z)|<\pi$. Here A_1,A_2,A_3 are numerical constants.

LEMMA 1. Let $\lambda,a,b,c,d,\alpha,\beta,p\in C$ with $c,d\neq 0,-1,-2,\ldots;k>0,\omega>0$ and $Re(vp)>0$, then S has the representation

$$S\left(\begin{matrix}\lambda,a,b;c,d;\alpha,\beta; \\ u,v,\omega\end{matrix}\ p,k\right):=v^{u-\lambda}\int_0^{\infty}t^{u-1}\emptyset^{\omega}(\alpha,\beta;-pvt)\quad {}_3R_2(\lambda,a,b;c,d;k;-t)\,dt,\qquad(15)$$

where $u\in\Omega_u, v\in\ \Omega_v$.

Proof. The proof is trivial.
For $b=d\ and\ \omega=1$,equation (15) reduces to a result given by Saxena, Ram, Naresh and Kalla [5,p.682]
When $\alpha=\beta,b=d\ and\ \omega=1$, equation (15) reduces to a result given by Virchenko et.al [10,p.97]

LEEMA 2. The partial derivatives of S are

$$\frac{\partial^n}{\partial u^n}S=v^{-\lambda}\int_0^{\infty}t^{u-1}\emptyset^{\omega}(\alpha,\beta;-pt)(logt)^n\quad {}_3R_2\left(\lambda,a,b;c,d;k;-\frac{t}{v}\right)dt,\qquad(16)$$

and

$$\frac{\partial^n}{\partial v^n} S = (-1)^n (\lambda)_n \, S\begin{pmatrix} \lambda+n, a, b; c, d; \alpha, \beta; & p, k \\ u, v, \omega & \end{pmatrix} \quad (17)$$

The proof of equations (16) and (17) is straight forward.

LEEMA 3. Let $\lambda, a, b, c, d, \alpha, \beta, p \in C$ with $c, d \neq 0, -1, -2, \dots; k > 0, \omega > 0$ and $Re(p) > 0$, then thefollowing relation holds

$$S\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & p, k \\ u, v, \omega & \end{pmatrix} = \frac{p \propto}{u\,\beta} \, S\begin{pmatrix} \lambda, a, b; c, d; \alpha+1, \beta+1; & p, k \\ u+1, v, \omega & \end{pmatrix}$$

$$+\frac{\lambda\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)} \, S\begin{pmatrix} \lambda+1, a+k, b+k; c+k, d+k; \alpha, \beta; & p, k \\ u+1, v, \omega & \end{pmatrix}, \quad (18)$$

Proof: If we use [1,equation (3.23)] for $\frac{d}{dz} \; {}_3R_2{}^k(z)$ and integrate by parts ,then equation (15) leads to equation (18).

Incomplete S-functions

A generalized incomplete gamma function corresponding to the S-function (15) is defined in the form

$$S_0^x\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & p, k \\ u, v, \omega & \end{pmatrix} := v^{-\lambda} \int_0^x t^{u-1} \emptyset^{\omega}(\alpha, \beta; -pt) \;\; {}_3R_2\left(\lambda, a, b; c, d; k; -\frac{t}{v}\right) dt, \quad (19)$$

where $x, k, \omega > 0$, $Re(u) > 0, Re(p) > 0 \; and \; |arg v| < \pi$.

The generalized complementary incomplete gamma function is defined by

$$S_x^{\infty}\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & p, k \\ u, v, \omega & \end{pmatrix} := v^{-\lambda} \int_x^{\infty} t^{u-1} \emptyset^{\omega}(\alpha, \beta; -pt) \;\; {}_3R_2\left(\lambda, a, b; c, d; k; -\frac{t}{v}\right) dt \quad (20)$$

where $x, k, \omega > 0$, $Re(u) > 0, Re(p) > 0 \; and \; |arg v| < \pi$.

Thus, the definitions (19) and (20) yield

$$S\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & p, k \\ u, v, \omega & \end{pmatrix} = S_0^x\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & p, k \\ u, v, \omega & \end{pmatrix} + S_x^{\infty}\begin{pmatrix} \lambda, a, b; c, d; \alpha, \beta; & p, k \\ u, v, \omega & \end{pmatrix} \quad (21)$$

For $\alpha = \beta, b = d$, (19) and (20) reduce to the generalized incomplete gamma functions developed by Virchenko et al. [10,p.98]. Further for $\alpha = \beta, b = d$, and $k = \omega = 1$, equations (19) and (20) yield the incomplete gamma functions given by Al-Musallam and Kalla [9].

Remark. If we set $\alpha = \beta, a = c, b = d, p = k = \omega = 1$ in (19) and (20) and $\lambda \to 0$, then we find that

$$\lim_{\lambda \to 0} S_0^x\begin{pmatrix} \lambda, a, b; a, b; \alpha, \alpha; & 1,1 \\ u, v, 1 & \end{pmatrix} = \gamma(u, x) = \int_0^x t^{u-1} e^{-t} dt \quad ,(22)$$

where $\gamma(u,x)$ is the incomplete gamma function of the first kind, and

$$\lim_{\lambda\to 0} S_x^{\infty}\begin{pmatrix}\lambda, a, b; a, b; \alpha, \alpha; & \\ u, v, 1 & 1,1\end{pmatrix} = \Gamma(u,x) = \int_x^{\infty} t^{u-1}e^{-t}dt\,, \tag{23}$$

where $\Gamma(u,x)$is the incomplete gamma function of the second kind.

The following theorem enumerates some properties and recurrence relations satisfied by $S_0^x\ \ and\ S_x^{\infty}$.The proofs are straight forward.

A class of probability density function

Let $m\ and\ \delta$ represent the shape parameters. Further, let $\sigma\ and\ \gamma$ denote the scale parameters. Then, by taking

$t = \sigma\, x^{\delta}$ and $dt = \sigma\delta x^{\delta-1}dx$, the definition (7) with $p = \frac{\gamma}{\sigma}(\gamma > 0; \sigma > 0), u = \frac{m+\delta}{\delta}(m+\delta > 0)$, and $v = n(n > 0)$ yields

$$\delta\sigma^{\frac{(m+\delta)}{\delta}}\int_0^{\infty} x^{m+\delta-1}\emptyset^{\omega}\left(\alpha,\beta;-\gamma x^{\delta}\right)\ {}_3R_2\left(\lambda, a, b; c, d; k; -\frac{\sigma x^{\delta}}{n}\right)dx \tag{24}$$

$$= n^{\lambda} S\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; & \\ 1+\frac{m}{\delta}, n, \omega & \frac{\gamma}{\sigma}, k\end{pmatrix}(min\{\gamma, \sigma, m+\delta, n\} > 0).$$

By virtue of integral formula (24), a class of probability density functions associated with the S-function can be defined by

$$f(x) \coloneqq \delta\sigma^{\frac{(m+\delta)}{\delta}} n^{-\lambda}x^{m+\delta-1}\emptyset^{\omega}\left(\alpha,\beta;-\gamma x^{\delta}\right)\ {}_3R_2\left(\lambda, a, b; c, d; k; -\frac{\sigma x^{\delta}}{n}\right)$$
$$\times\left\{S\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta; & \\ 1+\frac{m}{\delta}, n, \omega & \frac{\gamma}{\sigma}, k\end{pmatrix}\right\}^{-1}\quad (x > 0),$$

$=0$, elsewhere (25)

Provided that the various parameters and variable x occurring in equation (25) are so constrained that the density function is always non-negative. It is evident that

$$\int_{-\infty}^{\infty} f(x)dx = 1, \tag{26}$$

$$f(0)=\begin{cases}\delta\sigma^{\frac{1}{\delta}}n^{-\lambda}\left\{S\begin{pmatrix}\lambda,a,b;c,d;\alpha,\beta; & \frac{\gamma}{\sigma},k\\ 1+\frac{m}{\delta},n,\omega & \end{pmatrix}\right\}^{-1}, & (m+\delta=1)\\ 0, & (m+\delta)>1\end{cases} \quad (27)$$

$f(x)\to\infty\ as\ x\to 0^{+}\ when\, m+\delta<1$ (28)

$\lim_{x\to\infty} f(x)=0\ \ (\delta>0)$, (29)

and $f^{'}(x)=\left(\frac{m+\delta-1}{x}-\gamma\delta x^{\delta-1}-\frac{\sigma\delta}{n}x^{\delta-1\psi}\right)f(x),$ (30)

where for convenience,

$$\psi:=\frac{\lambda\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)}\frac{{}_3R_2\left(\lambda+1,a+k,b+k;c+k,d+k;k;-\frac{\sigma x^{\delta}}{n}\right)}{{}_3R_2\left(\lambda,a,b;c,d;-\frac{\sigma x^{\delta}}{n}\right)} \quad (31)$$

The formula (30) can be derived ,if we differentiate both sides of the equation(25) with respect to x logarithmically and apply the following formula$\frac{d}{dx}\left\{\ {}_3R_2\left(\lambda,a,b;c,d;-\frac{\sigma x^{\delta}}{n}\right)\right\}$

$$=-\frac{\lambda\sigma\delta\,\Gamma(c)\Gamma(d)\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(d+k)}x^{\delta-1}\ \ {}_3R_2\left(\lambda+1,a+k,b+k;c+k,d+k;k;-\frac{\sigma x^{\delta}}{n}\right)(32)$$

If we set $\alpha=\beta,\omega=1\ and\ b=d$ the results of this section reduce to the results due to Virchenko etal. [10].

Some statistical functions

In this section, serval basic statistical functions associated with the probability density function $f(x)$ defined by equation (25) will be evaluated explicitly.

The r^{th}moment

The r^{th} moment$\mu_r^{'}$ about the origin of a continuous real random variable X with the probability density function $f(x)$ is given by

$\mu_r^{/}:=\int_{-\infty}^{\infty}x^r f(x)dx=:E[x^r]\ (r\in N)$,(33)

which on using equation(24) and definition (25),gives

$$\mu_r^{/}=\sigma^{-\frac{r}{\delta}}S\begin{pmatrix}\lambda,a,b;c,d;\alpha,\beta; & \frac{\gamma}{\sigma},k\\ 1+\frac{(m+1)}{\delta},n,\omega & \end{pmatrix}\left\{S\begin{pmatrix}\lambda,a,b;c,d;\alpha,\beta; & \frac{\gamma}{\sigma},k\\ 1+\frac{m}{\delta},n,\omega & \end{pmatrix}\right\}^{-1} \quad (34)$$

In particular for r=1 the excepted value of random variable x (also referred to as the mean or first moment of X) is obtained as

$$E[x] := \int_0^\infty x f(x)\, dx$$

$$=\sigma^{-\frac{1}{\delta}} S\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta & \frac{\gamma}{\sigma}, k\\ 1+\frac{(m+1)}{\delta}, n, \omega & \end{pmatrix}\left\{S\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta & \frac{\gamma}{\sigma}, k\\ 1+\frac{m}{\delta}, n, \omega & \end{pmatrix}\right\}^{-1} \quad (35)$$

The moment generating function

The moment generating function $M(t,\delta)$ of a continuous random variable X having the probability density function $f(x)$ is defined by

$$M(t,\delta) = E[e^{tx}] := \int_{-\infty}^{\infty} e^{tx}\, f(x) dx$$

$$= \delta\sigma^{\frac{(m+\delta)}{\delta}} n^{-\lambda} \int_0^\infty x^{m+\delta-1} \emptyset^{\omega}(\alpha, \beta; -\gamma x^{\delta}) \quad {}_3R_2\left(\lambda, a, b; c, d; k; -\frac{\sigma x^{\delta}}{n}\right) dx$$

$$\times \left\{S\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta & \frac{\gamma}{\sigma}, k\\ 1+\frac{m}{\delta}, n, \omega & \end{pmatrix}\right\}^{-1} \quad (36)$$

Which itself the generalization of the result given by Saxena, Chena Ram, Naresh and Kalla [5] to which is reduce for $\omega = 1\ and\ b = d$.

For $\alpha = \beta, k = \omega = 1\ and\ b = d$the result (36) reduce to Kalla et.al.[11].

For $\alpha = \beta, k = \omega = n = 1\,, b = d\ and\ a = c$result (36) reduce Ghitany[12].

The Hazard rate function

For a continuous random variable X having the probability density function $f(x)$, the commulative distribution function $F(t)$ is given by

$F(t) := \int_{-\infty}^{t} f(x)dx =: prob\{X \in (-\infty, t)\}$(37)

That is by

$$F(t) = S_0^{\sigma\, t^{\delta}}\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta & \frac{\gamma}{\sigma}, k\\ 1+\frac{m}{\delta}, n, \omega & \end{pmatrix}\left\{S\begin{pmatrix}\lambda, a, b; c, d; \alpha, \beta & \frac{\gamma}{\sigma}, k\\ 1+\frac{m}{\delta}, n, \omega & \end{pmatrix}\right\}^{-1} \quad (38)$$

where we use the definition of (25) and (19). Thus by virtue of the relationship (21),the hazard (or failure)rate function h(t) is given by

$$h(t) := \frac{f(t)}{1-F(t)} =$$
$$\delta \sigma^{\frac{(m+\delta)}{\delta}} n^{-\lambda} t^{m+\delta-1} \emptyset^{\omega}\left(\alpha,\beta;-\gamma x^{\delta}\right) \;\; {}_3R_2\left(\lambda,a,b;c,d;k;-\frac{\sigma x^{\delta}}{n}\right) \times$$
$$\left\{ S^{\infty}_{\sigma t^{\delta}} \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta & \frac{\gamma}{\sigma},k \\ 1+\frac{m}{\delta},n,\omega & \end{pmatrix} \right\}^{-1} \qquad (t>0) \;\; (39)$$

in terms of the complementary incomplete S-function defined by equation (20).

In passing, we remark that the above derivations would also give the survival (or reliability) function $S^*(t)$ in the form

$$S^*(t) := 1 - F(t) = \int_t^{\infty} f(x)dx$$

$$= S^{\infty}_{\sigma t^{\delta}} \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta & \frac{\gamma}{\sigma},k \\ 1+\frac{m}{\delta},n,\omega & \end{pmatrix} \left\{ S \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta & \frac{\gamma}{\sigma},k \\ 1+\frac{m}{\delta},n,\omega & \end{pmatrix} \right\}^{-1} \quad (t>0)(40)$$

The mean residual life (or remaining life expectancy) function

For a continuous random variable X, the mean residual life (or remaining life expectancy) function K(t) is given by

$$K(t) := E[X-t \mid X \geq t] = \frac{1}{S^*(t)} \int_t^{\infty} S^*(x)dx = \frac{1}{S^*(t)} \int_t^{\infty} (x-t) f(x)\, dx, \qquad (41)$$

that is , by

$$K(t) = \frac{1}{S^*(t)} \int_t^{\infty} x f(x)\, dx - t \;, \qquad (42)$$

since

$$\frac{t}{S^*(t)} \int_t^{\infty} f(x)dx = \frac{t}{S^*(t)} \left(1 - \int_{-\infty}^{t} f(x)\, dx\right) = t \;, \qquad (43)$$

where $S^*(t)$ denotes the survivor (or reliability) function denoted by equation (40).

By virtue of the definition (25), if we use substitution$z = \sigma\, x^{\delta}$ and $dz = \sigma\, \delta\, x^{\delta-1} dx$,

The integral in equation (42) can be evaluated as

$$\int_t^{\infty} f(x)dx = \sigma^{-1/\delta} \;\; S^{\infty}_{\sigma t^{\delta}} \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta; & \frac{\gamma}{\sigma},k \\ \frac{m+1}{\delta}+1,n,\omega & \end{pmatrix} \left\{ S \begin{pmatrix} \lambda,a,b;c,d;\alpha,\beta; & \frac{\gamma}{\sigma},k \\ 1+\frac{m}{\delta},n,\omega & \end{pmatrix} \right\}^{-1} \qquad (44)$$

so that

$$K(t) = \sigma^{-1/\delta} \quad S_{\sigma t^{\delta}}^{\infty} \left(\begin{matrix} \lambda, a, b; c, d; \alpha, \beta; \\ \frac{m+1}{\delta} + 1, n, \omega \end{matrix} \; \frac{\gamma}{\sigma}, k \right) \left\{ S_{\sigma t^{\delta}}^{\infty} \left(\begin{matrix} \lambda, a, b; c, d; \alpha, \beta; \\ 1 + \frac{m}{\delta}, n, \omega \end{matrix} \; \frac{\gamma}{\sigma}, k \right) \right\}^{-1} - t \qquad (45)$$

In terms of the complementary incomplete S-function defined by equation (20).

Finally, it is expected that the results of this paper may find some applications in physical problems due to occurrence of the $\omega -$ confluent hypergeometric function in the density function (25),which includes various density functions that are useful in those problems arising in probability models.

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