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Z- transform and Sun's binomial inversion formulae

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Abstract: We employ the Z-transform to give elementary proofs of the Sun's binomial inversion formulae.

Keywords: Stirling numbers, Z-transform, Sun's binomial inversion, Bernoullinumbers.

1.- Introduction

The generating function for the Bernoulli numbers is given by [1-4]:

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1}, \tag{1}$$

which for $t = \frac{1}{z}$ allows to obtain the following Z-transform [5-7] for the sequence $\{\frac{B_0}{0!}, \frac{B_1}{1!}, \frac{B_2}{2!}, \dots\}$:

$$\frac{1}{z}(e^{1/z} - 1)^{-1} = \mathcal{Z}\{\frac{B_n}{n!}\}. \tag{2}$$

In Sec. 2 we determine the sequence $\{\frac{Q_n}{n!}\}$ such that:

$$(e^{1/z} - \lambda)^{-1} = \mathcal{Z}\left\{\frac{Q_n}{n!}\right\}, \qquad \lambda \neq 1, \tag{3}$$

and in Sec. 3 we use (2) and (3) to deduce the Sun's binomial inversion formulae [8-10].

2.- Determination of the quantities Q_n with the property (3)

It is evident the identity:

$$\frac{1}{z}(e^{1/z} - 1)^{-1} - \frac{1}{z}(e^{1/z} - \lambda)^{-1} = (1 - \lambda)\frac{1}{z}(e^{1/z} - 1)^{-1} \cdot (e^{1/z} - \lambda)^{-1}, \tag{4}$$

where we can apply Z^{-1}, (2) and (3) to obtain the recurrence relation:

$$B_n - n\,Q_{n-1} = (1-\lambda)\sum_{k=0}^{n}\binom{n}{k} Q_k B_{n-k}\,, \qquad n \geq 0, \tag{5}$$

which for $n = 0$ implies the value $Q_0 = (1-\lambda)^{-1}$. The expression (5) is equivalent to:

$$n\,Q_{n-1} = -\,(1-\lambda)\sum_{k=1}^{n}\binom{n}{k} Q_k B_{n-k}\,, \qquad n \geq 1, \tag{6}$$

and it allows determine the quantities Q_m , in fact:

$$Q_1 = -(1-\lambda)^{-2}\,, \qquad Q_2 = (1+\lambda)(1-\lambda)^{-3}\,, \qquad Q_3 = -\,(\lambda^2+4\lambda+1)(1-\lambda)^{-4}, \dots \tag{7}$$

If we remember the property [1]:

$$\frac{n}{j} S_{n-1}^{[j-1]} = \sum_{k=j}^{n}\binom{n}{k} S_k^{[j]} B_{n-k}\,, \qquad 1 \leq j \leq n, \tag{8}$$

involving the Stirling numbers of the second kind [1-3], then it is easy to obtain the following explicit solution of (6):

$$Q_n = \sum_{j=0}^{n} \frac{(-1)^j\; j!}{(1-\lambda)^{j+1}} S_n^{[j]}\,, \qquad n \geq 0, \qquad \lambda \neq 1, \tag{9}$$

in agreement with the values (7);hence from (3) and (9):

$$\mathcal{Z}\left\{\frac{1}{n!}\sum_{j=0}^{n} \frac{(-1)^j\; j!}{(1-\lambda)^{j+1}} S_n^{[j]}\right\} = \left(e^{1/z}-\lambda\right)^{-1}. \tag{10}$$

3.- Sun's binomial inversion formulae

If we consider the binomial expression:

$$F(n) = \sum_{k=0}^{n}\binom{n}{k} f(k) - \lambda\, f(n), \qquad n \geq 0, \tag{11}$$

then Sun [8-10] obtained the corresponding inversion formulae:

$$f(n) = \frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k} F(k) B_{n+1-k}\,, \qquad \lambda = 1, \tag{12}$$

$$f(n) = -\sum_{m=0}^{n}\binom{n}{m} F(m) \sum_{k=0}^{n-m} \frac{k!}{(\lambda-1)^{k+1}} S_{n-m}^{[k]}\,, \qquad \lambda \neq 1; \tag{13}$$

with the results deduced in Sec. 2 we can give elementary proofs of (12) and (13), in fact, from (11) for the case $\lambda = 1$:

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) - f(n) \quad \therefore \quad F(0) = 0, \tag{14}$$

where we apply the Z-transform to obtain:

$$\mathcal{Z}\left\{\frac{F(n)}{n!}\right\} = \mathcal{Z}\left\{\frac{f(n)}{n!}\right\}\left(\mathcal{Z}\left\{\frac{1}{n!}\right\} - 1\right), \qquad \mathcal{Z}\left\{\frac{1}{n!}\right\} = e^{1/z}, \tag{15}$$

then:

$$\mathcal{Z}\left\{\frac{f(n)}{n!}\right\} = (e^{1/z} - 1)^{-1}\,\mathcal{Z}\left\{\frac{F(n)}{n!}\right\} \overset{(2)}{=} z\,\mathcal{Z}\left\{\frac{B_n}{n!}\right\}\,\mathcal{Z}\left\{\frac{F(n)}{n!}\right\}, \tag{16}$$

such that:

$$z\,\mathcal{Z}\left\{\frac{F(n)}{n!}\right\} = z\left(\frac{F(1)}{1!\,z} + \frac{F(2)}{2!\,z^2} + \frac{F(3)}{3!\,z^3} + \cdots\right) = \mathcal{Z}\left\{\frac{F(n+1)}{(n+1)!}\right\}, \tag{17}$$

hence (16) implies the binomial transform $\frac{f(n)}{n!} = \sum_{k=0}^{n} \frac{F(k+1)}{(k+1)!}\frac{B_{n-k}}{(n-k)!}$ which is equivalent to (12), *q.e.d.*

The Z-transform of (11), for the case $\lambda \neq 1$, gives the relation:

$$\mathcal{Z}\left\{\frac{F(n)}{n!}\right\} = \mathcal{Z}\left\{\frac{f(n)}{n!}\right\}\left(\mathcal{Z}\left\{\frac{1}{n!}\right\} - \lambda\right) \overset{(15)}{=} \left(e^{1/z} - \lambda\right)\mathcal{Z}\left\{\frac{f(n)}{n!}\right\},$$

therefore:

$$\mathcal{Z}\left\{\frac{f(n)}{n!}\right\} = (e^{1/z} - \lambda)^{-1}\,\mathcal{Z}\left\{\frac{F(n)}{n!}\right\} \overset{(10)}{=} \mathcal{Z}\left\{\frac{1}{n!}\sum_{j=0}^{n} \frac{(-1)^j\, j!}{(1-\lambda)^{j+1}} S_n^{[j]}\right\}\mathcal{Z}\left\{\frac{F(n)}{n!}\right\},$$

which implies (13), *q.e.d.*

Remark.- We can exhibit an alternative proof of (12), in fact,we know the relation [14]:

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} B_{n-k} = \binom{n}{m}\left(B_{n-m} + \delta_{n-m,1}\right), \qquad n \geq m \geq 0, \tag{18}$$

thus:

$$\sum_{m=0}^{n}\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} B_{n-k}\, f(m) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m}\, f(m) + \binom{n}{n-1} f(n-1),$$

therefore:

$$n\, f(n-1) = \sum_{k=0}^{n} \binom{n}{k}\left[\sum_{m=0}^{k} \binom{k}{m} f(m) - f(k)\right] B_{n-k}\,,$$

in agreement with (11) and (12) for $\lambda = 1$.

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LRS Bianchi I Model with a Variable Λ-term in Self - Creation Theory of Gravitation in General Relativity

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Abstract

LRS Bianchi I cosmological model with a variable Λ-term under the framework of Barber's self-creation theory of gravitation is investigated. To get a physical model of the universe, we have assumed that the metric potentials are functions of x and t both. The equation of state is considered as $\rho = 3p$. The results of the model are consistent within the observational limits. The physical and kinematical aspects of the model are also discussed.

Keywords LRS Bianchi I ·Variable Λ-term · Self-creation · General relativity

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1 Introduction

It is well known that the Bianchi type-I space-time has a fundamental role in constructing cosmological models suitable for describing the early stages of evolution of universe. The present-day universe is satisfactorily described by homogeneous and isotropic models given by the FRW space-time. The universe in a smaller scale is neither homogeneous nor isotropic nor do we expect the universe in its early stages to have these properties. To unify gravitation and many other effects in the universe, several modifications of Einstein's general theory of relativity have been proposed and extensively studied by many cosmologists.

Ellis and MacCallum [1] have studied a class of homogeneous space-time in general relativity. Roy and Singh [2] have investigated Bianchi type-I non-static cosmological model filled with disordered radiation of perfect fluid. Jain et al. [3] have presented Bianchi type-I cosmological model with a varying Λ- term in self-creation theory. Bali et al. [4] have investigated a locally rotationally symmetric (LRS) Bianchi type-II space-time filled with string dust fluid where the shear is proportional to the expansion. Bianchi type-I magnetized radiating cosmological model in self-creation theory of gravitation is investigated by Jain and Jain [5].

The cosmological term-Λ provides a repulsive force opposing the gravitational pull between the galaxies. Linde [6] has suggested that Λ is a function of temperature and is related to the spontaneous symmetry breaking process, and therefore it could be a function of time. Recent investigations on the cosmological constant problem and consequence on cosmology with a time –varying cosmological constant are presented by many authors such as Ratra and Peebles [7], Sahni and Starobinsky [8] and Dolgov [9] etc. It is suggested that in the absence of any interaction with matter or radiation, the cosmological term-Λ remains a constant. The existence of the cosmological term-Λ is favourable to recent supernovae (SNe) Ia observations [10, 11] and it is also consistent with the recent anisotropy measurements of the cosmic microwave background (CMB) made by WMAP experiment [12].

In an attempt to produce a continuous creation theory, Barber [13] proposed two cosmological theories. In these theories the universe is seen to be created out of self-contained gravitational, scalar and matter fields. Barber's theories allow the scalar field to interact with the particle and photon momentum four vectors, which cannot happen in the Brans-Dicke theory **[14]** which develops Mach's principle in a relativistic framework, by assuming interaction of inertial masses of fundamental particles with some cosmic scalar field, coupled with the large-scale distribution of matter in motion. Thus,the Barber's first theory is a modified Brans-Dicke theory. The first theory was rejected on the grounds of a gross violation of the equivalence principle, which resulted in disagreement with experiment. Later Brans [15] also showed that it was internally inconsistent. Barber's second theory retains the attractive features of the first theory and overcomes previous objections. These modified theories create the universe out of self-contained gravitational and matter fields. In the second theory, the gravitational coupling of the Einstein's field equations is allowed to be a variable scalar on the space-time manifold.

In recent years, Sahu and Mohanty [16], Singh and Kumar [17] and Venkateshwarlu et al. [18] have studied Barber's second self-creation theory of gravitation in various contexts. Pradhan et al. [19] have evaluated LRS Bianchi type-I universe in Barber's second self-creation theory. Adhav et al. [20] have obtained axially symmetric Bianchi type-I model with massless scalar field and cosmic strings in Barber's self-creation cosmology. Jain and Jain [21] have investigated Bianchi type-I radiating model in Lyra Geometry and self-creation cosmology with constant deceleration parameter. Astankar et al. **[22]** and Tyagi **[23]** have found that the physical parameters are dominated by Barber's scalar function ϕ while studying LRS Bianchi type-II with bulk viscous string cosmological model.

Dark energy plays important role in cosmological model in current scenario which is studied by many authors in the framework of self - creation cosmology. Ram et al. **[24]** have discussed Kantowski-Sachs cosmological model. Rao and Prasanthi **[25]** for Bianchi type V, Katore and Kapse **[26]** for Bianchi type - I with polytropic equation of state. Jain and Jain **[27]** have studied self - creation cosmology in the context of Bianchi type - VI_0 with Dark energy and constant deceleration parameter. They find universe exhibits transition from deceleration to acceleration. Singh and Beesham **[28]** have investigated LRS Bianchi type I models with deceleration parameter $q = \alpha - 1$, for various values of α.

Bertolami **[29]** have studied the second order phase transition and inflation that is caused by scalar field.

In this paper LRS Bianchi type-I cosmological model for disordered radiation is investigated for the time-dependent cosmological term-Λ, in Barber's second self-creation theory of gravitation.

This paper is organized as follows: The metric and field equations are considered in sect. 2. Solutions of field equations are obtained in the sect. 3. The sect. 4 deals with some important physical and geometrical features of the model. In the last section i.e., sect. 5, conclusions are given.

2 The Metric and Field Equations

The FRW model has the disadvantage of being unstable near the singularity and it fails to describe early universe. Therefore, Bianchi Type I models are undertaken to understand the early universe on present day observations, hence considering LRS Bianchi type–I metric in the form

$$ds^2 = dt^2 - A^2 dx^2 - B^2(dy^2 + dz^2) \tag{2.1}$$

Where the metric potentials A and B are functions of x and $t$both.The field equations given by Barber [13] are

$$R_i^j - \frac{1}{2} R g_i^j = 8\pi\phi^{-1} T_i^j + \Lambda g_i^j, \tag{2.2}$$

and $\phi_{ik}^k = \frac{8\pi}{3}\lambda T,$ (2.3)

where ϕ_{ik}^k is the invariant d'Alembertian and the contracted tensor T is trace of the energy momentum tensor, that describes all non-gravitational and non-scalar field matter and energy. Here λ is a coupling constant to be determined from experiments.

The energy momentum tensor has the form

$$T_i^j = (\rho + p) v_i v^j - p g_i^j, \tag{2.4}$$

where v^i is the four-velocity vector of the fluid and p and ρ are the pressure and energy density, respectively.

Corresponding to metric (2.1), the four-velocity vector satisfies the relation

$$g_i^j v_i v^j = 1 \tag{2.5}$$

By adoption of co-moving coordinates, the field equations (2.2) and (2.3) for line element (2.1) is written as

$$2\frac{B_{44}}{B} + \frac{{B_4}^2}{B^2} - \frac{{B_1}^2}{A^2 B^2} = -8\pi\phi^{-1} p + \Lambda, \tag{2.6}$$

$$B_{14} - \frac{B_1 A_4}{A} = 0 \tag{2.7}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4 B_4}{AB} - \frac{B_{11}}{A^2 B} + \frac{A_1 B_1}{A^3 B} = -8\pi\phi^{-1} p + \Lambda \tag{2.8}$$

$$2\frac{B_{11}}{A^2 B} - 2\frac{A_1 B_1}{A^3 B} + \frac{{B_1}^2}{A^2 B^2} - 2\frac{A_4 B_4}{AB} - \frac{{B_4}^2}{B^2} = 8\pi\phi^{-1}\rho + \Lambda \tag{2.9}$$

and, $\phi_{44} + \frac{A_4 \phi_4}{A} + 2\frac{B_4 \phi_4}{B} + \frac{A_1 \phi_1}{A^3} - 2\frac{B_1 \phi_1}{A^2 B} - \frac{\phi_{11}}{A} = (\rho - 3p)\frac{8\pi\lambda}{3}.$ (2.10)

Here 1 and 4 indicate partial differentiation with respect to x and t respectively.

3 Solutions of the Field Equations

Equation (2.7) on integration leads to

$$A = \frac{1}{g} B_1, \tag{3.1}$$

where g is an arbitrary function of x only.

Using equation (3.1) in equations (2.7) and (2.9) we get

$$\frac{B}{B_1}\frac{d}{dx}\left(\frac{B_{44}}{B}\right) + \frac{B_4}{B_1}\frac{d}{dt}\left(\frac{B_1}{B}\right) + \frac{g^2}{B^2}\left(1 - \frac{g_1 B}{g B_1}\right) = 0 \tag{3.2}$$

To get the deterministic solution, we assume that $\frac{B_1}{B}$ as a function of x alone, then A and B are separable in x and t. Therefore, the equation (3.2), after integration, yields

$B = g(x)S(t),$ (3.3)
where S is an arbitrary function of t.
From equation (3.1) and (3.3), we have
$A = \frac{g_1}{g}S.$ (3.4)
Therefore, the metric (2.1) leads to
$ds^2 = dt^2 - S^2(t)[dx^2 + e^{2x}(dy^2 + dz^2)]$ (3.5)
Using equations (3.3) and (3.4) in equations (2.6), (2.9) and (2.10) we have
$\frac{2S_{44}}{S} + \frac{{S_4}^2}{S^2} - \frac{1}{S^2} = -8\pi\phi^{-1}p + \Lambda$ (3.6)
$\frac{3}{S^2} - \frac{3{S_4}^2}{S^2} = 8\pi\phi^{-1}\rho + \Lambda$ (3.7)
$\phi_{44} + \frac{3\phi_4 S_4}{S} - \frac{3\phi_4 g}{g_1 S^2} - \frac{g^2}{g_1 S^2}\frac{d}{dx}\left(\frac{\phi_1}{g_1}\right) = \frac{8\pi\lambda}{3}(\rho - 3p).$ (3.8)
For disordered radiation, equation of state is,
$\rho = 3p$ (3.9)
We assume that the scalar field (ϕ)depends on cosmic time t only, then equation (3.8) with the use of equation (3.9) leads to
$\phi_{44} + \frac{3\phi_4 S_4}{S} = 0$ (3.10)
This on twice integration leads
$\phi = k_1 \int \frac{1}{S^3} dt + k_2$ (3.11)
Where k_1 and k_2 are constants of integration.
To get the function $S(t)$ we assume the deceleration parameter to be constant i.e.
$q = -\frac{SS_{44}}{{S_4}^2} = k_3(\text{constant})$ (3.12)
Where $S(t) = (AB^2)^{1/3}$ is the overall scale factor. Here the constant is taken as negative i.e. it represents an accelerating universe.
The equation (3.12) can be written as
$\frac{S_{44}}{S} + k_3\frac{S_4}{S} = 0$ (3.13)
The solution of equation (3.13) yields
$S(t) = (at + b)^{\frac{1}{1+k}}$ (3.14)
Where
$a = k_4(1 + k_3)$ and $b = k_5(1 + k_3)$ (3.15)
Here k_4 and k_5 are constants of integration. Clearly equation (3.14) implies that the condition of expansion is $1 + k_3 > 0$.
Using equation (3.14) in equation (3.11) we get
$\phi = \alpha(at + b)^\beta + k_2 = \alpha T^\beta + k_2$ (3.16)
where
$\alpha = \frac{k_1}{\alpha\beta}$ and $\beta = \frac{k_3 - 2}{k_3 + 1}$ (3.17)
Clearly the scalar field remains finite throughout the evolution of universe. It has a singularity at $T = 0$ with $\beta < 0$ or $k_3 < 0$.
Metric (3.5) reduces to
$ds^2 = dt^2 - (at + b)^{2/1+k_3}[dx^2 + e^{2x}(dy^2 + dz^2)]$ (3.18)
After suitable transformation of coordinates, equation (3.18) reduces to
$dS^2 = \frac{1}{a^2}dT^2 - T^c[dX^2 + e^{2X}(dY^2 + dZ^2)]$ (3.19)

Equation (3.19) represents the LRS Bianchi type-I spatially homogeneous and anisotropic model for disordered radiation in Barber's self-creation theory of gravitation. The space time exhibits POINT-TYPE singularity (MacCallum, 30) at $T = 0$ i.e. $t = -b/a$ with $c > 0$, where $c = \frac{2}{1+k_3}$.

4 Some Physical and Geometrical Features

After using equation (3.9) in equations (3.6) and (3.7), the pressure, energy density and time dependent cosmological term are given by

$$p = \frac{1}{8\pi T^2}\left(\alpha T^\beta + k_2\right)\left[{k_4}^2\left(\frac{k_3}{2} - 1\right) + T^{\frac{2k_3}{1+k_3}}\right], \tag{4.1}$$

The energy density,

$$\rho = \frac{3}{8\pi T^2}\left(\alpha T^\beta + k_2\right)\left[{k_4}^2\left(\frac{k_3}{2} - 1\right) + T^{\frac{2k_3}{1+k_3}}\right], \tag{4.2}$$

The energy conditions [31]

$\rho + \mathrm{p} \geq 0$, $\rho - \mathrm{p} \geq 0$ and $\rho + 3\mathrm{p} \geq 0$, leads to

$$\rho + p = \frac{1}{2\pi T^2}\left(\alpha T^\beta + k_2\right)\left[{k_4}^2\left(\frac{k_3}{2} - 1\right) + T^{\frac{2k_3}{1+k_3}}\right] \geq 0 \tag{4.3}$$

$$\rho - p = \frac{1}{4\pi T^2}\left(\alpha T^\beta + k_2\right)\left[{k_4}^2\left(\frac{k_3}{2} - 1\right) + T^{\frac{2k_3}{1+k_3}}\right] \geq 0 \tag{4.4}$$

$$\rho + 3p = \frac{3}{4\pi T^2}\left(\alpha T^\beta + k_2\right)\left[{k_4}^2\left(\frac{k_3}{2} - 1\right) + T^{\frac{2k_3}{1+k_3}}\right] \geq 0 \tag{4.5}$$

The cosmological term,

$$\Lambda = -\frac{3}{2}k_3{k_4}^2\frac{1}{T^2}. \tag{4.6}$$

The scalar of expansion calculated for the flow vector v^i is given by

$$\theta = 3\frac{k_4}{T} \tag{4.7}$$

The three components of Hubble parameters (H_i, $i = 1,2,3$) are given by

$$H_1 = H_2 = H_3 = \frac{k_4}{T} \tag{4.8}$$

Hence the Hubble parameter H is given by

$$H = 3\frac{k_4}{T} \tag{4.9}$$

The anisotropy parameter is defined by

$$\bar{A} = \frac{1}{3}\sum_{i=1}^{3}\left[\frac{\Delta H_i}{H}\right]^2, \tag{4.10}$$

Where $\Delta H_i = H_i - H$ $(i = 1,2,3)$

For our model the anisotropy parameter is given by

$$\bar{A} = \frac{4}{9} \tag{4.11}$$

Which is constant.

The spatial volume (V) is evaluated as

$$V = S^3 = \sqrt{-g}AB^2 = (at + b)^{\frac{3}{1+k_3}} = T^{\frac{3}{1+k_3}} \tag{4.12}$$

The expansion velocity S_4 is given by

$$S_4 = \frac{a}{(1+k_3)}\frac{1}{T^{\frac{k_3}{1+k_3}}} \tag{4.13}$$

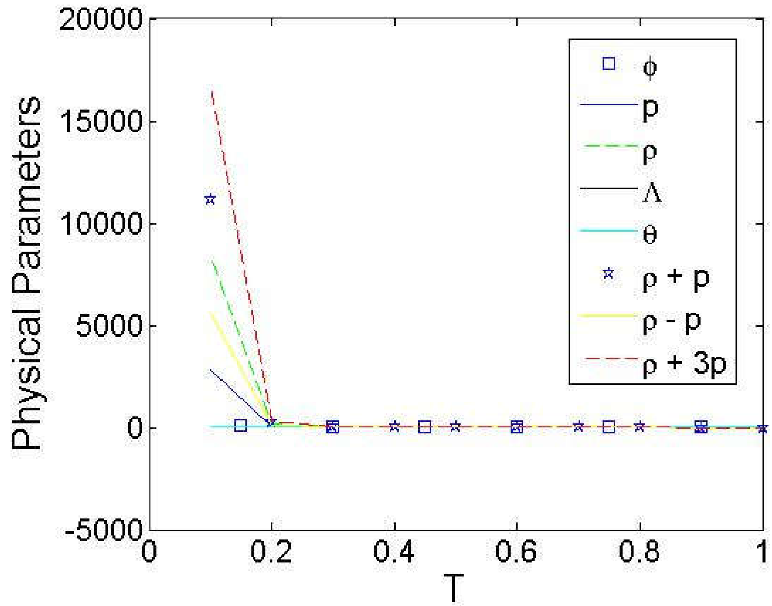
For the model (3.17), the particle horizon exist because

$$\int_{T_0}^{T} \frac{dt}{S(t)} = \frac{(1+k_3)}{ak_3}\left[(at+b)^{\frac{k_3}{1+k_3}}\right]_{T_0}^{T} \tag{4.14}$$

is a convergent integral.

The ratio of energy density to the square of the expansion scalar is calculated as

$$\frac{\rho}{\theta^2} = \frac{1}{24\pi {k_4}^2} T^{2k_3}\left(\alpha T^{\beta} + k_2\right)\left[{k_4}^2\left(\frac{k_3}{2} - 1\right) + T^{\frac{2k_3}{1+k_3}}\right] \tag{4.15}$$



In the model we observe that the spatial volume V is zero at $T = 0$ i.e. at $t = t_0$ and scalar expansion θ is infinite at initial singularity $t = t_0$ which shows that the universe starts evolving with zero volume and reaches infinite rate of expansion at $t = t_0$. Initially at $t = t_0$ the pressure p and energy density ρ are infinite. As t increases, the spatial volume V increases but the scalar expansion decreases. Thus, the expansion rate decreases as time increases. As $T \to \infty$, the spatial volume V becomes infinitely large.

It is worth mentioning that Λ is inversely proportional to the square of time T. Clearly $T \to 0$, gives $\Lambda \to \infty$, and $T \to \infty$, gives $\Lambda \to 0$. The cosmological term- Λ has constant value with in the range $0 < T < \infty$. The value of cosmological constant is in an excellent agreement with observations [32,11] of type Ia Supernovae (SNe). The main conclusion of these observations is, that the expansion of the universe is accelerating and the cosmological term was very large at initial times which relaxes to a genuine cosmological constant with due course of time.

The expansion velocity S_4 is given by diverges as $T \to 0$, i.e., $t \to t_0$. Hence the expansion of the universe is infinite as we approach towards $t \to t_0$. When k_3 = -1, i.e., q = -1 (accelerating universe), $\frac{\rho}{\theta^2}$ becomes constant, which means the energy density is proportional to the square of the scalar expansion. Hence the model approaches isotropy.

All the parameters $p, \rho, H_1, H_2, H_3, H$ and θ tend to zero when $T \to \infty$. Therefore, the model essentially gives an empty universe for large values of T. All the physical quantities remain finite and physically significant at finite region of the universe. In case $\lambda \to 0$, the solutions approach Einstein's general theory of relativity in all respects and the model represents non-rotating and expanding universe with a big - bang start.

Concluding Remarks

In this paper, we have obtained LRS Bianchi type-I cosmological model with a varying Λ-term for disordered radiation within the framework of Barber's second self-creation theory. From equation (4.7), we conclude that the model will represent an expanding universe. While solving Barber's field equations for disordered radiation, we have assumed deceleration parameter as constant. It is also observed that the Barber's scalar field ϕ is constant at $t \rightarrow t_0$i.e., $T \rightarrow 0$ with $\beta > 0$ or $k_3 > 2$ for disordered radiation universe model.

The anisotropic expansion of the universe with time is evident from the model (3.19). The value of cosmological constant Λ is in an excellent agreement with observations [32, 11] of type Ia Supernovae (SNe). The main conclusion of these observations is that the expansion of the universe is accelerating and the cosmological term was very large at initial times which relaxes to a genuine cosmological constant with due course of time.

The model obtained in this paper is of considerable interest and may be useful in Barber's self-creation theory to study an accelerating model of the universe. It is found that if $\lambda \rightarrow 0$ the Barber's self-creation theory tends to general theory of relativity in all respects.

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Dynamics of Evolution of Bogdanov system: Analysis of chaos and complexity By

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Abstract:
Dynamic Evolution of discrete 2-D nonlinear Bogdanov system has been studied in detail. Characteristic Bogdanov–Takens bifurcation observed by varying a certain parameter of the map. Pitchfork bifurcation observed at certain parameter space. During evolution this map shows chaotic behavior and interesting chaotic attractors emerged. Chaotic motionidentified through patterns of attractors and also, by positivity of Lyapunov exponents. Presence of complexity observed indicating internal multicomponent structure, the map shows complexity property which is different from chaos. All measurable quantities obtained through numerical simulations are represented through graphics and the results obtained are discussed with proper justification.

1. Introduction:

Almost all real systems are nonlinear and their evolution do not follow any definite rule. While solving nonlinear problems one has to adopt some specific rules.Internally many nonlinear systems structurally multicomponent and in such systems individual elementsevolve by their own rule which cannot be determined by deterministic rule. Only probabilistic rule can explain evolution criteria of such particles.Such systems termed as complex systems. During evolution a complex system exhibits chaos in some parameter space but also other phenomena called complexity. Complexity is due to the interaction among multiple agents within the system displayed in the form complex-patterns within periodic windows in bifurcation diagrams, form of existence of multiple attractors, bi-stability, intermittencyetc., [1–7].Study of complexity means investigating the dynamics that emerging from a collection of interacting parts.

Appearance of chaos most appropriately measured by Lyapunov exponents, (LCEs), [8–12]: if at any state measurement of LCE > 0 the evolution becomes chaotic and if LCE < 0 the evolution becomes regular. Presence of complexity in any system measured by increment in topological entropy: more increase, (or fluctuations), in topological entropy signifies the system is more complex [13 - 17].

The Bogdanov map, named after Russian mathematician *Rifkat Ibragimovich Bogdanov*. Bogdanov was known for his contributions to nonlinear dynamical systems, bifurcation theory and differential geometry.The Bogdanov map is a 2D planar quadratic map, conjugate to the Hénon-area-preserving map in its conservative limit, [18–20]. In dynamical systems theory, the map displays interesting chaotic attractorsand show bifurcations that indicate presence of complexity within the system.
Objective of the present article is to dynamic investigation of Bogdanov map and explore its evolutionary property. In the process of study bifurcation diagrams drawn by varying certain parameter while assigning values of other parameters. Analysis performed on bifurcations

including those of periodic windows appearing within chaotic region indicating complexity within the system. Numerical calculations extended to obtain some interesting chaotic attractors and to calculate Lyapunov exponents (LCEs). Simulation works performed to calculate topological entropies as measure of complexities. Results obtained displayed through different graphics. Finally, a brief discussion added on the study done.

2. Description of Bogdanov Map:

The map is a planer two-dimensional quadratic map and conjugate to the Hénon map in its non-dissipative limit. It is given by

$$\begin{aligned} x_{n+1} &= x_n + y_{n+1} \\ y_{n+1} &= y_n + \in y_n + k x_n (x_n - 1) + \mu x_n y_n \end{aligned} \tag{1}$$

Here, ϵ and μ are related to the Bogdanov vector field, while k plays the role of step length in the discretization, such that for a small k, the map behavior will resemble the original vector field.

Fixed points are real steady state solutions of a dynamical system. System (1) has fixed points $P_0^*(0,0)$ and $P_1^*(1,0)$irrespective of whatever values parameters$\in, k, \mu$ may assume.

Jacobian matrix of map (1) is $$J = \begin{pmatrix} 1 + 2kx - k + \mu y & 1 + \in + \mu x \\ 2kx - k + \mu y & 1 + \in + \mu x \end{pmatrix}. \tag{2}$$

(i)At $P_0^*(0,0)$ this Jacobian matrix is $$J_0 = \begin{pmatrix} 1 - k & 1 + \in \\ -k & 1 + \in \end{pmatrix} \tag{3}$$

Then, we find,Trace$(J_0) = 2 + \in - k$ and $|J_0| = 1 + \in$. Hence the fixed point $P_0^*(0,0)$ is non-hyperbolic on the line $(\in, k)$, [20]. Eigenvalues corresponding to fixed point $P_0^*(0,0)$obtained as

$$\lambda_{1,2} = \frac{1}{2}\left[(2 + \in - k) \pm \sqrt{(\in - k)^2 - 4k}\right] \tag{4}$$

(ii) At $P_1^*(1,0)$, Jacobian matrix (2) reduces to

$$J_1 = \begin{pmatrix} 1 + k & 1 + \in + \mu \\ k & 1 + \in + \mu \end{pmatrix} \tag{5}$$

Here, again we have Trace$(J_1) = 2 + \in + k + \mu$and $|J_1| = 1 + \in + \mu$. This is equivalent to the earlier case of the fixed point that$P_0^*(0,0)$ by certain adjustment within parameters, [19], and so $P_1^*(1,0)$ is non-hyperbolic. Attractors and orbits of different initial values around the fixed points are interesting and play significant role defining dynamics of map (1).

For values of $\in = 0, \mu = 0$ and$k = 1.2$, the fixed point $P_0^*(0,0)$shows neutral type of stability and behaves like a center. A set of orbits around this point provide very interesting criteria. For 20 chosen initial point around $(0,0)$, evolving orbits are drawn and shown in Figure 1.

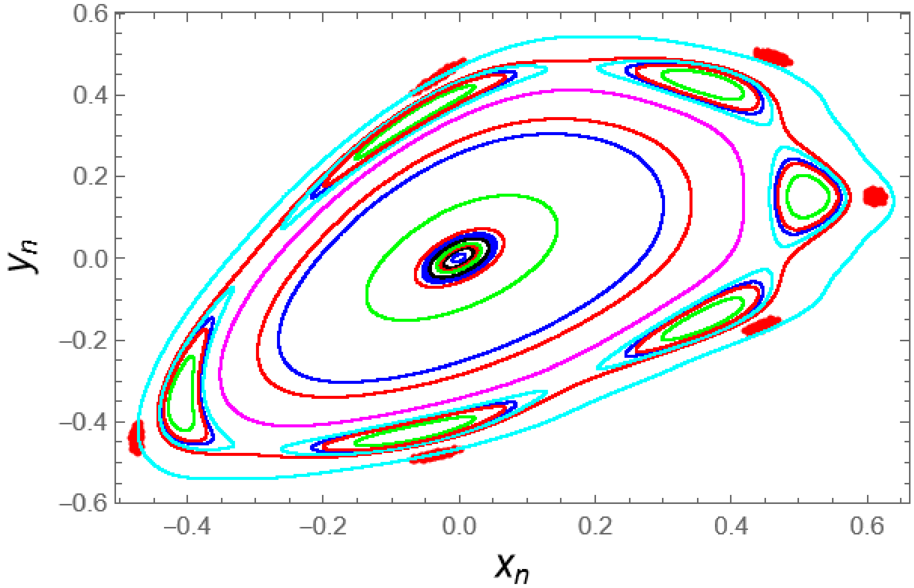

Figure 1: Orbits around fixed point $P_0^*(0,0)$ when $\in = 0, \mu = 0$ and $k = 1.2$.

For specific values of parameters $\in, k, \mu$ evolutionary dynamics of map (1) follows:

3. Bifurcation Analysis:
For k = 0.2, μ = 3 and $-2.5 \leq \epsilon \leq -1.0$, the system evolves into Pitchfork type of bifurcation, Figure $2(a)$, which is regular and two periodic. Regularity justifies from the time series plots and phase plot for $k = 0.2, \mu = 3, \epsilon = -2.0$, Figure $2(b)$. The phase plot shows only two points that corresponds to the 2-periodic motion.

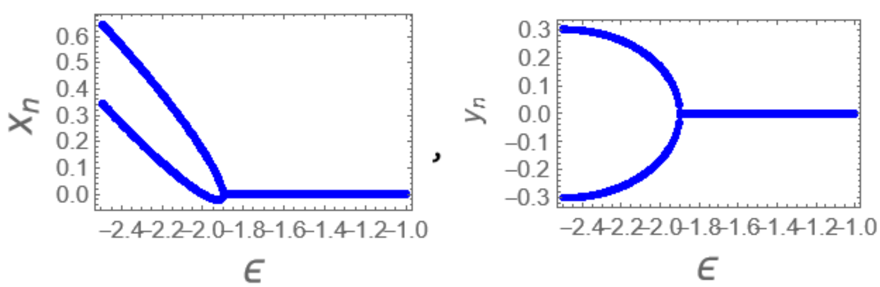

Figure 2(a): Pitchfork Bifurcation of map (1) for $k = 0.2, \mu = 3$ and $-2.5 \leq \in \leq -1.0$.

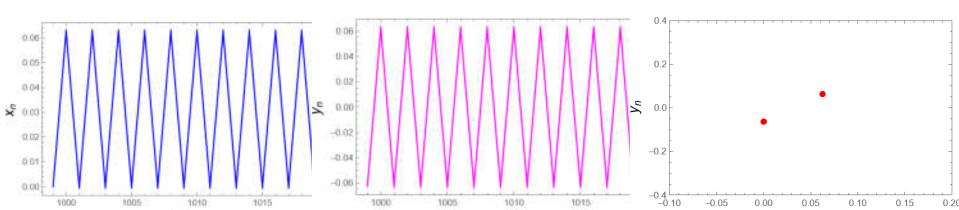

Figure 2(b):Time series curves and a phase plot $k = 0.2, \mu = 3, \epsilon = -2.0$.

Then, bifurcation diagrams of map (1) drawn for values $k = 1.44, \mu = -0.1$ and varying parameter $\in$ as$0.0001 \leq \in \leq 0.0045$ and shown in Figure $3(a)$. Also, by varying parameter k, $0.1 \leq k \leq 1.8$ and fixing values $\in = 0.02, \mu = -0.3$ bifurcation diagram drawn and shown in Figure $3(b)$. In both of these cases, periodic windows appearing within chaotic region; period-5 window in Figure $3(a)$ and period-6 window in Figure $3(b)$. These are indication of presence of complexity within the system.

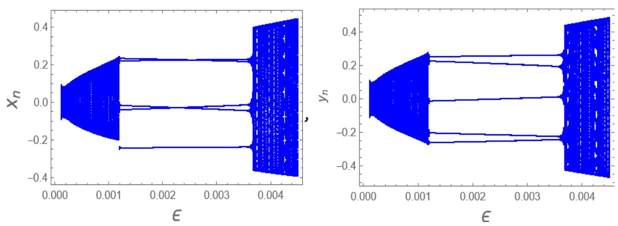

Figure $3(a)$: Bifurcation diagrams of map (1) for$k = 1.44, \mu = -0.1$.

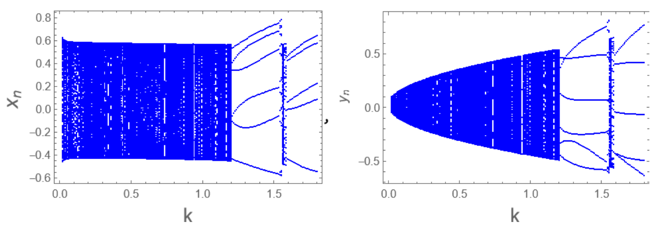

Fig. 3(b): Bifurcation of map (1) for $\in = 0.02, \mu = -0.3$ and $0.1 \leq k \leq 1.8$.

4. Attractors:

Regular 2 −periodic attractors already shown in Figure 2(b). Fork = 1.44, μ = - 0.1 and varyingϵ as shown, interesting chaotic attractors drawn for Bogdanov system (1) and presented in Figure 4 and in Figure 5.

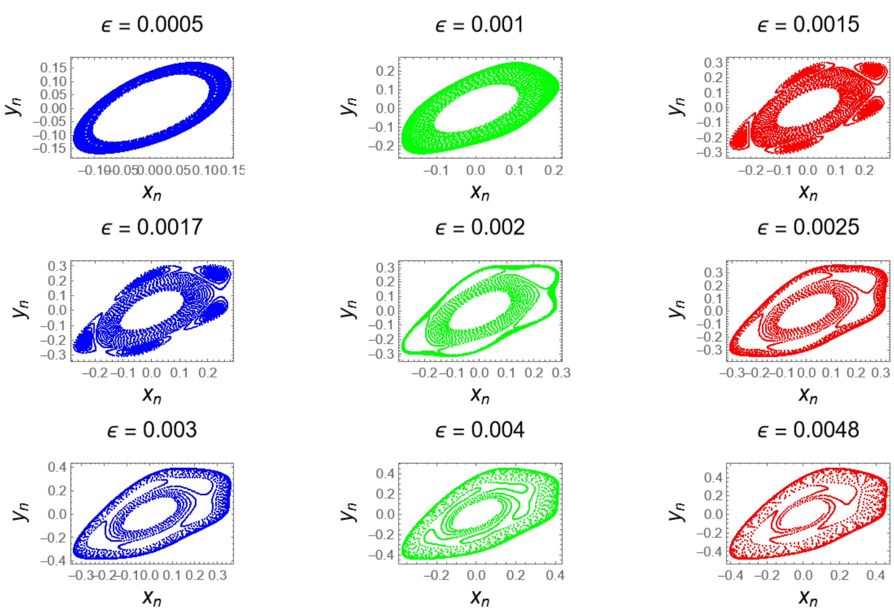

Figure 4: Attractors of map (1) for k = 1.44, μ = - 0.1 and different values of ε as shown.

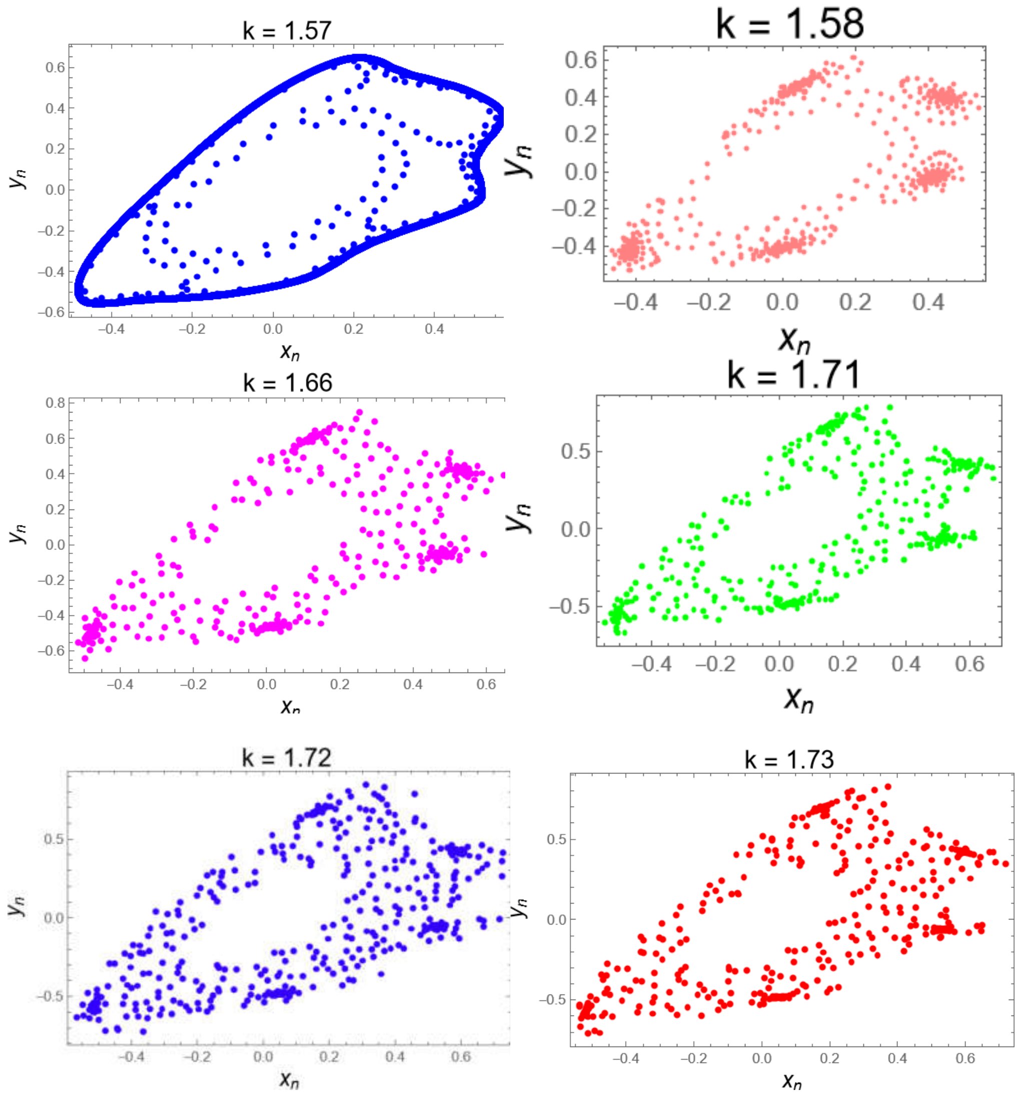

Figure 5: Chaotic attractors for $\in = 0.2$, $\mu = -0.3$ and varying values of k.

5. Lyapunov Exponents:

In case of chaos, system shows *sensitivity to initial conditions*, i.e., two trajectories originated extremely close to each other show divergence behavior during long term evolution.Lyapunov exponents (LCEs) proposed to measure exponential separations of such orbits,$[21-24]$. If at any state LCE > 0 the evolution becomes chaotic and if LCE < 0 the evolution becomes regular.

In case of Bogdanov system (1), there are cases when the initial evolution is chaotic (LCEs > 0), but in the long-term evolution, such motion converted into regularity (LCE< 0). This shown in Figure 7, for $k = 1.76, \mu = -0.1$ and $\in = 0.02$. Here,a chaotic attractor, figure (a), shown in the phase plane and figures (b), (c), (d) are LCE plots; figure (c) plotted in certain short range of evolution whereas (d) during long-term evolution.

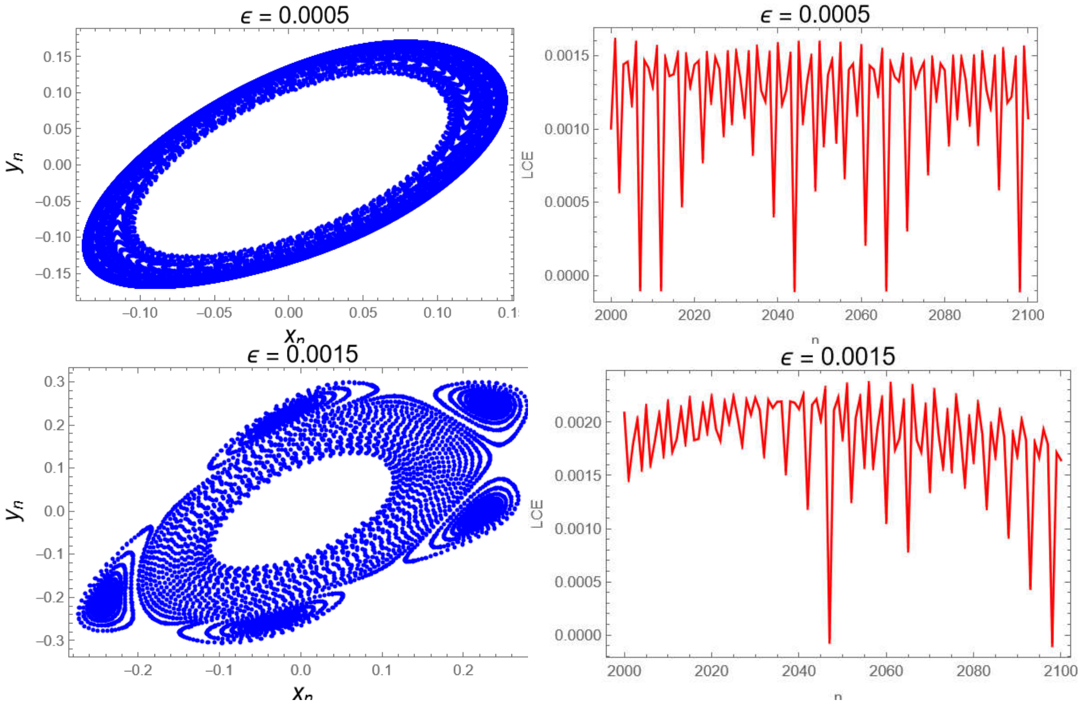

Figure 6: Plots of Lyapunov exponents (LCEs) for chaotic attractors for $k = 1.44, \mu = -0.1$ and values of $\in = 0.0005$ and $\in = 0.0015$.

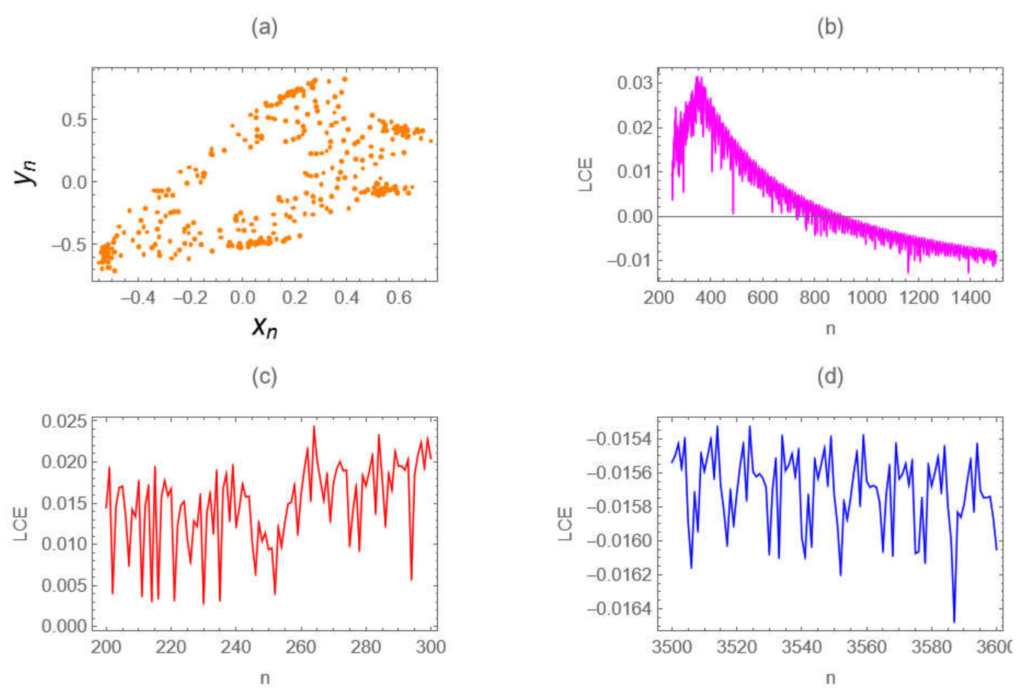

Figure 7:Showing initial chaotic motionfor $k = 1.76, \mu = -0.1$ and $\in = 0.02$ regularizes after long term evolution.

6. Topological Entropies: Measure of complexity.

Periodic windows appearing in bifurcation diagrams, Figure 3(a) and Figure 3(b), clearly indicating presence of complexity in Bogdanov system (1). During evolution their individual elements evolve in their own independent way and display mixed properties nonlinearity, chaos as well as complexity. Earlier studies made on this system, [20, 21], also indicating occurrence of Hopf bifurcation and multistability within very small parameter ranges. So, the study must be performed with some new perspectiveand the law of probability be applied to describe the rate of mixing of evolutions.By this way one measures the exponential growth rate of the number of distinguishable orbits as time advances.The topological entropy, (also called Kolmogorov – Sinai entropy, [25]),provides measure the presence of complexity in the system.We define *topological entropy*as a nonnegative number which measures the *complexity* of the system.

In order to explain method to measure topological entropy, consider a finite partition of a state space X denoted by$P = \{A_1, A_2, A_3, \ldots, A_N\}$. Then a measure μ on X with total measure $\mu(X) = 1$ defines the probability of a given reading as

$$p_i = \mu(A_i), i = 1, 2, .., N. \tag{6}$$

Then the entropy of the partition be given by

$$H(P) = -\sum_{i=0}^{N} p_i \text{Log} p_i \tag{7}$$

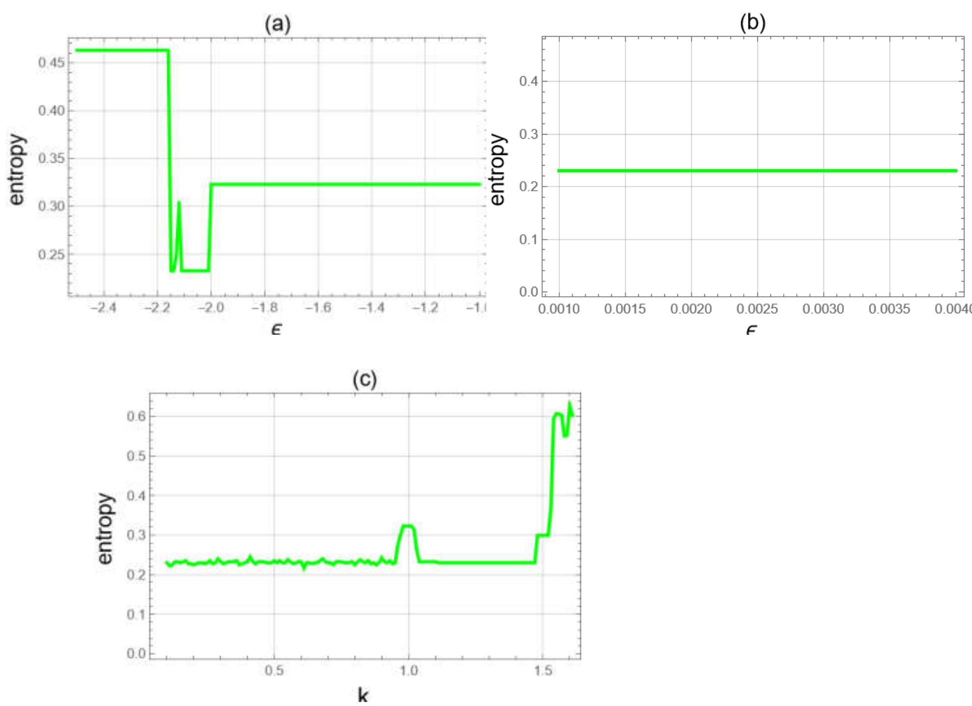

Figure 8: Plots of topological entropies; (a): $\mu = 1.0, k = 0.2$ and $-2.5 \leq \epsilon \leq -1.0$, (b) $\mu = -0.1, k = 1.44$ and$0.001 \leq \epsilon \leq 0.0045$ and (c) $\mu = -0.1, \epsilon = 0.02$ and $0.1 \leq k \leq 1.7$.

Above figure, Figure 6, represents plots of topological entropies for three cases of map (1); (a): $\mu = 1.0, k = 0.2$ and $-2.5 \leq \epsilon \leq -1.0$, (b) $\mu = -0.1, k = 1.44$ and $0.001 \leq \epsilon \leq 0.0045$ and (c) $\mu = -0.1, \epsilon = 0.02$ and $0.1 \leq k \leq 1.7$. One observes in figure (a) significant increase of topological entropy, in figure (b) significant but constant increase of topological entropy and in figure (c) significant and fluctuating increase in topological entropy.

7. Concluding Remarks:

In this article, specific bifurcations in quadratic type of Bogdanov map investigated. The evolution of chaotic attractors, which appear or disappear by contact bifurcations, with their own basin boundary has been observed. The system evolves into pitchfork bifurcation for values of $k = 0.2, \mu = 3$ and $-2.5 \leq \in \leq -1.0$, Figure 2(a).Interesting attractors, Figure 4 & Figure 5, drawn for map (1) in some parameter space. In some cases, the motion of the map display chaos, but in long term evolution, such motion turns into regularity, (see Figure 7). This shown by calculating LCEs by using Mathematica codes by Martelli, [28]. Periodic windows, of period *five* and *six* (Figure 3(a) and Figure 3(b)), appearing within chaotic region indicating presence of complexity in the system. Topological entropies calculated as measure of complexity and presented in Figure 8. Topological entropy has significant value even if the system is not chaotic.

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On the Gallego-Torromé's identity relating Ramanujan sums and the Euler totient function

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Abstract: We give an elementary proof of an identity involving the Euler totient function and Ramanujan sums, obtained by Gallego-Torromé, and we realize its application to Möbius and Pillai functions.

Keywords: Euler totient function, Ramanujan sums, Pillai's and Möbius functions.

1.- Introduction

Here we show the relation:

$$\varphi(n) = C(m,n) + 2\sum_{k=1,\ (k,n)=1}^{n-1} Sin^2\left(\frac{\pi k m}{n}\right), \qquad m \geq 0, \quad n \geq 1, \tag{1}$$

involving the Euler's totient function [1-6] and Ramanujan sums [2, 5, 7-9]. For the case $m = 2M$, (1) gives the identity obtained by Gallego-Torromé [10].

2.- Proof of the property (1)

In fact, by the definition of Ramanujan sums [2, 5]:

$$C(m,n) = \sum_{k=1,\ (k,n)=1}^{n-1} e^{i\,2\pi k m/n} = \sum_{k=1,\ (k,n)=1}^{n-1} Cos\left(\frac{2\pi k m}{n}\right) = \sum_{k=1,\ (k,n)=1}^{n-1} \left(1 - 2\,Sin^2\left(\frac{\pi k m}{n}\right)\right), \tag{2}$$

but $\varphi(n) = \sum_{k=1,\ (k,n)=1}^{n-1} 1$, then (2) implies (1), *q.e.d.*

Now we accept that *n* and *m* are relatively prime, that is, $(m,n) = 1$, then (1) takes the form:

$$\varphi(n) = \mu(n) + 2\sum_{k=1,\ (k,n)=1}^{n-1} Sin^2\left(\frac{\pi k m}{n}\right), \qquad m \geq 1, \quad n \geq 1, \tag{3}$$

with the presence of Möbius function [2, 4, 5, 11, 12]. If d/n then $(m,d) = 1$ and (3) gives the expression:

$$\varphi(d) = \mu(d) + 2\sum_{k=1,\ (k,d)=1}^{d-1} Sin^2(\frac{\pi k m}{d}), \quad m \geq 1, \tag{4}$$

where we can apply the Gauss identity $\sum_{d/n}\varphi(d) = n$ and $\sum_{d/n}\mu(d) = e_0(n)$ to obtain the property:

$$n = 2\sum_{d/n}\sum_{k=1,\ (k,d)=1}^{d-1} Sin^2(\frac{\pi k m}{d}), \quad m \geq 1, \quad n \geq 2, \quad (m,n) = 1; \tag{5}$$

if *n* is a prime number, then from (5):

$$p = 2\sum_{k=1,\ (k,p)=1}^{p-1} Sin^2(\frac{\pi k m}{p}), \quad m \geq 1, \quad p \geq 2, \quad (m,p) = 1. \tag{6}$$

This result (6) also can be deduced from (3)because $\varphi(p) = p - 1$and $\mu(p) = -1$.

We multiply (4) by $\frac{n}{d}$and after we applyv$\sum_{d/n}$ to obtain the following expression for the Pillai's function [2, 13]:

$\beta(n) := \sum_{j=1}^{n}(j,n) = \sum_{d/n}\frac{n}{d}\,\varphi(d) = \varphi(n) + 2\sum_{d/n}\frac{n}{d}\sum_{k=1,\ (k,d)=1}^{d-1} Sin^2(\frac{\pi k m}{d}), \quad (m,n) =$ 1,(7)

because $\varphi(n) = \sum_{d/n}\frac{n}{d}\,\mu(d)$.

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Varying G and Λ Models for a Barotropic Fluid with Magnetic Field in General Relativity

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Abstract

The Bianchi type V cosmological model for Bulk viscous barotropic fluid with variable gravitational constant [G(t)], and the variable cosmological constant [$\Lambda(t)$], in presence of magnetic field is investigated. To get a determinate model, we impose a physically viable condition between metric potentials. We have also used, $p = \gamma\rho$, and $\eta = \eta_0\rho^s$ where ρ is energy density, η is shear viscosity,H is the Hubble parameter, and $P = p - 3\eta H$ with $0 \le \gamma \le 1$. Some physical and kinematical characteristics of the model are also discussed .

Keywords: Bianchi Type V models; Varying G;Varying cosmological constant; Magnetic field

1 Introduction

The sign of curvature for Bianchi type-V space time is negative so it represents a model of open universe **[1]**.The Open universes (k = - 1) are examples of the low density models. The natural generalization of Friedmann – Robertson – Walker models (k = - 1) lead to the Bianchi type – V space-time [**2**]. As a generalization of open universe, Bianchi type – V models are interesting to study because these have richer structure, physically as well as geometrically [**3**].A barotropic fluid has pressure and density that are connected by a state equation which does not include temperature as a dependent variable. The equation of state of perfect fluid can be written as $p = p(\rho)$ or $\rho = \rho(p)$. A specific example of barotropic fluid [**4**] is one with a linear equation of state such as$p = \gamma\rho,\ 0 \le \gamma \le 1$.
The magnetic field is known to be present in galactic and intergalactic areas. According to Melvin [**5**], matter was strongly ionized during the evolution of the cosmos, but as the universe expanded, it became smoothly linked with the field and formed neutral matter. The current magnetic field strength is very low. This strength might have been noticeable in the early universe. So at that time,isotropy break down resulted, due to the presence of significant magnetic field [**6**]. Therefore it is evident that the matter fields, such as magnetic fields have a profound influence on the evolution of the universe **[7].**

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The incorporation of an electromagnetic field and matter into the space-time of Bianchi type-V with equation of state $p = \epsilon$, has been studied by Ftaclas and Cohen**[8]**. Lorenz**[9]**hasstudied an exact Bianchi type-V tilted cosmological model with matter and electromagnetic field.Singh [**10**] has investigated the Bianchi type-V cosmological solutions of massive strings in the presence and absence of the magnetic field. Bali and Jain **[11]**have studied the Bianchi type V magnetized string dust cosmological model for perfect fluid distribution. Bali **[12]**has investigated Bianchi Type V magnetized string dust universe with variable magnetic permeability. Kumar and Srivastava**[13]**have studied some new aspects of the Bianchi type-V space time.Billyard et al. **[14]**have studied scalar field cosmologies with barotropic matter models of Bianchi class B.Bali and Sharma **[15]**have analyzed tilted Bianchi type I cosmological models for barotropic perfect fluid in general relativity.The Bianchi type-V models have also been investigated by a large number of authors viz. [1, 16, 17, 18].

In Einstein's field equations, the Newtonian gravitational constant G, and the cosmological constant Λ both are allowed to be included. As a consequence, the G acts as a coupling constant for geometry and matter. The "Zwicky Theory", was proposed by Milne [**19,20**] in which the red shift spectra of distant galaxies were considered as a function of time scale factor variation with the assumption that the G depends on cosmological time t . Dirac's "Large Number Hypothesis'' [**21**] is the basis for cosmologies in which G decreases as time increases. In 1917, Einstein introduced the cosmological constant Λ, as a motivation to consider universal repulsion necessary to keep the universe static. The cosmological constant problem [**22**] has a solution through Weinberg's [**23**] suggestion that the Λ should be a function of temperature so that it may be made related to the spontaneous symmetry breaking. FRW cosmology is studied in such a way that the models with a cosmological constant seem more interesting **[24]**.Beesham **[25]** points out that the observations suggests that the varying Gravitational constant G is inversely proportional with respect to time t. Rahman **[26]** has studied varying Gravitational constant G and cosmological constant Λ, and obtained that G increases, and Λ decreases with time. Berman **[27]** and Kallinga **[28]**have also found that the value of cosmological term at early universe was very high. Bali and Tinker **[29]**haveinvestigated the Bianchi type-V bulk viscous barotropic fluid cosmological model with variable gravitational constant G and the cosmological constant Λ. Borkar et al. **[30]** have studied Bianchi type I bulk viscous barotropic fluid cosmological model with varying Λ involving a functional relation on Hubble parameter in self-creation theory of gravitation. Chaubey and Shukla **[31]**have discovered that the cosmological constant Λ is a positive decreasing function of time. This is supported by recent Supernovae Ia observations.Chaubey et al. **[32]**have discussed a general class of Bianchi cosmological models in the presence of dark energy with variable Λand G under the framework of viscous cosmology and found that Λand G are linear functions of time t with negative slope.Recently Dixit et al.**[33]**have studied the particle creation in FLRW higher dimensional universe with gravitational and cosmological constants. Tiwari et al. **[34]**have studied accelerating universe with varying Λ in *f*(*R, T*) theory of gravity. Many author'shave discussed that the gravitational constant G and cosmological constant Λ are $\sim R^{-2}$ $(also \sim t^{-2})$.Recently Bali and Tinker **[29]**have studiedBianchi type-V bulk viscous barotropic fluid cosmological model with variable G and Λ.Naidu et al. **[35]**have investigated Bianchi type-V bulk viscous string cosmological model in f (R,T) gravity with the bulk viscosity. Tiwari and Singh [**36**] have analyzed Bianchi type - V cosmological models with perfect fluid in presence of varying G and Λ and observe that the solutions favor the Λ CDM model. Goswami et al. [**37**] have also studied the existence of Λ- dominated anisotropic

universe filled with magnetized strings. Motivated by the above, we are presenting a varying G and Λ models in presence of Magnetic Field.
The organization of this paper is as follows: In section 2, we have established Einstein field equation. In next section 3 we have solved them using$B = C^n$, $\eta = \eta_0 \rho^s$ and $p = \gamma\rho\,, (0 \leq \gamma < 1)$.In section 4 we finds physical parameters, and geometrical features of the model. The physical behavior of the modelsare analyzed in section 6. Conclusion of the paper is presented section 7.

2. Bianchi Type-V Model and Field Equations:

Let us consider Bianchi Type V model representation in the form

$$ds^2 = -dt^2 + A^2 dx^2 + e^{2x}(B^2 dy^2 + C^2 dz^2)\,. \tag{1}$$

Einstein's Field Equation is given by

$$R_i^j - \frac{1}{2} R g_i^j = -8\pi G T_i^j + \Lambda g_i^j \quad , \tag{2}$$

where G ,and Λ are gravitational constant and cosmological constant respectively. Both are considered as function of time t.
The energy momentum tensor, T_i^j is

$$T_i^j = (\rho + P) v_i v^j + P g_i^j + E_i^j. \tag{3}$$

Here E_i^j is defined as follows [**12**, **38**],

$$E_i^j = \bar{\mu}\left[|h|^2 \left(v_i v^j + \frac{1}{2} g_i^j\right) - h_i h^j\right], \tag{4}$$

with

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}} \epsilon_{ijkl} F^{kl} v^j \tag{5}$$

The non-vanishing component of electromagnetic field tensor is, $F_{23} = K$, where K is constant.
We assume

$$P = p - 3\eta H, \tag{6}$$

where p is the equilibrium pressure, η is the coefficient of viscosity and ρ is the energy density, together with $v_i v^j = -1$.

To get deterministic model, we assume that the magnetic permeability $\bar{\mu}$ is a variable quantity and assume that$\bar{\mu} = e^{-4x}$.
Einstein's field equation (2) for the Bianchi Type-V metric (1) with equations (3), (4) and (5) reduces to,

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4 C_4}{BC} - \frac{1}{A^2} = -8\pi G\left(P - \frac{K^2}{2B^2C^2}\right) + \Lambda \tag{7}$$

$$\frac{A_{44}}{A}+\frac{C_{44}}{C}+\frac{A_4C_4}{AC}-\frac{1}{A^2}=-8\pi G\left(P+\frac{K^2}{2B^2C^2}\right)+\Lambda \tag{8}$$

$$\frac{B_{44}}{B}+\frac{A_{44}}{A}+\frac{B_4A_4}{BA}-\frac{1}{A^2}=-8\pi G\left(P+\frac{K^2}{2B^2C^2}\right)+\Lambda \tag{9}$$

$$\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}+\frac{B_4C_4}{BC}-\frac{3}{A^2}=8\pi G\left(\rho+\frac{K^2}{2B^2C^2}\right)+\Lambda \tag{10}$$

$$\frac{2A_4}{A}-\frac{B_4}{B}-\frac{C_4}{C}=0 \tag{11}$$

In the above field equations, suffix 4 represents differentiation with respect to the time variable.

The divergence of Einstein tensor $\left(R_i^j-\frac{1}{2}g_i^j\right)_{;j}=0$, gives one more equation.

It leads to $\left(8\pi GT_i^j-\Lambda g_i^j\right)_{;j}=0$, then from equation (3), we get

$$8\pi G\left[\frac{2K^2}{B^2C^2}+(P+\rho)\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)+\rho_4\right]+8\pi G_4\left(\rho+\frac{K^2}{2B^2C^2}\right)+\Lambda_4=0. \tag{12}$$

The conservation of energymomentum tensor gives us

$$\frac{2K^2}{B^2C^2}+(p+\rho)\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)+\rho_4=0. \tag{13}$$

Now concluding,

$$\eta=\eta_0\rho^s, \tag{14}$$

where η_0 is a positive number and s is a constant.

To find the complete solution of the model, wealso assume the condition

$$p=\gamma\rho\,,(0\le\gamma<1). \tag{15}$$

3. Solutions of the Field Equations:

Here we solve the Einstein field equations analytically. These are five nonlinear ordinary differential equations in seven unknowns, so we need at least two constraints to solve them exactly. We take,

$$B=C^n \tag{16}$$

Where n is a positive constant.

Equations (8) and (9) lead to

$$\frac{C_{44}}{C}+\frac{A_4C_4}{AC}=\frac{B_{44}}{B}+\frac{A_4B_4}{AB} \tag{17}$$

Using equation (11) in equation (16) we obtain,

$$\frac{B_{44}}{B}+\frac{1}{2}\left(\frac{B_4}{B}\right)^2=\frac{C_{44}}{C}+\frac{1}{2}\left(\frac{C_4}{C}\right)^2 \tag{18}$$

Using equation (16) in equation (18) we get,

$$\frac{C_{44}}{C}=\frac{\left(n-\frac{3}{2}n^2+\frac{1}{2}\right)}{(n-1)}\cdot\left(\frac{C_4}{C}\right)^2 \tag{19}$$

Integrating equation (19) with respect to the time variable we get,

$$C=[(1-\alpha)(k_1t+k_2)]^{\frac{1}{(1-\alpha)}} \tag{20}$$

Here $\alpha = \frac{2n-3n^2+1}{2(n-1)}$, k_1 and k_2 are constants of integration.
Putting the value of C from equation (20) in equation (16) we obtain

$$B = [(1-\alpha)(k_1 t + k_2)]^{\frac{n}{(1-\alpha)}} \tag{21}$$

and

$$A = k_3[(1-\alpha)(k_1 t + k_2)]^{\frac{n+1}{2(1-\alpha)}} \quad , \tag{22}$$

where k_3 is a constant of integration.
Hence the metric (1) leads to the following form

$$ds^2 = -dt^2 + {k_3}^2[(1-\alpha)(k_1 t + k_2)]^{\frac{(n+1)}{(1-\alpha)}}dx^2 + e^{2x}\Big([(1-\alpha)k_1 t + k_2]^{\frac{2n}{(1-\alpha)}}dy^2 + [(1-\alpha)k_1 t + k_2]^{\frac{2}{(1-\alpha)}}dz^2\Big) \tag{23}$$

After using suitable transformation the metric (23) reduces to

$$dS^2 = -\frac{1}{{k_1}^2}dT^2 + [(1-\alpha)T]^{\frac{n+1}{(1-\alpha)}}dX^2 + e^{2x}[(1-\alpha)T]^{\frac{2n}{(1-\alpha)}}dY^2 + e^{2x}[(1-\alpha)T]^{\frac{2}{(1-\alpha)}}dZ^2 \tag{24}$$

Where $k_3 x = X$,
$y = Y$,
$z = Z$,
$s = S$, and
$k_1 t + k_2 = T$.

4. Some physical parameters

Now we evaluate some parameters to characterize our model.
Subtracting equation (7) from (10) we get,

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{A_4 B_4}{AB} - \frac{A_4 C_4}{AC} + \frac{2}{A^2} = -8\pi G(P+\rho) \tag{25}$$

Using equations (20),(21),(22) and (15) in the equation (25) we obtain

$$8\pi G = \frac{\left\{(n+1)\frac{C_{44}}{C} - \left(\frac{(n+1)^2}{2}\right)\left(\frac{C_4}{C}\right)^2 + \frac{8}{C^{(2n+2)}}\right\}}{3\eta H - (1+\gamma)\rho}. \tag{26}$$

From equation (13)

$$\rho_4 + \frac{3(1+\gamma)(n+1)}{2}.\frac{C_4}{C}.\rho = -\frac{2K^2}{C^{(2n+2)}}. \tag{27}$$

Using first order linear differential equations techniques we get density of the model is,

$$\rho = \frac{1}{T^b}.\left\{\frac{-2K^2}{d}.\frac{1}{k_1}.\frac{T^{(c+1)}}{(c+1)} + L\right\} \tag{28}$$

Where L is a constant of integration with,

$$b = \frac{3(1+\gamma)(n+1)}{2(1-\alpha)}, \tag{29}$$

$$c = \frac{(3\gamma-1)(n+1)}{2(1-\alpha)}, \tag{30}$$

$$d = (1-\alpha)^{\left(\frac{2(n+1)}{1-\alpha}\right)}. \tag{31}$$

Equation (15) gives pressure as,

$$p = \frac{\gamma}{T^b}.\left\{\frac{-2K^2}{d}.\frac{1}{k_1}.\frac{T^{(c+1)}}{(c+1)} + L\right\}. \tag{32}$$

Using the equation (20) and equation (28) in equation (26), we obtain

$$8\pi G = -\frac{\left\{(n+1){k_1}^2\alpha - \left(\frac{(n+1)^2}{2}\right){k_1}^2 + 8[(1-\alpha)T]^{\frac{-2(n+1)}{(1-\alpha)}+2}\right\}}{\left\{\frac{(1-\alpha)^2(1+\gamma)}{T^{b-2}}.\left(\frac{-2K^2}{d}.\frac{1}{k_1}.\frac{T^{(c+1)}}{(c+1)} + L\right) - 3\eta_0\left(\frac{1}{T^b}.\left(\frac{-2K^2}{d}.\frac{1}{k_1}.\frac{T^{(c+1)}}{(c+1)} + L\right)\right)^S.\frac{(n+1)}{2}.k_1[(1-\alpha)T]\right\}} \tag{33}$$

From equation (10), we obtain

$$\Lambda(t) = \frac{\left\{(n+1){k_1}^2\alpha - \left(\frac{(n+1)^2}{2}\right){k_1}^2 + 8[(1-\alpha)T]^{\frac{-2(n+1)}{(1-\alpha)}+2}\right\}.\left\{\frac{1}{T^b}.\left(\frac{-2K^2}{d}.\frac{1}{k_1}.\frac{T^{(c+1)}}{(c+1)} + L\right) + \frac{K^2}{2[(1-\alpha)T]^{\frac{2(n+1)}{(1-\alpha)}}}\right\}}{\left\{\frac{(1-\alpha)^2(1+\gamma)}{T^{b-2}}.\left(\frac{-2K^2}{d}.\frac{1}{k_1}.\frac{T^{(c+1)}}{(c+1)} + L\right) - 3\eta_0\left(\frac{1}{T^b}.\left(\frac{-2K^2}{d}.\frac{1}{k_1}.\frac{T^{(c+1)}}{(c+1)} + L\right)\right)^S.\frac{(n+1)}{2}.k_1[(1-\alpha)T]\right\}}$$

$$+\frac{(n^2+4n+1){k_1}^2}{2[(1-\alpha)T]^2} - \frac{12}{[(1-\alpha)T]^{\frac{2(n+1)}{(1-\alpha)}}} \tag{34}$$

As we have taken bulk viscosity as

$$\eta = \eta_0.\left[\frac{1}{T^b}.\left(\frac{-2K^2}{d}.\frac{1}{k_1}.\frac{T^{(c+1)}}{(c+1)} + L\right)\right]^S \tag{35}$$

Also from equation (6), the total pressure is defined as

$$P = p - 3\eta H.$$

So we obtain

$$P = \frac{\gamma}{T^b}.\left\{\frac{-2K^2}{d}.\frac{1}{k_1}.\frac{T^{(c+1)}}{(c+1)} + L\right\} - 3.\eta_0.\left\{\frac{1}{T^b}.\left(\frac{-2K^2}{d}.\frac{1}{k_1}.\frac{T^{(c+1)}}{(c+1)} + L\right)\right\}^S.\frac{(n+1)}{2}.\frac{k_1}{[(1-\alpha)T]}. \tag{36}$$

Hubble parameter is defined as,

$$H = \frac{1}{3}\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right),$$

$$H = \frac{(n+1)}{2}.\frac{k_1}{[(1-\alpha)T]}. \tag{37}$$

Volume of the universe is defined as follows,

$$V = ABC = k_3.[(1-\alpha)T]^{\frac{3(n+1)}{2(1-\alpha)}}. \tag{38}$$

Scale factor turns out to be

$$a(t) = {k_3}^{1/3}.[(1-\alpha)T]^{\frac{(n+1)}{2(1-\alpha)}}. \tag{39}$$

The anisotropy parameter is

$$A_m = \frac{2(n-1)^2}{3(n+1)^2}. \tag{40}$$

The shear scalar is given by

$$\sigma^2 = \frac{(n-1)^2}{4}.\frac{{k_1}^2}{[(1-\alpha)(k_1 t+k_2)]^2}. \tag{41}$$

The expansion scalar is

$$\theta = \frac{3(n+1)k_1}{[(1-\alpha)(k_1 t+k_2)]}. \tag{42}$$

So, the ratio of the shear scalar, and the expansion scalar is

$$\frac{\sigma}{\theta} = \frac{(n-1)}{3(n+1)}. \tag{43}$$

5. The model in absence of magnetism:

Now we discuss the model in absence of magnetism.

Substituting K=0 in equation (28), we get

$$\rho = \frac{L}{(k_1 t + k_2)^b} \tag{44}$$

From equation (32) gives pressure of the model as,

$$p = \frac{\gamma L}{(k_1 t + k_2)^b}, \tag{45}$$

equation (33) reduces to,

$$8\pi G = -\frac{\left\{(n+1){k_1}^2\alpha - \left(\frac{(n+1)^2}{2}\right){k_1}^2 + 8[(1-\alpha)T]^{\frac{-2(n+1)}{(1-\alpha)}+2}\right\}}{\left\{\frac{(1-\alpha)^2(1+\gamma)}{T^{b-2}}L - 3\eta_0\left(\frac{L}{T^b}\right)^S . \frac{(n+1)}{2} . k_1[(1-\alpha)T]\right\}} \ . \tag{46}$$

The value of cosmological constant Λ is

$$\Lambda(t) = \frac{\left\{(n+1){k_1}^2\alpha - \left(\frac{(n+1)^2}{2}\right){k_1}^2 + 8[(1-\alpha)T]^{\frac{-2(n+1)}{(1-\alpha)}+2}\right\}}{\left\{\frac{(1-\alpha)^2(1+\gamma)}{T^{b-2}}L - 3\eta_0\left(\frac{L}{T^b}\right)^S . \frac{(n+1)}{2} . k_1[(1-\alpha)T]\right\}} . \frac{L}{T^b} + \frac{(n^2+4n+1){k_1}^2}{2[(1-\alpha)T]^2} - \frac{12}{[(1-\alpha)T]^{\frac{2(n+1)}{(1-\alpha)}}} .$$

The bulk viscosity is

$$\eta = \eta_0 . \left(\frac{L}{T^b}\right)^S$$

(47)

The total pressure is

$$P = \frac{\gamma L}{T^b} - 3.\eta_0 . \left\{\frac{L}{T^b} . \right\}^S . \frac{(n+1)}{2} . \frac{k_1}{[(1-\alpha)T]} \tag{48}$$

6. Graphical Representation

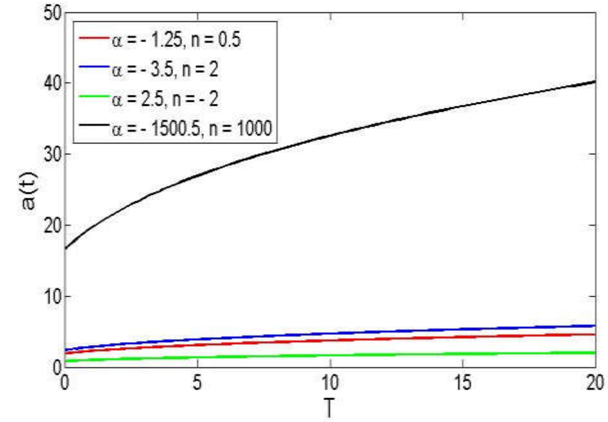


Fig. 1 The variation of scale factor along with cosmic time

From equations (39) we have plotted fig. 1, which shows that the variations of scale factors with cosmic time T for $k_1 = 1, k_2 = 1.5, k_3 = 2$, the scale factor shown with black line increases faster than others whereas green line varies slower than others.

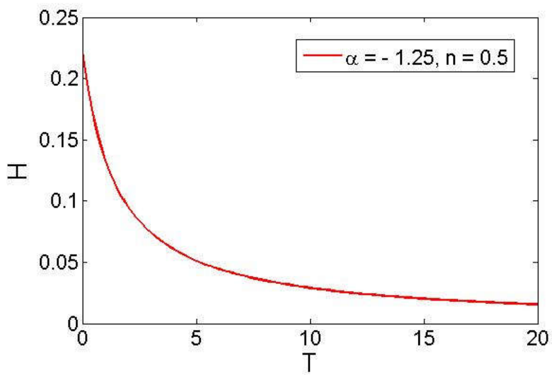


Fig. 2 Variation of Hubble parameter along with cosmic time

From equations (37) we have plotted fig. 2, which shows that the Hubble parameter decreases with cosmic time for $\alpha = -1.25, k_1 = 1, k_2 = 1.5,$ and $n = 0.5$. We also observe the same nature for different values of α, shown as overlapping over red line.

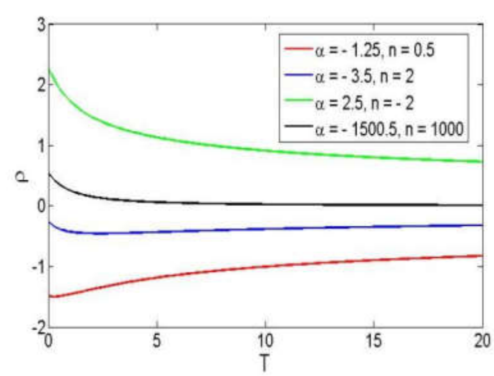


Fig. 3 Variation ofthe density with time**(2D)**

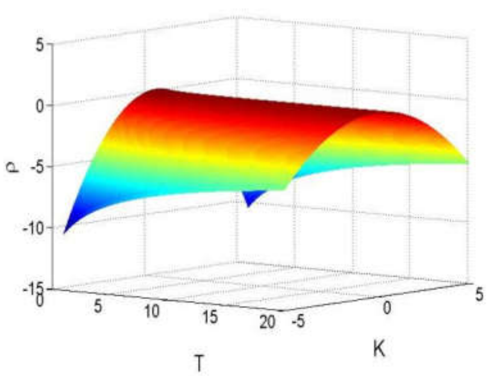
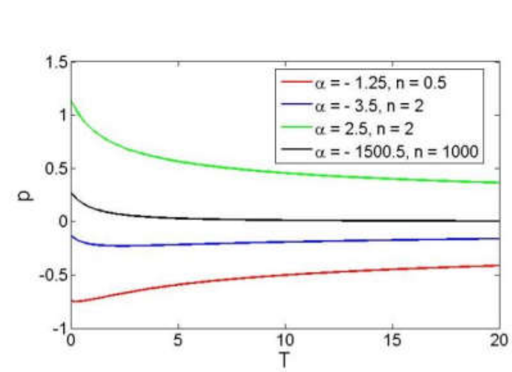


Fig. 4 Variation of density with time (3D)

Using equation (28) we have plotted fig. 3, It shows that the density reaches to zero from negative value, shown by red and blue line whereas green and black lines represents decreasing density, at late times to zero for K=2 , $k_1 = 1, k_2 = 1.5, k_3 = 2$, $L = 1$ $\gamma = 0.5$ (in 2-D diagram). Fig. 4 is plotted for different values of K from -5 to 5. It also gives symmetry with center K=0, for $\alpha = 1.25, k_1 = 1, k_2 = 1.5, k_3 = 2,\ K = 2, L = 1$ $\gamma = 0.5$ and $n = 0.5$.



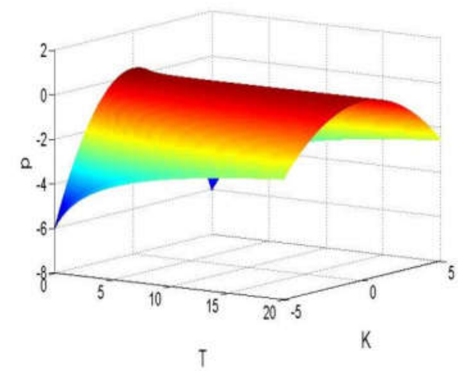
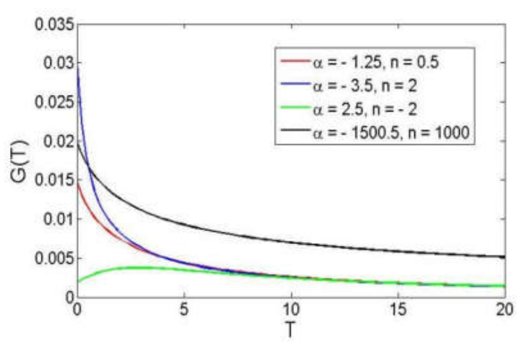


Fig. 5 Variation of pressure with time**(2D)** **Fig. 6** Variation of pressure with time (3D)

From equations (32) we have plotted fig. 5. Here red and black line overlapped, green line shows the decreasing equilibrium pressure to zero, starting from positive value for K=2 ,$k_1 = 1, k_2 = 1.5, k_3 = 2$, $L = 1$ $\gamma = 0.5$ (in 2-D diagram). Fig. 6, for different values of K from -5 to 5 also gives symmetry nature with center K=0, for $\alpha = -1.25, k_1 = 1, k_2 = 1.5, k_3 = 2,\ K = 2, L = 1$ $\gamma = 0.5$ and $n = 0.5$.



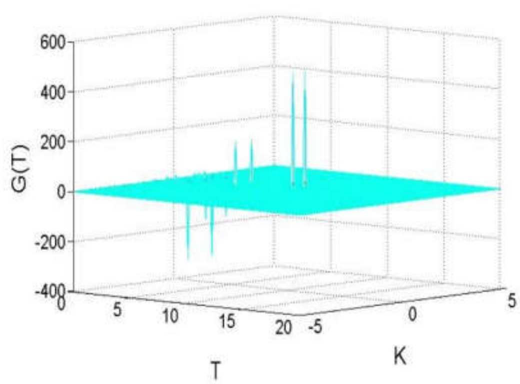
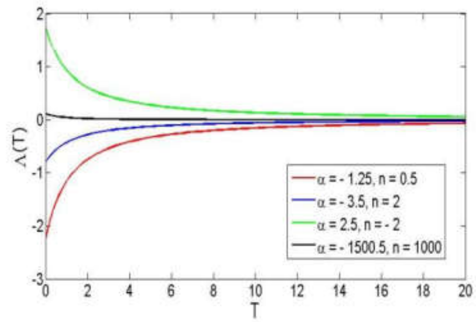


Fig. 7 Variation of [G(T)] with time**(2D)** **Fig. 8** Variation of [G(T)] with time (3D)

From equations (33) we have plotted fig. 7. It shows that all the lines of the gravitational constant [G(t)] reaches to zero in late times for K=2 (in 2-D diagram) . Fig. 8, for different values of K from -5 to 5 shows sometimes upwards to zero and sometime downwards movement for different values of K, here$\alpha = -1.25, k_1 = 1, k_2 = 1.5, k_3 = 2$, $K = 2, L = 1, \gamma = 0.5, s = 4, \pi = \frac{22}{7}, \eta = 0.4$ and $n = 0.5$.



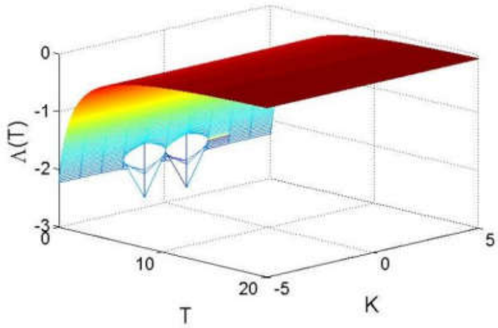
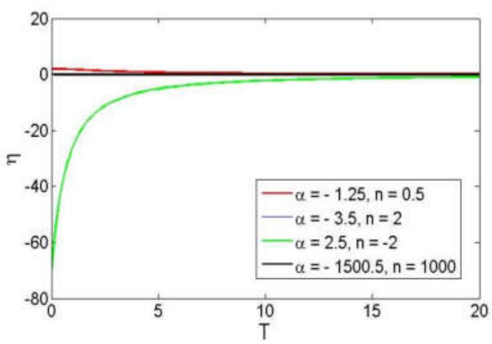


Fig. 9 Variation of $\Lambda(T)$ with time**(2D)** **Fig. 10** Variation of $\Lambda(T)$ with time (3D)

From equations (34) we have plotted fig. 9. It shows that green and black lines representing the cosmological constant $\Lambda(T)$ reaches positive to zero whereas red and blue lines decreases negative to zero in late times for K=2 (in 2-D diagram). In fig. 10, we have a symmetry for different values of K from -5 to 5 for $\alpha = -1.25,\ k_1 = 1,\ k_2 = 1.5, k_3 = 2\ , K = 2, L = 1, \gamma = 0.5, s = 4, \pi = \frac{22}{7}, \eta = 0.4$ and $n = 0.5$.



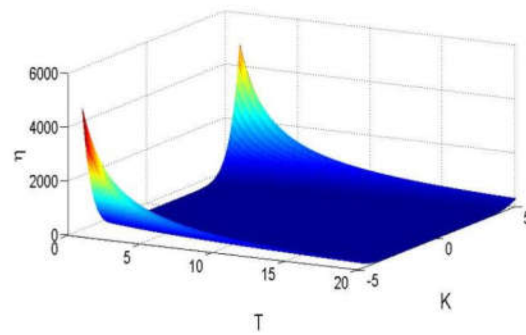
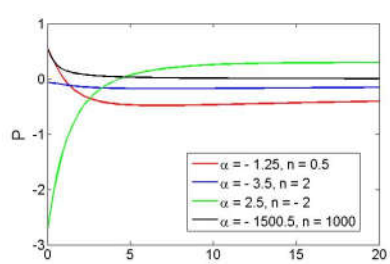


Fig. 11 Variation of η with time**(2D)** **Fig. 12** Variation of η with time (3D)

From equations (35) we have plotted fig. 11. It shows that the bulk viscosity ηrepresented by red , blue and black lines reaches positive uptoto zero, whereas green line reaches negative to zero but blue line constantly remain zero in late times for K=2 (in 2-D diagram). In fig. 12, we have plotted graphs for different values of K from -5 to 5 with $\alpha = -1.25, k_1 = 1, k_2 = 1.5, k_3 = 2,\ K = 2, L = 1, \gamma = 0.5, s = 4, \pi = \frac{22}{7}, \eta = 0.4$ and $n = 0.5$.



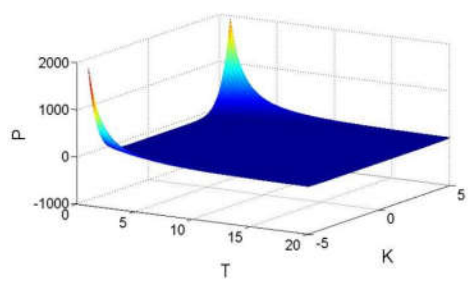


Fig. 13 Variation of pressure with time**(2D)** **Fig.14**Variation of pressure with time (3D)

Using equations (36) we have plotted fig. 13.Here we see that only the black line shows positive nature of the total pressure P, but red line moves from positive to negative.The blue always remains negative, and green moves from negative to positive in late times for K=2 (in 2-D). From fig. 14,we havegraphs for different values of K ranging -5 to 5. It also represents the same behavior for different values of K, with$\alpha = -1.25, k_1 = 1, k_2 = 1.5, k_3 = 2\ , K = 2, L = 1, \gamma = 0.5, s = 4, \pi = \frac{22}{7}, \eta = 0.4$ and $n = 0.5$.

7. Conclusion

In this paper we have studied the Bianchi Type V barotropic fluid with magnetic field in general relativity. The model (24) starts with a big bang at $= 0$, and the expression in the model decreases,as the time increases. The spatial volume (V) increases, as time (T) increases(for $n \neq -1$ or $\alpha \neq 1$). The matter density $\rho \to \infty$ when $T \to 0$, and $\rho \to 0$ when $T \to \infty$, provided $\gamma > -1$ and $n > -1$.

The model (24) has a point type singularity at $T = 0$ [Mac Callum (39)]. Shear scalar (σ) increases as $n > \rho$, $\alpha < 1$. Time (t) decreases and σ increases as T increases. Since $\frac{\sigma}{\theta} \neq 0$, hence anisotropy is maintained throughout. However at $n = 1$, the model (24) isotropizes. Hence the model (24) represents an inflationary scenario.From equation (46) it is evident that the gravitational constant G(t) increases with time for (K=0) . When $t \to \infty$ then $\eta \to 0$, and when $t \to 0$ then $\eta \to \infty$ is consistent with previously obtained results by Bali et al. [**29**].

It is observed that in presence of the magnetic field, pressure$p$, energy density ρ and cosmological constant Λ varies from negative to zero at late times. The gravitational constant G approaches to zero, when cosmic time tends to infinity.

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p-k Generalised C-Series and Its Properties

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Abstract

In this paper, we introduce a new $p-k$ generalised C- series. We find the integral representation of "$p-k$ generalised C- series and its properties". The Riemann-Liouville fractional order integral and derivative of the function have been derived in the paper. Some results previously given by Ali, M.F. et al.(2023), Gehlot et al.(2012), Sharma, M. et al.(2012)and Prabhakar, T.R.(1971) follow as particular cases of $p-k$ generalised C series.

Mathematics Subject Classification (2020) : 26A33, 33C60,33E20.

Keywords:C-series, R- series, M- series, k-series and special functions.

1 Introduction

The main aim of this paper is to introduce "$p-k$ generalised C series and find its integral representation.

Section 2 describes C-series, R- series, M- series, k-series.

Section3 Introduces new $p-k$ generalised C series and shows its integral representation, fractional order integral and derivative of the $p-k$ generalised C- series and some basic properties of it.

2 Preliminaries

2.1 Definition

Fractional Integral Operator:

The left sided Riemann-Liouville fractional integral of order α which is defined and denoted [12, P.4] for $Re(\alpha) > 0$ is as below

$$_{a}I_{x}^{\alpha} = _{a}D_{x}^{-\alpha} = \frac{1}{\Gamma\alpha}\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt, \tag{2.1}$$

here α could be real or fraction or complex.

Special case for $a = 0$

$$_{0}I_{x}^{\alpha} = _{0}D_{x}^{-\alpha} = \frac{1}{\Gamma\alpha}\int_{0}^{x}(x-t)^{\alpha-1}f(t)\,dt. \tag{2.2}$$

The right-sided Riemann-Liouville fractional integral of order α is defined and denoted [12, P.5] for $Re(\alpha) > 0$ is as below

$$ {}_xI_b^\alpha = {}_x D_b^{-\alpha} = \frac{1}{\Gamma\alpha}\int_x^b (t-x)^{\alpha-1} f(t)\, dt. \tag{2.3}$$

Special case for $b = \infty$

$$ {}_xI_\infty^\alpha = {}_x D_\infty^{-\alpha} = \frac{1}{\Gamma\alpha}\int_x^\infty (t-x)^{\alpha-1} f(t)\, dt. \tag{2.4}$$

Fractional Derivative Operator

The left sided Riemann-Liouville fractional derivative of order α is defined and denoted [12, P.52] for $Re(\alpha) > 0$ is as below

$$ {}_aD_x^\alpha f(x) = {}_a D_x^m \, {}_aD_x^{-(m-\alpha)} f(x)$$

$$\begin{cases} \frac{1}{\Gamma(m-\alpha)}(\frac{d}{dx})^m \int_a^x (x-t)^{m-\alpha-1} f(t)\, dt, & m-1 < \alpha < m \\ (\frac{d}{dx})^{m-1} f(x), \quad if & \alpha = m-1, \end{cases} \tag{2.5}$$

here $m = [\alpha] + 1$ where $[\alpha]$ denotes the integer part of $Re(\alpha)$ not exceeding α.
Special case for $a = 0$

$$ {}_0D_x^\alpha f(x) = {}_0 D_x^m \, {}_0D_x^{-(m-\alpha)} f(x)$$

$$\begin{cases} \frac{1}{\Gamma(m-\alpha)}(\frac{d}{dx})^m \int_0^x (x-t)^{m-\alpha-1} f(t)\, dt, & m-1 < \alpha < m \\ (\frac{d}{dx})^{m-1} f(x), \quad if & \alpha = m-1, \end{cases} \tag{2.6}$$

The Right sided Riemann-Liouville fractional derivative of order α is defined and denoted [12, P.63] for $Re(\alpha) > 0$ is as below

$$ {}_xD_b^\alpha f(x) = \begin{cases} \frac{(-1)^m}{\Gamma(m-\alpha)}(\frac{d}{dx})^m \int_x^b (t-x)^{m-\alpha-1} f(t)\, dt, & m-1 < \alpha < m \\ (\frac{d}{dx})^{m-1} f(x), \quad if & \alpha = m-1, \end{cases} \tag{2.7}$$

here $m = [\alpha] + 1$ where $[\alpha]$ denotes the integer part of $Re(\alpha)$ not exceeding α.
Special case for $b = \infty$

$$ {}_xD_\infty^\alpha f(x) = \begin{cases} \frac{(-1)^m}{\Gamma(m-\alpha)}(\frac{d}{dx})^m \int_x^\infty (t-x)^{m-\alpha-1} f(t)\, dt, & m-1 < \alpha < m \\ (\frac{d}{dx})^{m-1} f(x), \quad if & \alpha = m-1, \end{cases} \tag{2.8}$$

Special functions of Fractional Calculus

In 2009, **M-series** defined by Sharma and Jain [12] (see also [11]), as

$${}_pM_q^{\alpha,\beta}(a_1,\cdots a_p;c_1\cdots c_q;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_1)_p}{(c_1)_k\cdots(c_q)_k}\frac{z^k}{\Gamma(\alpha k+\beta)}={}_p\,M_q^{\alpha,\beta}(z), \quad (2.9)$$

where $\alpha,\beta,z\in\mathbb{C}$, $Re(\alpha)>0a_i,c_j\neq 0,-1,-2\cdots(i=1,2,\cdots p;j=1,2,\cdots q)$, $(a_i)_k$ and $(c_j)_k$ are known Pochhammer symbols.

M series is absolutely convergent for all z if $p\leq q$, it is convergent if $p=q+1$ and divergent if $p>q+1$.

In 2012, the **K- series** defined by Gehlot et al [3], as

$${}_pK_q^{(\beta,\eta)_m}(a_1,\cdots a_p:b_1,\cdots b_q,(\beta,\eta)_m;z)={}_p\,K_q^{(\beta,\eta)_m}(z)=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{n=1}^{q}(b_n)_k}\frac{z^k}{\prod_{i=1}^{m}\Gamma(\eta_i k+\beta_i)}, \quad (2.10)$$

where $a_j,b_n,\beta_i\in\mathbb{C}$; $\eta_i\in\mathbb{R}$, $(j=1,2\cdots p;n=1,2\cdots q;i=1,2\cdots m)$.

The series (2.10) is defined when none of the parameter $b_n(n=1,2\cdots q)$ is negative integer or zero. If any numerator parameter $a_j(j=1,2\cdots p)$ is negative integer or zero, the series terminates into polynomial in z.

The convergence/ divergence of the series is subject to the following conditions:

(i) if $p<q+\sum_{i=1}^{m}\eta_i$, then the series is absolutely convergent for all $z\in\mathbb{C}$.

(ii) if $p=q+\sum_{i=1}^{m}\eta_i$, then the series is absolutely convergent

if $|z|<\prod_{i=1}^{m}(|\eta_i|)^{\eta_i}$, and if $|z|=\prod_{i=1}^{m}(|\eta_i|)^{\eta_i}$ then $Re(\sum_{n=1}^{q}(b_n)+\sum_{i=1}^{m}(\beta_i)-\sum_{j=1}^{p}(a_j))>\frac{2+q+m-p}{2}$.

In 2016, the **R- series** defined by M.F. Ali et. al [1], as

$${}_pR_q^{\alpha,\beta}(a_1,\cdots a_p;b_1,\cdots b_q;z)={}_p\,R_q^{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{\Gamma(\alpha k+\beta)k!}, \quad (2.11)$$

here, p upper parameters $a_1,a_2\cdots a_p$ and q lower parameters $b_1,b_2,\cdots b_q,\alpha\in\mathbb{C}$, $Re(\alpha)>0,m>0$ and $(a_j)_k,(b_j)_k$ are pochhammer symbols.

In 2023, the **C-series** was introduced and defined by Mohd.Farman Ali et al.[2]. The C-series is defined as:

$${}_{p,m}C_{q,n}^{(\alpha,\beta)l}(a_1,\cdots a_p;b_1,\cdots b_q,(\alpha,\beta)_l;z)={}_{p,m}\,C_{q,n}^{(\alpha,\beta)l}(z)$$
$$=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{z^k}{\prod_{r=1}^{l}\Gamma(\alpha_r k+\beta_r)},$$

$${}_{p,m}C_{q,n}^{(\alpha,\beta)l}(a_1,\cdots a_p;b_1,\cdots b_q,(\alpha_1,\beta_1),\cdots(\alpha_l,\beta_l);z)=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{km}}{\prod_{j=1}^{q}(b_j)_{kn}}\frac{z^k}{\prod_{r=1}^{l}\Gamma(\alpha_r k+\beta_r)}, \quad (2.12)$$

here, $\alpha_i,\beta_i\in\mathbb{C}$, $Re(\alpha_i)>0,Re(\beta_i)>0$, $(a_j)_{mk},(b_j)_{nk}$ are pochhammer symbols.

The series (2.12) is defined when none of the parameter $b_j(j=1,2\cdots q)$ is negative integer or zero. If any numerator parameter $a_i(i=1,2\cdots p)$ is negative integer or zero, the series terminates into polynomial in z.

By applying ratio test, the series is
(i) convergent for all z, when $p \leq q + \sum_{r=1}^{l} \alpha_r$,
(ii) convergent for $|z| = 1$ when $|\prod_{r=1}^{l} \Gamma(\alpha_r)^{\alpha_r}| > 1$.

2.2 p-k generalised C-series

The authors have introduced and defined the $p-k$ generalised C-series as

$${}_t^p\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1,\cdots a_r; b_1,\cdots b_q; (\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l); z)\right]_s^k = {}_t^p\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(z)\right]_s^k$$

$$= \sum_{n=0}^{\infty} \frac{{}_{p_1}(a_1)_{nm,k_1}\cdots{}_{p_r}(a_r)_{nm,k_r}}{{}_{t_1}(b_1)_{nd,s_1}\cdots{}_{t_q}(b_q)_{nd,s_q}} \frac{z^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)},$$

$${}_t^p\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(z)\right]_s^k = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q} {}_{t_j}(b_j)_{nd,s_j}} \frac{z^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}, \tag{2.13}$$

where $p_i, k_i, t_j, s_j, \alpha_\delta \in \mathbb{R}^+ - (0)$, $z, a_i, b_j, \beta_\delta \in \mathbb{C}$, $(i = 1,2\cdots r; j = 1,2\cdots q; \delta = 1,2\cdots l)$, $Re(a_i) > 0$, $Re(b_j) > 0$, m, d are non negative integers.

The series (2.13) is **not** defined when either of the parameters b_j or $s_j (j = 1,2\cdots q)$ is negative integer or zero. The series terminates into polynomial in z if any parameter of either a_i or k_i, $(i = 1,2\cdots r)$ is negative integer or zero.

Convergence criteria of $p-k$ generalised C-series when applying ratio test is

(i) if $qd + \sum_{i=1}^{l} \alpha_i > mr$, then the series is absolutely convergent for all values of $z \in \mathbb{C}$,

(ii)if $qd + \sum_{i=1}^{l} \alpha_i = mr$, then the series is absolutely convergent for all values of $|z| < \frac{(t_1\cdots t_q)^d}{(p_1\cdots p_r)^m} d^{dq} m^{-mr} \prod_{\delta=1}^{l} (\alpha_\delta)^{\alpha_\delta}$,

(iii)if $qd + \sum_{i=1}^{l} \alpha_i = mr$ and $|z| = |\frac{(t_1\cdots t_q)^d}{(p_1\cdots p_r)^m} d^{dq} m^{-mr} \prod_{\delta=1}^{l} (\alpha_\delta)^{\alpha_\delta}|$,then series is absolutely convergent when $\sum_{j=1}^{q} \frac{b_j}{s_j} + \sum_{\delta=1}^{l} \beta_\delta - \sum_{i=1}^{r} \frac{a_i}{k_i} > \frac{2+q+l-r}{2}$.

Particular Cases of p-k generalised C-series When some particular values given to parameters of equation (2.13)

(i) Putting $p = k = t = s = 1$ and $m = 1 = d$ in the equation (2.13), the equation reduces to the K-series defined by Kuldeep Singh Gehlot [3],

$${}_1^1\left[\ {}_{r,1}C_{q,1}^{(\alpha,\beta)l}(z)\right]_1^1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)} = {}_r K_q^{(\alpha,\beta)_l}(z). \tag{2.14}$$

(ii) Putting $p = k = t = s = 1$, $m = 1 = d$ and $l = 1$ in the equation (2.13), it reduces to M-series introduced by Sharma [12],

$${}_1^1\left[\ {}_{r,1}C_{q,1}^{(\alpha,\beta)1}(z)\right]_1^1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{\Gamma(\alpha n+\beta)} = {}_r M_q^{(\alpha,\beta)}(z). \tag{2.15}$$

(iii) Putting $p = k = t = s = 1$, $m = 1 = d$, $l = 2$ and $\alpha_1 = 1, \alpha_2 = \alpha, \beta_1 = 1, \beta_2 = \beta$ in the equation (2.13), the equation reduces to the R-series defined by M.F.Ali et.al [1],

$${}_1^1\left[\ {}_{r,1}C_{q,1}^{(\alpha,\beta)2}(z)\right]_1^1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{\Gamma(\alpha n+\beta)\Gamma(n+1)} = {}_r R_q^{(\alpha,\beta)}(z). \tag{2.16}$$

(iv) Putting $p=k=t=s=1$ in the equation (2.13), the equation reduces to the C series defined by Mohd.Farman Ali et al.[2],

$$ {}_1^1\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(z)\right]_1^1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_{nm}}{\prod_{j=1}^{q}(b_j)_{nd}} \frac{z^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)} = {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(z). \quad (2.17) $$

(v) Putting $p=k=t=s=1$, $m=1=d$, $l=2$, $\alpha_1=1, \alpha_2=\alpha, \beta_1=1, \beta_2=\beta$, $r=1$ and $q=0$ i.e no lower parameter q in the equation (2.13), the equation reduces to the generalised Mittag-Leffler function introduced by Prabhakar [9] in (1971),

$$ {}_1^1\left[\ {}_{1,1}C_{0,1}^{(\alpha,\beta)2}(z)\right]_1^1 = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{\Gamma(\alpha n+\beta)\Gamma(n+1)} = E_{\alpha,\beta}^{a}(z). \quad (2.18) $$

(vi) If there is no upper and lower parameter i.e $r=0=q$, $m=1=d$, $p=k=t=s=1$ and $l=1$ in the equation (2.13), the equation reduces to Mittag-Leffler function by Wiman [13] in 1905,

$$ {}_1^1\left[\ {}_{0,1}C_{0,1}^{(\alpha,\beta)1}(z)\right]_1^1 = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha_1 n+\beta_1)} = E_{\alpha_1,\beta_1}(z). \quad (2.19) $$

(vii) Putting $r=0=q, m=1=d, l=1, \alpha_1=\alpha, \beta_1=1$ and $p=k=t=s=1$ in in the equation (2.13), the equation reduces to Mittag-Leffler function defined by Gosta Mittag-Leffler [8] in 1903,

$$ {}_1^1\left[\ {}_{0,1}C_{0,1}^{(\alpha,1)1}(z)\right]_1^1 = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} = E_{\alpha}(z). \quad (2.20) $$

(viii) Putting $p=k=t=s=1$, $m=1=d, l=1, \alpha_1=1, \beta_1=1$ in the equation (2.13), the equation reduces to generalised hypergeometric function [7],

$$ {}_1^1\left[\ {}_{r,1}C_{q,1}^{(1,1)1}(z)\right]_1^1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{\Gamma(n+1)} = {}_r F_q(a_1, a_2 \cdots a_r; b_1, b_2 \cdots b_q; z). \quad (2.21) $$

(ix) Putting $p=k=t=s=1$, $m=1=d, l=1, \alpha_1=1, \beta_1=1$ and no upper and lower parameter i.e $r=0=q$ in the equation (2.13), the equation reduces to an exponential function,

$$ {}_1^1\left[\ {}_{0,1}C_{0,1}^{(1,1)1}(z)\right]_1^1 = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = e^z. \quad (2.22) $$

(x) Putting $p=k=t=s=1$, $l=1, \alpha_1=1, \beta_1=1$ in the equation (2.13), the equation reduces to the general Wright function [7],

$$ {}_1^1\left[\ {}_{r,m}C_{q,d}^{(1,1)1}(z)\right]_1^1 = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_{nm}}{\prod_{j=1}^{q}(b_j)_{nd}} \frac{z^n}{\Gamma(n+1)} = \theta\, {}_r\psi_q \left[\begin{matrix}(a_1,m)\cdots(a_r,m);\\ (b_1,d)\cdots(b_q,d)\end{matrix}; |z\right], \quad (2.23) $$

where $\theta = \frac{\prod_{j=1}^{q}\Gamma b_j}{\prod_{i=1}^{r}\Gamma a_i}$.

2.3 Integral Representation of uniform convergent p-k generalised C-series

Theorem 1. For $m = d, q = r, p = t;\ b_i, s_i, a_i, k_i > 0,\ (i = 1,2 \cdots r; j = 1,2 \cdots q)$, $\frac{b_i}{s_i} > \frac{a_i}{k_i}$, then

$$
{}_p^p\left[\ {}_{r,m}C_{r,m}^{(\alpha,\beta)l}(z)\right]_s^k = \sum_{n=0}^{\infty} \prod_{i=1}^{r} \frac{1}{\beta\left(\left(\frac{a_i}{k_i}\right),\left(\frac{b_i}{s_i}-\frac{a_i}{k_i}\right)\right)} \int_0^1 x^{\left(\frac{a_i}{k_i}-1\right)}(1-\ x)^{\left(\frac{b_i}{s_i}-\frac{a_i}{k_i}\right)-1}\ {}^p_p\left[\ {}_{0,m}C_{0,m}^{(\alpha,\beta)l}(zx^m)\right]_s^k dx. \tag{2.24}
$$

Proof: Substituting $m = d, q = r, p = t$ in the equation (2.13), we have

$$
{}_p^p\left[\ {}_{r,m}C_{r,m}^{(\alpha,\beta)l}(z)\right]_s^k = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{r} {}_{p_j}(b_j)_{nm,s_j}} \frac{z^n}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n+\beta_\delta)}.
$$

Using equation (2.19), (2.20) and (2.22) of [4] in the RHS of above equation, we get

$$
= \sum_{n=0}^{\infty} \prod_{i=1}^{r} \frac{p_i^{nm}\Gamma(\frac{b_i}{s_i})\Gamma(\frac{a_i}{k_i}+nm)}{p_i^{nm}\Gamma(\frac{b_i}{s_i}+nm)\Gamma(\frac{a_i}{k_i})} \frac{z^n}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n+\beta_\delta)}
$$

$$
= \sum_{n=0}^{\infty} \prod_{i=1}^{r} \frac{\Gamma(\frac{b_i}{s_i})}{\Gamma(\frac{a_i}{k_i})\Gamma(\frac{b_i}{s_i}-\frac{a_i}{k_i})} \frac{\Gamma(\frac{a_i}{k_i}+nm)\Gamma(\frac{b_i}{s_i}-\frac{a_i}{k_i})}{\Gamma(\frac{b_i}{s_i}+nm)} \frac{z^n}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n+\beta_\delta)}
$$

We know the first Eulerian integral is

$$
\beta(m,n) = \int_0^1 x^{(m-1)}(1-x)^{(n-1)}\, dx = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}
$$

$$
= \sum_{n=0}^{\infty} \prod_{i=1}^{r} \frac{\Gamma(\frac{b_i}{s_i})}{\Gamma(\frac{a_i}{k_i})\Gamma(\frac{b_i}{s_i}-\frac{a_i}{k_i})} \frac{z^n}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n+\beta_\delta)} \int_0^1 x^{(\frac{a_i}{k_i}+nm-1)}(1-x)^{(\frac{b_i}{s_i}-\frac{a_i}{k_i})-1}\, dx
$$

$$
= \sum_{n=0}^{\infty} \prod_{i=1}^{r} \frac{1}{\beta((\frac{a_i}{k_i}),(\frac{b_i}{s_i}-\frac{a_i}{k_i}))} \int_0^1 x^{(\frac{a_i}{k_i}-1)}(1-x)^{(\frac{b_i}{s_i}-\frac{a_i}{k_i})-1} \frac{(zx^m)^n}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n+\beta_\delta)}\, dx
$$

$$
= \sum_{n=0}^{\infty} \prod_{i=1}^{r} \frac{1}{\beta((\frac{a_i}{k_i}),(\frac{b_i}{s_i}-\frac{a_i}{k_i}))} \int_0^1 x^{(\frac{a_i}{k_i}-1)}(1-x)^{(\frac{b_i}{s_i}-\frac{a_i}{k_i})-1}\ {}^p_p\left[\ {}_{0,m}C_{0,m}^{(\alpha,\beta)l}(zx^m)\right]_s^k dx.
$$

Hence proved.

Corollary 1.1 For $m = d, q = r, p = t\ ;\ b_i, s_i, a_i, k_i > 0,\ (i = 1,2 \cdots r)$, $\frac{b_i}{s_i} > \frac{a_i}{k_i}$, then

$$ {}_{p}^{p}\Big[{}_{r,m}C_{r,m}^{(\alpha,\beta)l}(z)\Big]_{s}^{k} = $$

$$ \frac{\Gamma(\frac{b_1}{s_1})}{\Gamma(\frac{a_1}{k_1})\Gamma(\frac{b_1}{s_1}-\frac{a_1}{k_1})}\int_0^1 x^{(\frac{a_1}{k_1}-1)}(1-x)^{(\frac{b_1}{s_1}-\frac{a_1}{k_1})-1}\,{}_{p-1}^{p-1}\Big[{}_{r-1,m}C_{r-1,m}^{(\alpha,\beta)l}(zx^m)\Big]_{s-1}^{k-1} dx. \qquad (2.25) $$

Proof: Substituting $m=d, q=r, p=t$ in the equation (2.13), we have

$$ {}_{p}^{p}\Big[{}_{r,m}C_{r,m}^{(\alpha,\beta)l}(z)\Big]_{s}^{k} = \sum_{n=0}^{\infty}\prod_{i=1}^{r}\frac{{}_{p_i}(a_i)_{nm,k_i}}{{}_{p_i}(b_i)_{nm,s_i}}\frac{z^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)} \qquad (2.26) $$

Taking RHS of the equation(2.26), we have

$$ = \sum_{n=0}^{\infty}\prod_{i=2}^{r}\frac{{}_{p_i}(a_i)_{nm,k_i}}{{}_{p_i}(b_i)_{nm,s_i}}\frac{{}_{p_1}(a_1)_{nm,k_1}}{{}_{p_1}(b_1)_{nm,s_1}}\frac{z^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)} $$

By using equation (2.19),(2.20) and (2.22) of [4], we have

$$ = \sum_{n=0}^{\infty}\prod_{i=2}^{r}\frac{{}_{p_i}(a_i)_{nm,k_i}}{{}_{p_i}(b_i)_{nm,s_i}}\frac{{p_1}^{nm}(\frac{a_1}{k_1})_{nm}}{{p_1}^{nm}(\frac{b_1}{s_1})_{nm}}\frac{z^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)} $$

$$ = \sum_{n=0}^{\infty}\prod_{i=2}^{r}\frac{{}_{p_i}(a_i)_{nm,k_i}}{{}_{p_i}(b_i)_{nm,s_i}}\frac{\Gamma(\frac{a_1}{k_1}+nm)}{\Gamma(\frac{a_1}{k_1})}\frac{\Gamma(\frac{b_1}{s_1})}{\Gamma(\frac{b_1}{s_1}+nm)}\frac{\Gamma(\frac{b_1}{s_1}-\frac{a_1}{k_1})}{\Gamma(\frac{b_1}{s_1}-\frac{a_1}{k_1})}\frac{z^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)} $$

By using definition of Beta function

$$ = \sum_{n=0}^{\infty}\prod_{i=2}^{r}\frac{{}_{p_i}(a_i)_{nm,k_i}}{{}_{p_i}(b_i)_{nm,s_i}}\frac{\Gamma(\frac{b_1}{s_1})}{\Gamma\frac{a_1}{k_1}\Gamma(\frac{b_1}{s_1}-\frac{a_1}{k_1})}\frac{z^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\int_0^1 x^{(\frac{a_1}{k_1}+nm-1)}(1-x)^{(\frac{b_1}{s_1}-\frac{a_1}{k_1})-1}\,dx $$

$$ = \frac{\Gamma(\frac{b_1}{s_1})}{\Gamma\frac{a_1}{k_1}\Gamma(\frac{b_1}{s_1}-\frac{a_1}{k_1})}\int_0^1 x^{(\frac{a_1}{k_1}-1)}(1-x)^{(\frac{b_1}{s_1}-\frac{a_1}{k_1})-1}\,dx\sum_{n=0}^{\infty}\prod_{i=2}^{r}\frac{{}_{p_i}(a_i)_{nm,k_i}}{{}_{p_i}(b_i)_{nm,s_i}}\frac{(zx^m)^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)} $$

$$ = \frac{\Gamma(\frac{b_1}{s_1})}{\Gamma(\frac{a_1}{k_1})\Gamma(\frac{b_1}{s_1}-\frac{a_1}{k_1})}\int_0^1 x^{(\frac{a_1}{k_1}-1)}(1-x)^{(\frac{b_1}{s_1}-\frac{a_1}{k_1})-1}\,{}_{p-1}^{p-1}\Big[{}_{r-1,m}C_{r-1,m}^{(\alpha,\beta)l}(zx^m)\Big]_{s-1}^{k-1} dx. $$

Hence proved.

Theorem 2. For $\alpha_\delta\in\mathbb{R};\beta_\delta\in\mathbb{C}$; $(\alpha_\delta)>0$,$Re(\beta_\delta)>0$ for $\delta=1\cdots l$, $\beta,\gamma\in\mathbb{R}$, $(\beta-\gamma)>0, x\in\mathbb{C}, Re(x)>0$, $g\in I^{+}$ then we have

$$ {}_{t}^{p}\Big[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(z)\Big]_{s}^{k} = $$

$$ gz^{\gamma-\beta}\int_0^{\infty} e^{-\frac{x^g}{z^g}}x^{\beta-\gamma-1}\,{}_{t}^{p}\Big[{}_{r,m}C_{q,d}^{(\alpha,\beta)l+1}(a_1,\cdots a_r;b_1\cdots b_q;(\alpha,\beta)_l,(\tfrac{1}{g},\tfrac{\beta-\gamma}{g});x)\Big]_{s}^{k} dx. \quad (2.27) $$

Proof: Taking RHS of the equation (2.27), we have

$$= gz^{\gamma-\beta}\int_0^\infty e^{-\frac{x^g}{z^g}} x^{\beta-\gamma-1}{}_t^p\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l+1}(a_1,\cdots a_r; b_1\cdots b_q; (\alpha,\beta)_l, (\frac{1}{g},\frac{\beta-\gamma}{g}); x)\right]_s^k dx.$$

Using the equation (2.13), we get

$$= gz^{\gamma-\beta}\int_0^\infty e^{-\frac{x^g}{z^g}} x^{\beta-\gamma-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^r\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^q\ {}_{t_j}(b_j)_{nd,s_j}}\frac{x^n}{\prod_{\delta=1}^l\Gamma(\alpha_\delta n+\beta_\delta)}\frac{dx}{\Gamma(\frac{n}{g}+\frac{\beta-\gamma}{g})}.$$

Order of integration and summation can be interchanged under the conditions of uniform/ absolute convergence of the series as discussed in the subsection (2.2).

$$= gz^{\gamma-\beta}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^r\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^q\ {}_{t_j}(b_j)_{nd,s_j}}\frac{1}{\Gamma(\frac{n}{g}+\frac{\beta-\gamma}{g})}\frac{1}{\prod_{\delta=1}^l\Gamma(\alpha_\delta n+\beta_\delta)}\int_0^\infty e^{-\frac{x^g}{z^g}} x^{\beta-\gamma-1}x^n\,dx$$

Let $\frac{x^g}{z^g} = u$; $x = zu^{\frac{1}{g}}$; $dx = \frac{z}{g}u^{\frac{1}{g}-1}du$

$$= z^{\gamma-\beta}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^r\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^q\ {}_{t_j}(b_j)_{nd,s_j}}\frac{1}{\Gamma(\frac{n}{g}+\frac{\beta-\gamma}{g})}\frac{z^{\beta-\gamma+n}}{\prod_{\delta=1}^l\Gamma(\alpha_\delta n+\beta_\delta)}\int_0^\infty e^{-u}u^{\frac{\beta-\gamma+n}{g}-1}\,du$$

Using gamma function definition

$$= z^{\gamma-\beta}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^r\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^q\ {}_{t_j}(b_j)_{nd,s_j}}\frac{z^{\beta-\gamma+n}}{\prod_{\delta=1}^l\Gamma(\alpha_\delta n+\beta_\delta)}$$

$$= {}_t^p\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(z)\right]_s^k.$$

Hence proved.

Corollary 2.1 Putting $p = k = t = s = 1$ in the equation (2.27), we get the integral representation of C-series defined by Mohd. Farman Ali et al. [2].

$${}_1^1\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(z)\right]_1^1 =$$

$$gz^{\gamma-\beta}\int_0^\infty e^{-\frac{x^g}{z^g}} x^{\beta-\gamma-1}{}_1^1\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l+1}(a_1,\cdots a_r; b_1\cdots b_q; (\alpha,\beta)_l, (\frac{1}{g},\frac{\beta-\gamma}{g}); x)\right]_1^1 dx$$

$$= \left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(z)\right] \tag{2.28}$$

Corollary 2.2 Putting $p = k = t = s = 1$; $m = 1 = d$ in the equation (2.27), we get the integral representation of K-series defined by Kuldeep Sigh Gehlot et. al[3]

$${}_1^1\left[\ {}_{r,1}C_{q,1}^{(\alpha,\beta)l}(z)\right]_1^1 =$$

$$gz^{\gamma-\beta}\int_0^\infty e^{-\frac{x^g}{z^g}} x^{\beta-\gamma-1}{}_1^1\left[\ {}_{r,1}C_{q,1}^{(\alpha,\beta)l+1}(a_1,\cdots a_r; b_1\cdots b_q; (\alpha,\beta)_l, (\frac{1}{g},\frac{\beta-\gamma}{g}); x)\right]_1^1 dx$$

$$={}_p K_q^{(\alpha,\beta)_l}(z). \tag{2.29}$$

Corollary 2.3 Putting $p=k=t=s=1$; $m=1=dl=1$ in the equation(2.27), we get the integral representation of M-series

$$\frac{1}{1}\left[{}_{r,1}C_{q,1}^{(\alpha,\beta)1}(z)\right]_1^1 =$$

$$gz^{\gamma-\beta}\int_0^\infty e^{-\frac{x^g}{z^g}}x^{\beta-\gamma-1}\frac{1}{1}\left[{}_{r,1}C_{q,1}^{(\alpha,\beta)2}(a_1,\cdots a_r;b_1\cdots b_q;(\alpha,\beta)_1,(\frac{1}{g},\frac{\beta-\gamma}{g});x)\right]_1^1 dx$$

$$={}_r M_q^{(\alpha,\beta)_1}(z). \tag{2.30}$$

Corollary 2.4 Putting $p=k=t=s=1$; $m=1=d$, $l=1$, $\alpha_1=1$, $\beta_1=1$ in the equation (2.27), we get integral representation of ${}_rF_q(z)$

$$\frac{1}{1}\left[{}_{r,1}C_{q,1}^{(1,1)}(z)\right]_1^1 =$$

$$gz^{\gamma-\beta}\int_0^\infty e^{-\frac{x^g}{z^g}}x^{\beta-\gamma-1}\frac{1}{1}\left[{}_{r,1}C_{q,1}^{(1,1)2}(a_1,\cdots a_r;b_1\cdots b_q;(1,1),(\frac{1}{g},\frac{\beta-\gamma}{g});x)\right]_1^1 dx$$

$$={}_r F_q(z). \tag{2.31}$$

Corollary 2.5 Putting $p=k=t=s=1$; $d=1$, $l=l+1$, $\alpha_{l+1}=\beta_{l+1}=1$, $r=1$, $a_1=\rho$ and no lower parameter q in the eqation (2.27), we get integral representation of generalised Mittag Leffler function studied by Saxena R.K. et al.[10]

$$\frac{1}{1}\left[{}_{1,m}C_{0,1}^{(\alpha,\beta)l+1}(z)\right]_1^1 =$$

$$gz^{\gamma-\beta}\int_0^\infty e^{-\frac{x^g}{z^g}}x^{\beta-\gamma-1}\frac{1}{1}\left[{}_{1,m}C_{0,1}^{(\alpha,\beta)l+2}(\rho;-;(\alpha,\beta)_l,(1,1),(\frac{1}{g},\frac{\beta-\gamma}{g});x)\right]_1^1 dx$$
$$=E_{\rho,m}[(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l):z]=E_{\rho,m}[(\alpha_\delta,\beta_\delta)_{1,l}:z] \tag{2.32}$$

Corollary 2.6 Putting $p=k=t=s=1$; $m=1=d$, $l=1$ and there are no upper and lower parameter in the equation (2.27), we get the integral representation of generalised Mittag Leffler function $E_{\alpha,\beta}(z)$

$$\frac{1}{1}\left[{}_{0,1}C_{0,1}^{(\alpha,\beta)1}(z)\right]_1^1$$
$$=gz^{\gamma-\beta}\int_0^\infty e^{-\frac{x^g}{z^g}}x^{\beta-\gamma-1}\frac{1}{1}\left[{}_{0,1}C_{0,1}^{(1,1)2}(-;-,(\alpha,\beta)(\frac{1}{g},\frac{\beta-\gamma}{g});x)\right]_1^1 dx$$

$$=E_{\alpha,\beta}(z). \tag{2.33}$$

Theorem 3. For $x\in\mathbb{C}$, $\lambda,\mu,\gamma\in\mathbb{R}$, and $\lambda>0$, $\mu>0$, $\gamma>0$, $\gamma+\mu>\lambda$, then we have

$$\frac{p}{t}\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1,\cdots a_r;b_1,\cdots b_q;(\lambda,\gamma+\mu),(\alpha_2,\beta_2)\cdots(\alpha_l,\beta_l);z)\right]_s^k$$

$$= \frac{1}{\lambda\Gamma(\gamma+\mu-\lambda)} \int_0^1$$

$$(1-x^{\frac{1}{\lambda}})^{\gamma+\mu-\lambda-1} {}_t^p\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\lambda,\lambda), (\alpha_2,\beta_2) \cdots (\alpha_l,\beta_l); xz)\right]_s^k dx. \tag{2.34}$$

Proof: Taking RHS of the equation (2.34), we have

$$= \frac{1}{\lambda\Gamma(\gamma+\mu-\lambda)} \int_0^1$$

$$(1-x^{\frac{1}{\lambda}})^{\gamma+\mu-\lambda-1} {}_t^p\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\lambda,\lambda), (\alpha_2,\beta_2) \cdots (\alpha_l,\beta_l); xz)\right]_s^k dx$$

.

Using the equation (2.13), we have

$$= \frac{1}{\lambda\Gamma(\gamma+\mu-\lambda)} \int_0^1 (1 - x^{\frac{1}{\lambda}})^{\gamma+\mu-\lambda-1} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q} {}_{t_j}(b_j)_{nd,s_j}} \frac{(xz)^n}{\prod_{\delta=2}^{l} \Gamma(\alpha_\delta n+\beta_\delta)\Gamma(\lambda n+\lambda)} dx$$

Order of integration and summation can be interchanged under the conditions of uniform/absolute convergence of the series as discussed in the subsection (2.2).

$$= \frac{1}{\lambda\Gamma(\gamma+\mu-\lambda)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q} {}_{t_j}(b_j)_{nd,s_j}} \frac{z^n}{\prod_{\delta=2}^{l} \Gamma(\alpha_\delta n+\beta_\delta)\Gamma(\lambda n+\lambda)} \int_0^1 (1-x^{\frac{1}{\lambda}})^{\gamma+\mu-\lambda-1} x^n \, dx$$

Let $x^{\frac{1}{\lambda}} = u; x = u^\lambda; dx = \lambda u^{\lambda-1}$

$$= \frac{1}{\Gamma(\gamma+\mu-\lambda)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q} {}_{t_j}(b_j)_{nd,s_j}} \frac{z^n}{\prod_{\delta=2}^{l} \Gamma(\alpha_\delta n+\beta_\delta)\Gamma(\lambda n+\lambda)} \int_0^1 (1-u)^{\gamma+\mu-\lambda-1} u^{\lambda n+\lambda-1} \, du$$

Using definition of Beta function, we have

$$= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q} {}_{t_j}(b_j)_{nd,s_j}} \frac{z^n}{\prod_{\delta=2}^{l} \Gamma(\alpha_\delta n+\beta_\delta)\Gamma(\lambda n+(\gamma+\mu))}$$

$$= {}_t^p\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\lambda,\gamma+\mu), (\alpha_2,\beta_2) \cdots (\alpha_l,\beta_l); z)\right]_s^k.$$

Hence proved.

Corollary 3.1 Putting $p = k = t = s = 1$ in the equation (2.34), we get the integral representation of C-series studied by Mohd. Farman Ali et. al[2]

$${}_1^1\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\lambda,\gamma+\mu), (\alpha_2,\beta_2) \cdots (\alpha_l,\beta_l); z)\right]_1^1$$

$$= \frac{1}{\lambda\Gamma(\gamma+\mu-\lambda)} \int_0^1$$

$$(1 - x^{\frac{1}{\lambda}})^{\gamma+\mu-\lambda-1}\,{}_1^1\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\lambda,\lambda), (\alpha_2,\beta_2) \cdots (\alpha_l,\beta_l); xz)\right]_1^1 dx. \tag{2.35}$$

Corollary 3.2 Putting $p = k = t = s = 1$ and $m = 1 = d$ in the equation (2.34), we get the integral representation of K-series studied by Kuldeep Singh Gehlot et al.[3,P.393]

$$\begin{aligned}&{}_1^1\left[\ {}_{r,1}C_{q,1}^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\lambda,\gamma+\mu), (\alpha_2,\beta_2) \cdots (\alpha_l,\beta_l); z)\right]_1^1\\ &= \frac{1}{\lambda\Gamma(\gamma+\mu-\lambda)}\int_0^1\\ &(1 - x^{\frac{1}{\lambda}})^{\gamma+\mu-\lambda-1}\,{}_1^1\left[\ {}_{r,1}C_{q,1}^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\lambda,\lambda), (\alpha_2,\beta_2) \cdots (\alpha_l,\beta_l); xz)\right]_1^1 dx\\ &= {}_r K_q^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\lambda,\gamma+\mu), (\alpha_2,\beta_2) \cdots (\alpha_l,\beta_l); z).\end{aligned} \tag{2.36}$$

Theorem 4. For $\rho, \sigma, a \in \mathbb{R}$ and $\rho > 0$, $\sigma > 0$, then we have

$$\begin{aligned}&{}_t^p\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\alpha_1,\beta_1) \cdots (\alpha_l,\beta_l); \tfrac{a}{z^\sigma})\right]_s^k\\ &= \\ &z^\rho \int_0^\infty x^{\rho-1}e^{-xz}\ {}_t^p\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l+1}(a_1, \cdots a_r; b_1, \cdots b_q; (\sigma,\rho), (\alpha_1,\beta_1) \cdots (\alpha_l,\beta_l); ax^\sigma)\right]_s^k dx.\end{aligned} \tag{2.37}$$

Proof: Taking RHS of the equation (2.37), we have

$$= z^\rho \int_0^\infty x^{\rho-1}e^{-xz}\ {}_t^p\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l+1}(a_1, \cdots a_r; b_1, \cdots b_q; (\sigma,\rho), (\alpha_1,\beta_1) \cdots (\alpha_l,\beta_l); ax^\sigma)\right]_s^k dx$$

Using the equation (2.13), we have

$$= z^\rho \int_0^\infty x^{\rho-1}e^{-xz} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^q\ {}_{t_j}(b_j)_{nd,s_j}} \frac{(ax^\sigma)^n}{\prod_{\delta=1}^l \Gamma(\alpha_\delta n + \beta_\delta)\Gamma(\sigma n + \rho)}\, dx$$

Order of integration and summation can be interchanged under the conditions of uniform/absolute convergence of the series as discussed in the subsection (2.2).

$$= z^\rho \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^q\ {}_{t_j}(b_j)_{nd,s_j}} \frac{(a)^n}{\prod_{\delta=1}^l \Gamma(\alpha_\delta n + \beta_\delta)\Gamma(\sigma n + \rho)} \int_0^\infty x^{\sigma n+\rho-1}e^{-xz}\, dx$$

Using definition of Gamma function

$$= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^r\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^q\ {}_{t_j}(b_j)_{nd,s_j}} \frac{1}{\prod_{\delta=1}^l \Gamma(\alpha_\delta n + \beta_\delta)} (\frac{a}{z^\sigma})^n$$

$$= {}_t^p\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\alpha_1,\beta_1) \cdots (\alpha_l,\beta_l); \tfrac{a}{z^\sigma})\right]_s^k$$

Hence Proved.

Corollary 4.1: Putting $p = k = t = s = 1$ in the equation (2.37), we get the integral representation of C-series studied by Mohd. Farman Ali et. al[2]

$$
{}_1^1\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1,\cdots a_r;b_1,\cdots b_q;(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l);\tfrac{a}{z^\sigma})\right]_1^1
$$

$$
= z^\rho \int_0^\infty x^{\rho-1}e^{-xz}\ {}_1^1\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l+1}(a_1,\cdots a_r;b_1,\cdots b_q;(\sigma,\rho),(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l);ax^\sigma)\right]_1^1 dx. \tag{2.38}
$$

Corollary 4.2: Putting $p=k=t=s=1$, $m=1=d$ in the equation (2.37), we get the integral representation of K-series studied and introduced by Kuldeep Singh Gehlot et. al [3]

$$
{}_1^1\left[{}_{r,1}C_{q,1}^{(\alpha,\beta)l}(a_1,\cdots a_r;b_1,\cdots b_q;(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l);\tfrac{a}{z^\sigma})\right]_1^1
$$

$$
= z^\rho \int_0^\infty x^{\rho-1}e^{-xz}{}_1^1\left[{}_{r,1}C_{q,1}^{(\alpha,\beta)l+1}(a_1,\cdots a_r;b_1,\cdots b_q;(\sigma,\rho),(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l);ax^\sigma)\right]_1^1 dx
$$

$$
={}_r K_q^{(\alpha,\beta)l}[a_1,\cdots a_r;b_1,\cdots b_q;(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l);\tfrac{a}{z^\sigma}]. \tag{2.39}
$$

Corollary 4.3: Putting $p=k=t=s=1$; $d=1$, $l=l+1$, $\alpha_{l+1}=\beta_{l+1}=1$, $r=1$, $a_1=\rho$ and no lower parameter q in the eqation (2.37), we get the integral representation of generalised Mittag Leffler function defined by Saxena R.K. et al.[10]

$$
{}_1^1\left[{}_{1,m}C_{0,1}^{(\alpha,\beta)l+1}(\rho;-;(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l)(1,1);\tfrac{a}{z^\sigma})\right]_1^1
$$

$$
= z^\rho \int_0^\infty x^{\rho-1}e^{-xz}{}_1^1\left[{}_{1,m}C_{0,1}^{(\alpha,\beta)l+2}(\rho;-;(\sigma,\rho),(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l)(1,1);ax^\sigma)\right]_1^1 dx
$$

$$
= E_{\rho,m}[(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l);\tfrac{a}{z^\sigma}]. \tag{2.40}
$$

2.4 k-integral Laplace transform of uniform convergent p-k generalised C-series

Theorem:5. For $\alpha>0$; $\mu\geq 1$, $\theta, x\in\mathbb{C}$, then

$$
L_\mu\left[{}_t^p\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l+1}(a_1,\cdots a_r;b_1,\cdots b_q;(\sigma,\tfrac{1}{\mu})(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l);(x^{\sigma\mu}z)\right]_s^k;\theta\right]
$$

$$
=\frac{\theta^{-\frac{1}{\mu\alpha}}}{\mu}{}_t^p\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1,\cdots a_r;b_1,\cdots b_q;(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l);(\tfrac{z}{\theta^{\frac{\sigma}{\alpha}}})\right]_s^k. \tag{2.41}
$$

To avoid confusion between parameter k of $p-k$ generalised C-series and k-integral Laplace transform [5], μ is used in place of k in k-integral Laplace transform.

Proof: Taking LHS of (2.41) and using definition of k- integral Laplace transform, we get

$$
=\int_0^\infty e^{-\theta^{\frac{1}{\alpha}}x^\mu}\ {}_t^p\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l+1}(a_1,\cdots a_r;b_1,\cdots b_q;(\sigma,\frac{1}{\mu})(\alpha_1,\beta_1)\cdots(\alpha_l,\beta_l);(x^{\sigma\mu}z)\right]_s^k dx
$$

Using equation (2.13), we get

$$= \int_0^\infty e^{-\theta^{\frac{1}{\alpha}} x^\mu} \sum_{n=0}^\infty \frac{{}_{p_1}(a_1)_{nm,k_1} \cdots {}_{p_r}(a_r)_{nm,k_r}}{{}_{t_1}(b_1)_{nd,s_1} \cdots {}_{t_q}(b_q)_{nd,s_q}} \frac{(x^{\sigma\mu} z)^n}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n + \beta_\delta)} \frac{dx}{\Gamma(\sigma n + \frac{1}{\mu})},$$

Order of integration and summation can be interchanged under the conditions of uniform convergence of the series as discussed in the subsection (2.2).

$$= \sum_{n=0}^\infty \frac{{}_{p_1}(a_1)_{nm,k_1} \cdots {}_{p_r}(a_r)_{nm,k_r}}{{}_{t_1}(b_1)_{nd,s_1} \cdots {}_{t_q}(b_q)_{nd,s_q}} \frac{z^n}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n + \beta_\delta)} \frac{1}{\Gamma(\sigma n + \frac{1}{\mu})} \int_0^\infty e^{-\theta^{\frac{1}{\alpha}} x^\mu} x^{\sigma n \mu}\, dx,$$

Let $x^\mu = y$ i.e $x = y^{\frac{1}{\mu}}$ and $dx = \frac{1}{\mu} y^{\frac{1}{\mu}-1} dy$

$$= \sum_{n=0}^\infty \frac{{}_{p_1}(a_1)_{nm,k_1} \cdots {}_{p_r}(a_r)_{nm,k_r}}{{}_{t_1}(b_1)_{nd,s_1} \cdots {}_{t_q}(b_q)_{nd,s_q}} \frac{z^n}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n + \beta_\delta)} \frac{1}{\Gamma(\sigma n + \frac{1}{\mu})} \int_0^\infty \frac{1}{\mu} e^{-\theta^{\frac{1}{\alpha}} y} y^{\sigma n + \frac{1}{\mu} - 1}\, dy,$$

Using definition of Euler's second equation, we get

$$= \frac{\theta^{-\frac{1}{\mu\alpha}}}{\mu} \sum_{n=0}^\infty \frac{{}_{p_1}(a_1)_{nm,k_1} \cdots {}_{p_r}(a_r)_{nm,k_r}}{{}_{t_1}(b_1)_{nd,s_1} \cdots {}_{t_q}(b_q)_{nd,s_q}} \frac{z^n}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n + \beta_\delta)} \frac{1}{\theta^{\frac{\sigma n}{\alpha}}},$$

$$= \frac{\theta^{-\frac{1}{\mu\alpha}}}{\mu} {}_t^p\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1, \cdots a_r; b_1, \cdots b_q; (\alpha_1, \beta_1) \cdots (\alpha_l, \beta_l); (\tfrac{z}{\theta^{\frac{\sigma}{\alpha}}})\right]_s^k.$$

Hence proved.

2.5 Fractional order Integral and Differentiation of uniform convergent p-k generalised C-series

Theorem:6. Let $\gamma > 0, Re(\beta_i) > 0, \alpha_i > 0, \forall i = 1,2 \cdots l$. The special case (i.e $a = 0$) of left-sided Riemann-Liouville fractional integral given by equation (2.2), then

$${}_0I_z^\gamma [x^{\beta_1 - 1}\ {}_t^p\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a x^{\alpha_1})\right]_s^k = z^{\beta_1 + \gamma - 1}\, {}_t^p\left[{}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a z^{\alpha_1})\right]_s^k. \quad (2.42)$$

Proof: Using equations (2.2) and (2.13) in the LHS of the equation (2.42), we get

$$= \frac{1}{\Gamma\gamma} \int_0^z \frac{x^{\beta_1 - 1}}{(z-x)^{1-\gamma}} \sum_{n=0}^\infty \frac{\prod_{i=1}^r\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^q\ {}_{t_j}(b_j)_{nd,s_j}} \frac{(a x^{\alpha_1})^n}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n + \beta_\delta)}\, dx$$

Order of integration and summation can be interchanged under the conditions of uniform/absolute convergence of the series as discussed in the subsection (2.2).

$$= \frac{1}{\Gamma\gamma} \sum_{n=0}^\infty \frac{\prod_{i=1}^r\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^q\ {}_{t_j}(b_j)_{nd,s_j}} \frac{1}{\prod_{\delta=1}^{l} \Gamma(\alpha_\delta n + \beta_\delta)} \int_0^z a^n x^{\beta_1 + \alpha_1 n - 1} (z-x)^{\gamma - 1}\, dx$$

Let $x = zu$ and $dx = zdu$

$$=\frac{1}{\Gamma\gamma}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{a^n z^{\beta_1+\alpha_1 n+\gamma-1}}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\int_0^1 u^{\beta_1+\alpha_1 n-1}(1-u)^{\gamma-1}\,du$$

Using Beta function, we get

$$=\frac{z^{\beta_1+\gamma-1}}{\Gamma\gamma}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{a^n z^{\alpha_1 n}}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\frac{\Gamma(\beta_1+\alpha_1 n)\Gamma\gamma}{\Gamma(\beta_1+\gamma+\alpha_1 n)}$$

$$=z^{\beta_1+\gamma-1}\,{}_t^p\Big[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1,\cdots a_r;b_1,\cdots b_q;(\alpha_1,\beta_1+\gamma)(\alpha_2,\beta_2)\cdots(\alpha_l,\beta_l);az^{\alpha_1})\Big]_s^k$$

$$=z^{\beta_1+\gamma-1}\,{}_t^p\Big[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(az^{\alpha_1})\Big]_s^k.$$

Hence proved.

Theorem 7. Let $\gamma>0, Re(\beta_i)>0, \alpha_i>0, \forall\ i=1,2\cdots l$. The special case (i.e. $b=\infty$) of right-sided Riemann-Liouville fractional integral given by equation (2.4), then

$$ {}_zI_\infty^\gamma[x^{-\gamma-\beta_1}\ {}_t^p\Big[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(ax^{-\alpha_1})\Big]_s^k=z^{-\beta_1}\,{}_t^p\Big[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(az^{-\alpha_1})\Big]_s^k. \quad (2.43)$$

Proof: Using equations (2.4) and (2.13) in the LHS of the equation (2.43), we get

$$=\frac{1}{\Gamma\gamma}\int_z^\infty\frac{x^{-\gamma-\beta_1}}{(x-z)^{1-\gamma}}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{(ax^{-\alpha_1})^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\,dx$$

Order of integration and summation can be interchanged under the conditions of uniform/absolute convergence of the series as discussed in the subsection (2.2).

$$=\frac{1}{\Gamma\gamma}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{1}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\int_z^\infty a^n x^{-\gamma-\beta_1-\alpha_1 n}(x-z)^{\gamma-1}\,dx$$

Let $x=\frac{z}{u}$ and $dx=-\frac{z}{u^2}du$

$$=\frac{1}{\Gamma\gamma}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{a^n z^{-\beta_1-\alpha_1 n}}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\int_0^1 u^{\beta_1+\alpha_1 n-1}(1-u)^{\gamma-1}\,du$$

Using Beta function, we get

$$=\frac{1}{\Gamma\gamma}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{a^n z^{-\beta_1-\alpha_1 n}}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\frac{\Gamma(\beta_1+\alpha_1 n)\Gamma(\gamma)}{\Gamma(\gamma+\beta_1+\alpha_1 n)}$$

$$=z^{-\beta_1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{z^{-\alpha_1 n}a^n}{\prod_{\delta=2}^{l}\Gamma(\alpha_\delta n+\beta_\delta)\Gamma(\gamma+\beta_1)+\alpha_1 n)}$$

$$=z^{-\beta_1}\,{}_t^p\Big[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1,\cdots a_r;b_1,\cdots b_q;(\alpha_1,\beta_1+\gamma)(\alpha_2,\beta_2)\cdots(\alpha_l,\beta_l);az^{-\alpha_1})\Big]_s^k$$

$$= z^{-\beta_1}{}_{t}^{p}\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(az^{-\alpha_1})\right]_s^k.$$

Hence proved.

Theorem 8. Let $\gamma > 0, Re(\beta_i) > 0, \alpha_i > 0, \forall i = 1,2 \cdots l$. The special case (i.e. $a = 0$) of left-sided Riemann-Liouville fractional derivative given by equation (2.6), then

$${}_0D_z^{\gamma}[x^{\beta_1-1}\ {}_{t}^{p}\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(ax^{\alpha_1})\right]_s^k = z^{\beta_1-\gamma-1}\ {}_{t}^{p}\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(az^{\alpha_1})\right]_s^k. \tag{2.44}$$

Proof: Using equations (2.6) and (2.13) in the LHS of the equation (2.44),we get

$$= \frac{1}{\Gamma(\rho-\gamma)}\left(\frac{d}{dz}\right)^{\rho}\int_0^z \frac{x^{\beta_1-1}}{(z-x)^{1+\gamma-\rho}}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{(ax^{\alpha_1})^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\,dx,$$

where $\rho = [\gamma] + 1$

Order of integration and summation can be interchanged under the conditions of uniform/absolute convergence of the series as discussed in the subsection (2.2).

$$= \frac{1}{\Gamma(\rho-\gamma)}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{1}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\left(\frac{d}{dz}\right)^{\rho}\int_0^z a^n x^{n\alpha_1+\beta_1-1}(z-x)^{\rho-\gamma-1}\,dx$$

Let $x = zu$ and $dx = zdu$

$$= \frac{1}{\Gamma(\rho-\gamma)}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{a^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\left(\frac{d}{dz}\right)^{\rho} z^{n\alpha_1+\beta_1+\rho-\gamma-1}\int_0^1 u^{n\alpha_1+\beta_1-1}(1-u)^{\rho-\gamma-1}\,du$$

Using Beta function we get

$$= \frac{1}{\Gamma(\rho-\gamma)}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{a^n z^{n\alpha_1+\beta_1-\gamma-1}}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\frac{\Gamma(n\alpha_1+\beta_1+\rho-\gamma)}{\Gamma(n\alpha_1+\beta_1-\gamma)}\frac{\Gamma(n\alpha_1+\beta_1)\Gamma(\rho-\gamma)}{\Gamma(n\alpha_1+\beta_1+\rho-\gamma)}$$

$$= z^{\beta_1-\gamma-1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{a^n z^{n\alpha_1}}{\prod_{\delta=2}^{l}\Gamma(\alpha_\delta n+\beta_\delta)\Gamma(n\alpha_1+(\beta_1-\gamma))}$$

$$= z^{\beta_1-\gamma-1}{}_{t}^{p}\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1,\cdots a_r; b_1,\cdots b_q; (\alpha_1,\beta_1-\gamma)(\alpha_2,\beta_2)\cdots(\alpha_l,\beta_l); az^{\alpha_1})\right]_s^k$$

$$= z^{\beta_1-\gamma-1}\ {}_{t}^{p}\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(az^{\alpha_1})\right]_s^k.$$

Hence proved.

Theorem 9. Let $\gamma > 0, Re(\beta_i) > 0, \alpha_i > 0, \forall\ i = 1,2 \cdots l$. The special case (i.e. $b = \infty$) of right-sided Riemann-Liouville fractional derivative given by equation (2.8), then

$${}_zD_{\infty}^{\gamma}[x^{\gamma-\beta_1}\ {}_{t}^{p}\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(ax^{-\alpha_1})\right]_s^k = z^{-\beta_1}\ {}_{t}^{p}\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(az^{-\alpha_1})\right]_s^k. \tag{2.45}$$

Proof: Using equations (2.8) and (2.13) in the LHS of the equation (2.45),we get

$$=\frac{1}{\Gamma(\rho-\gamma)}(-\frac{d}{dz})^{\rho}\int_{z}^{\infty}\frac{x^{\gamma-\beta_1}}{(x-z)^{1+\gamma-\rho}}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{(ax^{-\alpha_1})^n}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\,dx,$$

where $\rho=[\gamma]+1$

Order of integration and summation can be interchanged under the conditions of uniform/absolute convergence of the series as discussed in the subsection (2.2).

$$=\frac{1}{\Gamma(\rho-\gamma)}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{1}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}(-\frac{d}{dz})^{\rho}\int_{z}^{\infty}a^n x^{\gamma-\beta_1-\alpha_1 n}$$
$$(x-z)^{\rho-\gamma-1}\,dx$$

Let $x=\frac{z}{u}$ and $dx=-\frac{z}{u^2}du$

$$=\frac{1}{\Gamma(\rho-\gamma)}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{a^n(-\frac{d}{dz})^{\rho}z^{\rho-\beta_1-\alpha_1 n}}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\int_{0}^{1}u^{\beta_1+\alpha_1 n-\rho-1}(1-u)^{\rho-\gamma-1}\,du$$

Using Beta function, we get

$$=\frac{1}{\Gamma(\rho-\gamma)}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{a^n z^{-\beta_1-\alpha_1 n}}{\prod_{\delta=1}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\frac{\Gamma(\alpha_1 n+\beta_1)}{\Gamma(\alpha_1 n+\beta_1-\rho)}\frac{\Gamma(\alpha_1 n+\beta_1-\rho)\Gamma(\rho-\gamma)}{\Gamma(\alpha_1 n+\beta_1-\gamma)}$$

$$=z^{-\beta_1}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}\ {}_{p_i}(a_i)_{nm,k_i}}{\prod_{j=1}^{q}\ {}_{t_j}(b_j)_{nd,s_j}}\frac{a^n z^{-\alpha_1 n}}{\prod_{\delta=2}^{l}\Gamma(\alpha_\delta n+\beta_\delta)}\frac{1}{\Gamma(\alpha_1 n+\beta_1-\gamma)}$$

$$=z^{-\beta_1}{}_{t}^{p}\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(a_1,\cdots a_r;b_1,\cdots b_q;(\alpha_1,\beta_1-\gamma)(\alpha_2,\beta_2)\cdots(\alpha_l,\beta_l);az^{-\alpha_1})\right]_{s}^{k}$$

$$=z^{-\beta_1}{}_{t}^{p}\left[\ {}_{r,m}C_{q,d}^{(\alpha,\beta)l}(az^{-\alpha_1})\right]_{s}^{k}.$$

Hence proved.

2.6 A differential equation

Theorem:10 The $p-k$ generalised C-series for $m=d=1$ and $\alpha_\delta=1$ where $\delta=1,2\cdots l$ satisfies the differential equation

$$\left[\prod_{j=1}^{q}(\theta+\frac{b_j}{s_j}-1)\prod_{\delta=1}^{l}(\theta+\beta_\delta-1)-zA\prod_{i=1}^{r}(\theta+\frac{a_i}{k_i})\right]w=0, \qquad (2.46)$$

where $\theta=z\frac{d}{dz}$, $A=\frac{\prod_{i=1}^{r}p_i}{\prod_{j=1}^{q}t_j}$ and $w=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}{}_{p_i}(a_i)_{n,k_i}}{\prod_{j=1}^{q}{}_{t_j}(b_j)_{n,s_j}}\frac{z^n}{\prod_{\delta=1}^{l}\Gamma(n+\beta_\delta)}$,

for $q>r$ and no $\frac{b_j}{s_j}$ is negative or zero.

Proof: Using the equation(2.13) and the equation (2.20) of [4], we get the desired result.

Hence proved.

3 Conclusion

We have introduced and studied a "p-k generalised C-series". The function/series is a generalisation of K-series, M-series, R-series and C-series. This series satisfies a differential equation (2.46), which shows that the series is useful in solution of any real word problem, which is expressed in the form of differential equation (2.46). Further, we obtained k-Integral Laplace transform of the series to connect the series with integral transform to make it relevant for future usage in the field of various research of science and engineering. This piece of research also shows relevance of "p-k generalised C-series" by way of finding fractional order integration and differentiationof the series. As it is well known the importance of the fractional calculus in the field of science and engineering.

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I hereby declare that the particulars given above are true to the best of my knowledge and belief.

DR. M. L. SUKHWAL
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